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Non-Monotonic Implies Very Oscillatory

Let f be a measurable real function defined on a measurable linear set E, and for each point $x_0 \in E$ let

$$A(x_0) = \{x \in E: f(x) = f(x_0)\} ,$$

$$A_+(x_0) = \{x \in E: (x - x_0)^{-1}(f(x) - f(x_0)) > 0\} ,$$

$$A_-(x_0) = \{x \in E: (x - x_0)^{-1}(f(x) - f(x_0)) < 0\} .$$

Khinchin [5] called f asymptotically directed (we shall write AD) at x_0 if one of the above sets (which are evidently measurable) has density 1 at x_0. He showed that for almost all points x_0 at which $A_+(x_0)$ or $A_-(x_0)$ has density 1, x_0 is a density point of a compact set of positive measure on which f is strictly monotonic; and f is approximately differentiable at almost all points at which it is AD.

As regards the points $x_0 \in E$ at which f is not AD, Good [4] showed that at almost all of them at least one of the sets $A_+(x_0)$, $A_-(x_0)$ has upper unilateral density 1 on both sides at x_0. In Theorem 1 we improve "at least one" to "both", using similar reasoning. (Related results of Császár [2] appear not quite to imply this.) After drawing some conclusions about oscillatory behaviour at non-AD points, we then discuss approximate maxima and generalize a result recently given by Pu and Pu [7].

Theorem 1. At almost all points $x_0 \in E$ at which f is not AD, both of the sets $A_+(x_0)$, $A_-(x_0)$ have upper unilateral density 1 on both sides at x_0.

Proof. There are only countably many values of α such that the set $E_\alpha = \{x: f(x) = \alpha\}$ has positive measure, and f is AD at almost all points of each set E_α (the density points). Hence we may suppose that f takes no value on a set of positive measure. We may also suppose that E is compact and (by Luzin's theorem) that f is continuous. Under these conditions we have the following result, which together with three similar results (obtained by interchanging left, right and $>$, $<$) clearly implies our theorem.

<u>Lemma</u>. At almost all points $x_0 \in E$ at which the set $\{x: f(x) > f(x_0)\}$ has lower unilateral density greater than zero from the right, the set $\{x: f(x) < f(x_0)\}$ has unilateral density 1 from the left.

<u>Proof of the Lemma</u>. Let B_n denote the set of points $x \in E$ such that

$$0 < h \leqslant n^{-1} \Rightarrow m([x, x+h] \cap \{y: f(y) > f(x)\}) \geqslant n^{-1}h \,. \quad (1)$$

It is sufficient to prove the assertion for B_n, which is compact. Take any point $x_0 \in B_n$ and $0 < \delta \leqslant n^{-1}$ such that

$$0 < h \leqslant \delta \Rightarrow m([x_0 - h, x_0] \cap B_n) > (1 - n^{-1})h \,; \quad (2)$$

almost all points of B_n have this property for some δ.

Consider f on the compact set $[x_0 - \delta, x_0] \cap B_n$; its supremum is attained, and I claim that it is attained at x_0. For suppose it is at a point $x_1 \neq x_0$ and $f(x_1) > f(x_0)$. Then by (1) applied to the point $x = x_1 \in B_n$,

$$m([x_1, x_0] \cap \{y: f(y) > f(x_1)\}) \geqslant n^{-1}(x_0 - x_1) \,. \quad (3)$$

But no points of the set $[x_1, x_0] \cap \{y: f(y) > f(x_1)\}$ can belong to B_n, by the maximality of $f(x_1)$. Hence by (3)

$m([x_1, x_0] \cap B_n) < (1 - n^{-1})(x_0 - x_1)$. This contradicts (2) for $h = x_0 - x_1$.

Hence the supremum is indeed attained at x_0 , and provided x_0 is a density point from the left of B_n it is a density point from the left of the set $\{y: f(y) < f(x_0)\}$, as required.

Remark. Our theorem shows that at almost all points $x_0 \in E$ at which f is not AD, it is oscillatory, in the sense that at x_0 the set $A(x_0)$ has density zero and both of the sets $A_+(x_0)$, $A_-(x_0)$ have upper unilateral density 1 on both sides. It is easy to see that almost all points x_0 at which f is oscillatory divide themselves into two subclasses:

I. Those at which f has approximate derivative zero, and the function $f(x) + \alpha x$ is AD at x_0 for every $\alpha \neq 0$.

II. Those at which f is not approximately differentiable, and each of the functions $f(x) + \alpha x$ is also oscillatory.

We might call f weakly and strongly oscillatory in these two cases. Only constant functions have approximate derivative zero everywhere, so no function is everywhere weakly oscillatory on $\mathbb{R}$.

Now let f be an arbitrary real function defined on an arbitrary linear set E , and let $M = M(f)$ denote the set of points $x_0 \in E$ at which f has an approximate strict maximum, that is, for which the set $\{x: f(x) < f(x_0)\}$ has density 1 at x_0 with respect to inner measure. Pu and Pu [7] showed, in the case when E is the whole line, that if f is measurable then M has measure zero, and if f is continuous then M is also meagre. Their first conclusion can be regarded as a corollary of the results of Khinchin and Good quoted earlier, and in fact Theorem 5.21 of Császár [2] implies that it is valid without the measurability assumption. In Theorem 2 we provide a slight generalization of this fact.

<u>Theorem 2</u>. For almost all points $x_0 \in E$, for every $\epsilon > 0$ there exist arbitrarily small intervals I containing x_0 such that

$$m^*[I \cap \{x \in E: f(x) \geqslant f(x_0)\}] > (1 - \epsilon)m(I) .$$

<u>Proof</u>. Suppose not; then for some $\epsilon > 0, \delta > 0$ there exists a subset E_0 of E of positive outer measure such that

$$x_0 \in E_0 \cap I \ \& \ 0 < m(I) < \delta \Rightarrow m^*[I \cap \{x \in E: f(x) \geqslant f(x_0)\}] \leqslant (1-\epsilon)m(I) . \quad (4)$$

Choose an interval I_0 with $m(I_0) < \delta$, such that

$$m^*(E_0 \cap I_0) > (1 - \epsilon)m(I_0) . \quad (5)$$

Let $\lambda = \inf\{f(x): x \in E_0 \cap I_0\}$ and choose a sequence (x_k) of points of $E_0 \cap I_0$ such that $f(x_1) \geqslant f(x_2) \geqslant \ldots \to \lambda$. Now

$$m^*[I_0 \cap \{x \in E: f(x) > \lambda\}] \leqslant \lim_k m^*[I_0 \cap \{x \in E: f(x) \geqslant f(x_k)\}] \leqslant$$
$$\leqslant (1 - \epsilon)m(I_0) < m^*(E_0 \cap I_0)$$

by (4) and (5), so

$$m^*[E_0 \cap I_0 \cap \{x \in E: f(x) \leqslant \lambda\}] > 0 .$$

In view of the definition of λ, this implies that the set

$$E_0 \cap I_0 \cap \{x \in E: f(x) = \lambda\}$$

is of positive outer measure; but at any point x_0 of this set at which it has upper density 1 (with respect to outer measure), it is clear that (4) is contradicted.

<u>Remark</u>. Various generalizations to $\mathbb{R}^n$ have been proved in [1], [3], [6], and [8], as a referee has pointed out. I am grateful for comments from him and the editor.

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