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Constructions Which Control Dimensions

Nowhere dense, perfect sets in $[0,1]$ are the focus of this investigation. They are analyzed in terms of the Hausdorff ($dim_{\mathcal{H}}$), packing ($dim_{\mathcal{P}}$), lower Minkowski ($dim_{\underline{M}}$) and upper Minkowski ($dim_{\overline{M}}$) dimension functions. For an arbitrary set F, it is necessary that

$$dim_{\mathcal{H}} F \le dim_{\mathcal{P}} F \le dim_{\overline{M}} F \text{ and } dim_{\mathcal{H}} F \le dim_{\underline{M}} F \le dim_{\overline{M}} F.$$

It is the objective of this investigation to produce a construction that realizes, after being given

$$0 < h < p < s < u < 1,$$

a set X with $h = dim_{\mathcal{H}} X$, $p = dim_{\mathcal{P}} X$, $s = dim_{\underline{M}} X$, and $u = dim_{\overline{M}} X$. The set $E = \{0\} \cup \{1/n : n \in \mathbb{N}\}$, is a simple set for which $dim_{\mathcal{H}} E = 0 = dim_{\mathcal{P}} E$, but $dim_{\underline{M}} E = 1/2 = dim_{\overline{M}} E$. Notice that the points $1/n$ approach 0 "more slowly" than the geometric series $\{2^{-n} : n \in \mathbb{N}\}$.

Lemma 1 *If $X = \cup_{n=1}^{\infty} W_n$, then $dim_{\mathcal{H}} X = \sup_n\{dim_{\mathcal{H}} W_n\}$ and $dim_{\mathcal{P}} X = \sup_n\{dim_{\mathcal{P}} W_n\}$. Likewise, if $X = \cup_{n=1}^{m} W_n$, then $dim_{\underline{M}} X = \sup_n\{dim_{\underline{M}} W_n\}$ and $dim_{\overline{M}} X = \sup_n\{dim_{\overline{M}} W_n\}$.*

Lemma 2 *If T is a similarity map, $T(E)$ has the same, respectively, Hausdorff and packing dimension as E.*

Eventually nonoverlapping dyadic intervals $\{L_{j,k}\}$ in $[0,1]$ will be picked and respectively subsets, $\{X_{j,k}\}$. Each $X_{j,k}$ will be a similar copy of a symmetric Cantor set, K^{ζ^j}, with $dim_{\mathcal{H}} K^{\zeta^j} = h$ and $dim_{\mathcal{P}} K^{\zeta^j} = p$. For $X = \cup_{j,k} X_{j,k}$, $dim_{\mathcal{H}} X = \sup_{j,k}\{dim_{\mathcal{H}} X_{j,k}\}$ and $dim_{\mathcal{P}} X = \sup_{j,k}\{dim_{\mathcal{P}} X_{j,k}\}$.

Let $\gamma > 0$ and $m_j = [\gamma^j]$. For large enough $j \ge j_0$ and an appropriate positive integer c, set $I_j = \left\{1, 2, \ldots, \left[2^{u\gamma^{j+c}}\right]\right\}$ and choose the dyadic intervals

$$L_{j,k} = \left[2^{m_j} + (k-1)2^{-m_{j+c}}, 2^{-m_j} + k2^{-m_{j+c}}\right],$$

where $k \in I_j$. For $k \in I_j$, place a reduced by $2^{-m_{j+c}}$ similar copy of K^{ζ^j} into $L_{j,k}$ and call it $X_{j,k}$. Finally, set

$$X = \{0\} \cup \bigcup_{j=j_0}^{\infty} \bigcup_{k \in I_j} X_{j,k}.$$