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On the Baire class of a mixed second derivative

1. Introduction. Let $F(x, y)$ be a real-valued function of two real variables, and suppose that the second order partial derivative $F_{xy}(x, y)$ exists everywhere. In [3], G. Petruska pointed out that F_{xy} is then a Baire 3 function, and he answered M. Laczkovich's question of whether F_{xy} is always Baire 1, by constructing an example in which F_{xy} is Baire 2 but not Baire 1. In the present note, after showing that a function constructed in [1] also leads immediately to such a counter-example, we give a simple proof that F_{xy} must always be Baire 2.

2. An example. Complementing a proof that if $f: R^2 \to R$ is separately approximately continuous then it must be Baire 2, in [1] a separately approximately continuous function that is not Baire 1 was constructed. It is of the form

$$f(x, y) = \Sigma_{n=1}^{\infty} g_n(x)g_n(y)$$

where the g_n's are approximately continuous functions with values in $[0, 1]$, having disjoint support. The following facts are easy consequences —— details are left to the reader. (I) For each fixed x, f is a bounded approximately continuous function of y. (II) For each x and y,

$$\int_0^y f(x, v)\, dv = \Sigma_{n=1}^{\infty} g_n(x)[\int_0^y g_n(v)\, dv] .$$

(III) For each fixed y, $\int_0^y f(x, v)\, dv$ is a locally bounded approximately continuous function of x. (IV) For each x and y,

$$\int_0^x \left[\int_0^y f(u, v)\, dv\right] du = \sum_{n=1}^{\infty} \left[\int_0^x g_n(u)\, du\right]\left[\int_0^y g_n(v)\, dv\right] .$$

(V) The function $F(x, y) = \int_0^x [\int_0^y f(u, v)\, dv] du$ is continuous, with $F_{xy} = F_{yx}$ equal to f everywhere and thus not Baire 1. We have the desired example. It is interesting that in his construction, found quite independently, Petruska applied exactly the same Lemma 12 of Zahorski ([4]; see also [2]) as I used in constructing the sequence of functions (g_n).

3. Theorem. If F_{xy} exists everywhere then it is a Baire 2 function.

Proof. Because F_x exists everywhere, the function $F(x, y_0)$ is continuous, for each fixed $y_0 \in R$. It follows that the function $\Phi_{mn}(x, y)$ that we are about to define is a continuous function of (x, y). For $m, n = 1, 2, \ldots$ let

$$\Phi_{mn}(x, \tfrac{k}{n}) = 2m[F(x + \tfrac{1}{m}, \tfrac{k}{n}) - F(x - \tfrac{1}{m}, \tfrac{k}{n})] \quad \text{for } k \in Z ,$$

and for other values of y define $\Phi_{mn}(x, y)$ by linear interpolation with respect to y, that is, for $0 < \lambda < 1$

$$\Phi_{mn}(x, \lambda\tfrac{k}{n} + (1 - \lambda)\tfrac{k+1}{n}) = \lambda.\Phi_{mn}(x, \tfrac{k}{n}) + (1 - \lambda).\Phi_{mn}(x, \tfrac{k+1}{n}) .$$

Consider the Baire 1 function

$$\Phi_n(x, y) = \lim_{m\to\infty} \Phi_{mn}(x, y) .$$

Clearly $\Phi_n(x, \frac{k}{n}) = F_x(x, \frac{k}{n})$ for $k \in Z$, while $\Phi_n(x, y)$ is the corresponding linear interpolation for other values of y, that is, for $0 < \lambda < 1$

$$\Phi_n(x, \lambda\tfrac{k}{n} + (1 - \lambda)\tfrac{k+1}{n}) = \lambda.F_x(x, \tfrac{k}{n}) + (1 - \lambda).F_x(x, \tfrac{k+1}{n}) .$$

The following function is necessarily also Baire 1:

$$\theta_n(x, y) = 2n[\Phi_n(x, y + \frac{1}{n}) - \Phi_n(x, y - \frac{1}{n})] ;$$

but it is easy to see that for all (x, y) we have

$$\lim_{n\to\infty}\theta_n(x, y) = F_{xy}(x, y) ,$$

and therefore F_{xy} is Baire 2 .

4. Problems. Several interesting questions raised by Petruska [3] still remain unanswered, in particular: if both F_{xy} and F_{yx} exist everywhere, must they agree at some points ? Also, as he points out, F_{xy} may be identically zero even for a nonmeasurable F (for example, $F(x, y) = H(y)$, where H is nonmeasurable). It is therefore natural to ask whether, if F_{xy} exists everywhere, there necessarily exists a function $G(x, y)$ with $G_{xy} = F_{xy}$ which is 'smooth', in the sense of being measurable or even of low Baire class.

References

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