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ON STRICT LOCAL EXTREMA OF DIFFERENTIABLE FUNCTIONS

For an arbitrary real-valued function f defined on $[0,1]$ denote

$$A(f) = \{x:\ f \text{ attains a strict local maximum at } x\},$$

$$B(f) = \{x:\ f \text{ attains a strict local minimum at } x\}.$$

In this paper we give a short proof of the following theorem.

<u>Theorem</u> /Z. Zalcwasser [4]/. Let A and B be arbitrary disjoint at most denumerable subsets of $(0,1)$. Then there exists a function F having a bounded derivative on $[0,1]$ such that $A(F) = A$ and $B(F) = B$.

The idea is based on a result of R. Fleissner and J. Foran [2]: if a function f satisfies a Lipschitz condition on $[0,1]$, then there exists an increasing homeomorphism h of $f([0,1])$ onto itself such that the function $F = h \circ f$ has a bounded derivative on $[0,1]$ /see also [1], Theorem 2.1, p. 133/. Then obviously $A(F) = A(f)$, $B(F) = B(f)$. Therefore it suffices to construct a function f satisfying a Lipschitz condition such that $A(f) = A$ and $B(f) = B$.

Since the proof of the result [2] depends on a theorem of Zahorski [3], our proof can be in some way an answer to a question posed by A. M. Bruckner in [1], p. 44.

Construction. Suppose that the set $C = A \cup B$ is infinite. Enumerate C as a sequence $\{c_n\}_{n=1}^{\infty}$ where $i \neq j$ implies $c_i \neq c_j$. Define a sequence $\{d_n\}_{n=1}^{\infty}$ by

$$d_n = \begin{cases} 1 & \text{if } c_n \in A \\ -1 & \text{if } c_n \in B \end{cases}.$$

We shall construct a sequence of functions $\{p_n\}_{n=0}^{\infty}$. We start with $p_0 \equiv 0$. Assume we have already defined functions $p_0, p_1, \ldots, p_{n-1}$ satisfying the conditions

/1/ the function $s_{n-1} = p_0+p_1+\ldots+p_{n-1}$ is piecewise linear on $[0,1]$;

/2/ s_{n-1} is differentiable at c_i for all $i \geqq n$;

/3/ $A(s_{n-1}) \cup B(s_{n-1}) = \{c_1, \ldots, c_{n-1}\}$;

/4/ s_{n-1} satisfies a Lipschitz condition with constant $L_{n-1} = 1 - 2^{-(n-1)}$

and for each j, $1 \leqq j \leqq n-1$

/5/ the set $\{x: p_j(x) \neq 0\}$ is an interval (u_j, v_j) containing c_j;

/6/ s_{n-1} is not differentiable at u_j, v_j;

/7/ if $x \in (u_j, v_j) \setminus \{c_j\}$, then $d_j s_{n-1}(c_j) > d_j s_{n-1}(x)$.

Now p_n will be defined in such a way that statements /1/ - /7/ are valid for n instead of $n-1$ and moreover, the following conditions are satisfied:

/8/ $0 \leqq d_n p_n(x) \leqq 2^{-n}$ for all $x \in [0,1]$;

/9/ $v_n - u_n < 2n^{-1}$;

/10/ $[u_n, v_n] \subset (a,b)$ where $[a,b]$ denotes the maximal interval containing c_n on which s_{n-1} is linear.

We describe a construction for $d_n = 1$ and $s'_{n-1}(c_n) \geqq 0$, the other cases being similar. Let ε be so chosen that $0 < \varepsilon \leqq 2^{-n}$ and $s_{n-1}(c_j) > \varepsilon + s_{n-1}(c_n)$ whenever

j satisfies $1 \leq j \leq n-1$, $d_j = 1$ and $c_n \in (u_j, v_j)$. Put $\delta = \min(n^{-1}, c_n - a, b - c_n)$. Choose $u \in (c_n - \delta, c_n) \setminus C$. Find $v \in (c_n, c_n + \delta) \setminus C$ so that $s_{n-1}(v) < s_{n-1}(c_n) + \varepsilon.(c_n - u)$. Choose $w \in (c_n, v) \setminus C$. Put

$$s_n(c_n) = \min(s_{n-1}(c_n) + \varepsilon.(c_n - u),\ s_{n-1}(v) + (1-2^{-n}).(w - c_n)),$$

$$s_n(x) = s_{n-1}(x) \quad \text{for} \quad x \in [0,u] \cup [v,1],$$

$$s_n(x) = s_{n-1}(v) \quad \text{for} \quad x \in [w,v]$$

and define s_n linearly on each of the intervals $[u, c_n]$, $[c_n, w]$. Put $p_n = s_n - s_{n-1}$.

Having defined the sequence $\{p_n\}_{n=0}^{\infty}$, set

$$f = \sum_{n=1}^{\infty} p_n .$$

The series converges uniformly because of /8/. From /4/ we see f satisfies a Lipschitz condition with constant $L = 1$. Using /1/ - /10/, it is not hard to prove that for each i, $x \in (u_i, v_i) \setminus \{c_i\}$ implies $d_i f(x) < d_i f(c_i)$, and if $x \in (0,1) \setminus C$, then $x \notin A(f) \cup B(f)$. Hence $A(f) = A$ and $B(f) = B$. This completes the proof.

References

[1] A. Bruckner: Differentiation of real functions, Lecture Notes in Math. 659, Springer 1978.

[2] R. Fleissner and J. Foran: Transformations of differentiable functions, Colloq. Math. 39 /1976/, 278-281.

[3] Z. Zahorski: Sur la première dérivée, Trans. Amer. Math. Soc. 69 /1950/, 1-54.

[4] Z. Zalcwasser: Sur les fonctions de Köpcke, Prace Mat. Fiz. 35 /1927-28/, 57-99.