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Richard Fleissner, Department of Mathematics, WIU,
Macomb, Illinois 61455 and
James Foran, Department of Mathematics, UMKC,
Kansas City, Missouri 64108

A Note on the Garsia-Sawyer Class.

It is known that the Garsia-Sawyer class, GS, defined by $\int \log^+(n_f(y))dy < \infty$, where $n_f(y)$ denotes the Banach indicatrix of f, is not closed under addition. (See [1, p. 73] for background information.) The purpose of this note is to give a simple example of this.

Let E_n^k; $n = 0,1,2,\ldots$; $k = 1,2,\ldots,2^n$ denote the closed intervals contiguous to the Cantor middle-thirds set. That is, E_0^1 is the middle third of $[0,1]$, E_1^1 and E_1^2 are the middle thirds of $[0,1]\setminus E_0^1$ and so forth. On E_n^k the Cantor function takes the value $c(x) = (2k-1)/2^{n+1}$ and for all but these countably many values, $n_{c(x)}(y) = 1$. Thus, $c(x) \in GS$.

For a fixed n, define f(x) on E_n^k, $k = 1,2,\ldots,2^n$, to be the piecewise linear function which joins the points $(a,0)$ to $((b+2a)/3,\ 2^{-n-1})$ to $((a+2b)/3,\ -2^{-n-1})$ to $(b,0)$ where $E_n^k = [a,b]$. Set $f(x) = 0$ for all other x in $[0,1]$.

For $y \in (1/4,\ 1/2)$, $n_f(y) = 2\cdot 1 = 2(2-1)$;

$y \in (1/8,\ 1/4)$, $n_f(y) = 2\cdot(1+2) = 2(2^2-1)$;

$y \in (1/16, 1/8)$, $n_f(y) = 2(1+2+4) = 2(2^3-1)$;

$\vdots$

$$\vdots$$

$$y \in (1/2^{n+1}, 1/2^n),\ n_f(y) = 2(2^n-1).$$

Therefore, $\int_{-1/2}^{1/2}\log^+(n_f(y))dy = 2\int_0^{1/2}\log^+(n_f(y))dy$

$$= 2\sum_{i=1}^{\infty} \log(2^{n+1}-2)\cdot 2^{-n-1} < \infty. \quad \text{Thus } f(x) \in GS.$$

Noting the values of f(x) and c(x) on $E_n^1, E_n^2, \ldots, E_n^{2^n}$, one sees that for each fixed n, $[f+g]\ \bigcup_{k=1}^{2^n} E_n^k = [0,1]$. So for each $y \in [0,1]$, $n_{f+g}(y) = \infty$ and the example is complete.

Reference

1. D. Waterman, λ-Bounded Variation: Recent Results and Unsolved Problems, Real Analysis Exchange 4(1978), 69-75.

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