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ON CHARACTERIZING CONNECTED FUNCTIONS

1. Introduction. In the paper [2], A.M. Bruckner introduced the following notion: a class K of real functions is said to be characterized by associated sets if there exists a family of sets P so that $f \in K$ if and only if for all real α the sets $E^{\alpha}(f)$, $E_{\alpha}(f)$ belong to P. ($E^{\alpha}(f) = \{x/f(x)<\alpha\}$ and $E_{\alpha}(f) = \{x/f(x) >\alpha\}$).

In that paper the author showed that some usual classes of real functions were not characterized by associated sets. Such classes are derivatives, approximate derivatives, functions of bounded variation, functions which satisfy Lusin's condition (N), Lebesgue, Denjoy and Denjoy-Hinčin primitives, Darboux functions.

In the following we shall show that the class of connected functions can not be characterized by associated sets.

2. Preliminaries and notations. Without loss of generality, we are working with functions $f:I\to I$, where $I = [0,1]$.

A function f is said connected if its graph Γ_f is a connected set in I^2. ($\Gamma_f=\{(x,y)\in I^2/y=f(x)\}$) .

A set $M \subset I$ is said <u>f-negligible</u> for a connected function f if every function which agrees with f on $I - M$ is a connected function [1].

For a function f denote $f(+) = \{(x,y) \in I^2 / y > f(x)\}$, $f(-) = \{(x,y) \in I^2 / y < f(x)\}$ and $[y]_f = f^{-1}(y)$; the set $[y]_f$ is called a level of the function f.

<u>Theorem 1</u>: A necessary and sufficient condition that a function f be connected is that every continuum $C \subseteq I^2$ intersects Γ_f whenever C intersects both $f(+)$ and $f(-)$. [3]

<u>Theorem 2</u>: If f is a connected function, then the following statements are equivalent:

(i) Γ_f is dense in I^2

(ii) every nowhere dense subset of I is f-negligible. [1]

By a <u>dense function</u> we mean a function whose graph is dense in I^2.

For $A \subseteq I^2$, $(A)_x$, $(A)_y$ will be X- and Y-projection of A and for $y \in I$ we shall note $h_y = I \times \{y\}$.

If f is a Darboux dense function then for any $y \in (0,1)$ the level $[y]_f$ is a boundary dense subset of I and hence for any closed set $F \subseteq I^2$ with $F \cap \Gamma_f$ empty, $F \cap h_y$ and $(F \cap h_y)_x$ are nowhere dense in h_y and R.

3. <u>Associated sets for connected functions</u>.

<u>Proposition</u>: Let f and g be functions such that:

(i) f is Darboux, dense and non-connected

(ii) there exist $y_1, \ldots y_n \in (0,1)$ such that

$[y]_f = [y]_g$ for any $y \in I - \{y_1, \ldots, y_n\}$.

Then g is dense and non-connected.

Proof: If $A = \bigcup_{i=1}^{n} [y_i]_f$, observe that f agrees g on I-A, $f(A) \subseteq \{y_1, \ldots, y_n\}$ and $g(A) \subseteq \{y_1, \ldots, y_n\}$.

Obviously g is dense.

Because f is non-connected, by theorem 1, a continuum C exists such that C intersects both f(+) and f(-) and $C \cap \Gamma_f = \emptyset$.

$(C)_y$ being an interval and g dense, C intersects both g(+) and g(-).

Suppose g connected. Then, by the same theorem, $C \cap \Gamma_g \neq \emptyset$ and moreover $C \cap \Gamma_g \subseteq \left(\bigcup_{i=1}^{n} h_{y_i} \right) \cap C$. Because $C \cap \Gamma_f = \emptyset$ the set $B = (C \cap \Gamma_g)_x$ is nowhere dense in I, hence B is a g-negligible set, by theorem 2. Define the function h by:

$$h(x) = \begin{cases} f(x) & \text{if } x \in B \\ g(x) & \text{if } x \in I - B \end{cases}$$

By theorem 2, h is connected. The continuum C intersects both h(+) and h(-), hence $C \cap \Gamma_h \neq \emptyset$. But, by definition of h, $C \cap \Gamma_h \subseteq \Gamma_f$ and that contradicts the choice of C. Hence g is non-connected.

Corollary: Let f and g be functions such that:

(i) f and g are both Darboux and dense

(ii) there exist $\{y_1, \ldots, y_n\} \in (0,1)$ such that

$[y]_f = [y]_g$ for any $y \in I - \{y_1,\dots,y_n\}$. Then f and g are both connected or non-connected.

Theorem: The class of connected functions is not characterizable by associated sets.

Proof: Let P be the family of associated sets of connected functions. Let f be a connected, dense function. Let $A = \{x/f(x) < 1/2\}$, $B = \{x/f(x) \geq 1/2\}$.

Then $A \cap B = \emptyset$, $A \cup B = I$ and $A \in P$.

For $y_0 \in (0,1/2)$ the level $[y_0]_f$ is a dense set. This set can be written as $H_1 \cup H_2$, where H_1, H_2 are dense boundary disjoint sets. Define the function g as

$$g(x) = \begin{cases} 1/2 & \text{if } x \in H_2 \\ 3/4 & \text{if } x \in [1/2]_f \\ f(x) & \text{if } x \in I - (H_2 \cup [1/2]_f) \end{cases}$$

Then g is a Darboux dense function, because it takes all values in (0,1) on any interval $J \subseteq I$, and we have $\{x/g(x)>1/2\} = B$. But $[y]_f = [y]_g$ for any $y \in I - \{y_0, 1/2, 3/4\}$, hence g is connected and $B \in P$.

Define the function h as:

$$h(x) = \begin{cases} 0 & \text{if } x \in B \\ 1 & \text{if } x \in A \end{cases}$$

Then for any real α, both $E_\alpha(h)$ and $E^\alpha(h)$ belong to P and h is non-connected.

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