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ON THE SYMMETRIC DERIVATIVE

One of the most interesting problems concerning symmetrically differentiable functions is whether an arbitrary symmetric derivative is in the first Baire class of functions. The purpose of this note is to indicate an affirmative answer to this question and to state some of the interesting results which are consequences of this theorem.

For f, a real-valued function defined on $\mathbb{R}$, we denote by C(f) the set of points at which f is continuous and by D(f) the set of points at which f'(x) exists and is finite. $\mathfrak{B}_1$ stands for the first Baire class of functions and $\mathcal{D}\mathfrak{B}_1$ is the set of all $f \in \mathfrak{B}_1$ such that f has the Darboux property. $f^s(x)$ is the symmetric derivative of f at x. The class Σ is defined to consist of all functions, f, such that $f^s(x)$ exists (finite or infinite) everywhere. In addition, we define $\sigma^* = \{f \in \Sigma\colon C(f) \text{ is dense}\}$ and $\sigma = \{f \in \sigma^*\colon f^s \text{ is finite everywhere}\}$. We denote $\Sigma^s = \{f^s\colon f \in \Sigma\}$. σ^{*s} and σ^s are defined similarly.

It follows at once from theorems of Khintchine [6] and Preiss [9] that all measurable symmetrically differentiable functions are in σ^* and $f \in \sigma$ iff f is measurable and $f^s(x)$ exists and is finite everywhere.

<u>Theorem 1</u>. $\Sigma^s \subset \mathfrak{B}_1$.

This extends a result of Filipczak [4], who showed that if f is approximately continuous and symmetrically differentiable, then $f^s \in \mathfrak{B}_1$.

Let $f \in \sigma^*$ and $x \in \mathbb{R}$. Since $C(f)$ is residual, there exists a sequence, $\{\rho_n\}$, decreasing to 0, such that both $x+\rho_n$ and $x-\rho_n$ are in $C(f)$ for each n. Using this observation, we define

$$f^{sc}(x) = \lim_{n\to\infty} \frac{f(x+\rho_n)-f(x-\rho_n)}{2\rho_n}$$

if this limit exists and is the same for all such sequences, $\{\rho_n\}$.

<u>Theorem 2</u>. Let $f \in \sigma^*$. Then there are two sets, A_1 and A_2, each with countable closure, and two functions, g_1 and g_2, both in $\mathfrak{B}_1$, satisfying:

(a) $g_i^{sc}(x) = f^s(x)$ everywhere, $i=1,2$;

(b) $g_i^s(x) = f^s(x)$ everywhere on $\overline{A}_i^c$, $i=1,2$;

(c) g_1 (g_2) is upper (lower) semicontinuous on $\overline{A}_1^c$ ($\overline{A}_2^c$);

(d) $C(f) \subset C(g_i)$ and $f(x) = g_i(x)$ for each $x \in C(f)$, $i=1,2$;

(e) $D(f) \subset D(g_i)$ and $f'(x) = g_i'(x)$ for each $x \in D(f)$, $i=1,2$;

(f) If I is a component of $\overline{A}_i^c$, then $g_i \in \mathcal{M}_{-1}(I)$ (see Evans [3]), $i=1,2$.

This is proved with the help of the following lemma.

<u>Lemma 3</u>. If $f \in \sigma^*$, then the sets

$A_1 = \{x: |\limsup_{t\to x} f(t)| = \infty\}$ and $A_2 = \{x: |\liminf_{t\to x} f(t)| = \infty\}$ both have countable closure.

Thus, given an arbitrary $f \in \sigma^*$, which may be badly behaved, theorem 2 gives us a means of associating f with another function which retains the desirable properties of f and does not possess some of the less desirable ones.

This association allows us to explore the properties of f^s with more precision. We call g_1 of the theorem the "nice copy" of f and A_1 the "essential set" for f. (This choice of g_1 and A_1 over g_2 and A_2 is entirely arbitrary.) The uniqueness of the nice copy is indicated by the following thoerem.

Theorem 4. Let f and g be functions in σ^* and suppose that D is any dense subset of $\mathbb{R}$. If $f(x)=g(x)$ for every $x\in D$, then the essential sets for f and g are equal and the nice copies of f and g are equal up to their values on the essential set.

If $f\in\sigma$, then the conclusions of theorem 2 may be strengthened.

Theorem 5. Let $f\in\sigma$ with A the essential set for f and g the nice copy of f. Then A is a symmetric set and g satisfies:

(a) $g^s(x)=f^s(x)$ everywhere;

(b) g is upper semicontinuous on $\overline{A^c}$;

(c) If I is a component of $\overline{A^c}$, then $g\in\mathcal{DB}_1(I)$.

Further, g is uniquely determined up to an additive constant and its values on A by (a), (b) and (c).

Corollary 6. Let $f\in\sigma^s$. Then there is a unique symmetric set, A, and a function, $F\in\sigma$, satisfying:

(a) $F^s(x)=f(x)$ everywhere;

(b) F is upper semicontinuous on $\overline{A^c}$;

(c) If I is a component of $\overline{A^c}$, then $F\in\mathcal{DB}_1(I)$;

(d) F is unique up to its values on A and an additive constant.

Given $f \in \sigma^s$, we call the function, F, of the theorem the "nice primitive" of f.

Theorem 7. Let $f \in \sigma^*$ such that $f^s(x) \geq 0$ a. e. and $f^s(x)$ is never $-\infty$. Then the nice copy of f is nondecreasing.

Corollary 8. Let $f \in \sigma^s$ such that $f(x) \geq 0$ a. e.. Then any nice primitive for f is continuous and nondecreasing.

Corollary 9. Let $f \in \sigma^{*s}$ such that f has the Darboux property and let F be the nice copy of a primitive for f. If $f(x) \geq 0$ a. e., then F is nondecreasing.

Corollary 10. If $f \in \sigma^{*s}$, then f is finite a. e..

Using corollary 10 in conjunction with the main theorem contained in [2], we obtain the following theorem.

Theorem 11. Let $f \in \sigma^*$. Then the set of points at which f is not differentiable is σ-porous and f has a finite ordinary derivative a. e..

Corollary 12. $f \in \sigma^*$ iff f is measurable and symmetrically differentiable.

The preceding corollary naturally leads to the question of whether there are any nonmeasurable functions in Σ. This question was posed as long ago as 1928 by Sierpinski and still remains open. The following theorem may be helpful in resolving this question.

Theorem 13. Let $f \in \Sigma$. Then the set $\{x: |f^s(x)| = \infty\}$ can contain no interval and f is symmetrically continuous on a dense set.

Using these results, the quasi-mean value theorems of Aull [1], Evans [3] and Kundu [6] can be extended to σ^*.

Theorem 14. Let $f\in\sigma^*$ and $\alpha,\beta\in C(f)$ with $\alpha<\beta$. Then there are nonempty G_δ sets, A and B, contained in (α,β) such that

$$f^s(a) \le \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \le f^s(b)$$

for all $a\in A$ and all $b\in B$. Further, if $f\in\sigma$, both A and B have positive measure.

Even though an arbitrary symmetric derivative need not satisfy the Darboux property, there is another "Darboux-like" condition which it must satisfy.

Theorem 15. Let $f\in\sigma^s$. Then for each $x\in\mathbb{R}$,

$$\liminf_{h\to 0}\frac{f(x+h)+f(x-h)}{2} \le f(x) \le \limsup_{h\to 0}\frac{f(x+h)+f(x-h)}{2}.$$

We can also state a condition sufficient to guarantee that $f^s\in\mathcal{DB}_1$.

Theorem 16. Let $f\in\Sigma$ such that f is symmetrically continuous and nonangular (see Garg [5]). Then $f^s\in\mathcal{DB}_1$.

The above results can be used to extend the generalized Zahorski class theorems of Kundu [8] to σ^*. Using Kundu's notation, the following theorems can be proved.

Theorem 17. $\mathcal{D}\sigma^{*s}\subset\mathcal{M}_2$.

Theorem 18. Let $f\in\sigma^{*s}$ and suppose F is a primitive for f. Then $f\in\mathcal{M}_3(D(F))$.

Actually, a generalized form of Weil's property Z ([10]) is shown to hold and theorem 18 follows as a corollary.

<u>Theorem 19</u>. Let $f \in \sigma^s$ such that f is bounded. If F is any primitive for f, then $f \in \mathcal{M}_4(D(F))$.

These Zahorski-type theorems are relatively sharp, as the following examples show.

<u>Example</u>. There is a continuous and nonangular $f \in \sigma$ such that $f^s \in \mathcal{DB}_1$ and f^s is bounded, but $f^s \notin \mathcal{M}_3$.

<u>Example</u>. There is a bounded symmetric derivative, $f \in \mathcal{M}_4$, which is not a derivative.

<u>Example</u>. There exists an $f \in \mathcal{M}_5$ which is a symmetric derivative, but not a derivative.

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