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An Introduction to Shell Porosity

Many results of the following form have recently appeared:

THEOREM: *A function $f : \mathbb{R} \to \mathbb{R}$ has SOME SPECIFIC PROPERTY except on a σ-porous (or σ-symmetrically porous) set.*

To understand this theorem we give these definitions:

Definition 1 *Let E be a set in $\mathbb{R}$ and let x be any point. Then the porosity of E at x is defined by*

$$p(E;x) = \limsup_{h\to 0} \frac{\lambda(E;x,x+h)}{|h|}$$

where $\lambda(E;a,b) = \lambda(E;b,a)$ is the length of the largest open subinterval in $(a,b)\cap E^c$.

Definition 2 *Let E be a set in $\mathbb{R}$ and let x be any point. For $R > 0$ define $\gamma(E;x,R)$ as the supremum over*

$$\{h > 0 : \exists t > 0 \text{ with } t+h < R,\ (x-t-h, x-t)\cap E = \emptyset$$

$$\text{and } (x+t, x+t+h)\cap E = \emptyset\}.$$

Furthermore, define the symmetric porosity of E at x as

$$p^s(E;x) = \limsup_{R\to 0+} \frac{\gamma(E;x,R)}{R}.$$

A set E is called σ-porous (σ-symmetrically porous) if it can be written as a countable union of sets E_n where for any $x \in E_n$ the porosity (symmetric porosity) of E_n at x is positive.

Shell porosity is an extension of symmetric porosity from the real line into a metric space. We start with the definitions of porosity and shell porosity in a metric space (X,d).

Definition 3 *Let E be a set in the metric space (X,d). By $B_x(r)$ we mean the open ball centered at x of radius r, i. e. $\{y \in X : d(x,y) < r\}$. For $x \in X$ and $R > 0$ define $\Lambda(E;x,R)$ as the supremum of*

$$\{h > 0 : \exists\, z \in X \text{ with } B_z(h) \subset B_x(R) \cap E^c\}.$$

Furthermore, define the porosity of E at x as

$$p(E;x) = 2 \limsup_{R \to 0+} \frac{\Lambda(E;x,R)}{R}.$$

Definition 4 *Let $x \in X$ and $0 < r_1 < r_2$. The open shell about x of radii r_1 and r_2 is given by*

$$S_x(r_1,r_2) = B_x(r_2) \setminus \overline{B_x(r_1)}.$$

For $R > 0$, we define $\Gamma(E;x,R)$ as the supremum of

$$\{h > 0 : \exists\, t > 0 \text{ with } t + h < R \text{ and } S_x(t,t+h) \cap E = \emptyset\}.$$

The shell porosity of E at x is given by

$$p^s(E;x) = \limsup_{R \to 0+} \frac{\Gamma(E;x,R)}{R}.$$

First in this talk we will note the abundance of shell porous sets in the following sense (An observation first made by P.M. Gruber):

Theorem 1 *Let (X,d) be a complete metric space. The collection of all strongly shell porous compact subsets of X is a dense $\mathbf{G}_\delta$ subset of the space of compact subsets of X endowed with the Hausdorff metric.*

We will next look at the similarities and differences between shell porosity and porosity by extending results of L. Zajíček and M. J. Evans, P.D. Humke and K. Saxe. Then we will add S. J. Agronsky and A. M. Bruckner's idea of a totally porous set and T. Zamfirescu's notion of a hyperporous set to the mixture to see that among these types of porosity, only shell porosity yields the following:

Theorem 2 *If E is a closed σ-shell porous set in a complete metric space then E is totally disconnected.*

Finally, we will give two open questions from this study of porosities.