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Uniform Ideas in Analysis

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IA. Scope of paper

During the last ten years there has been a renewed interest in the subject of uniform spaces. This renewed interest is due largely to the recognition that many of the mathematical ideas which are central to the foundations of real analysis can be discussed in a very natural manner using the language of uniform spaces. These ideas include measurable functions, sigma-fields, and the theory of measure and integration. Moreover, one can present a reasonable argument that analysts have been intuitively aware of this "connection" for a long time - at least since the introduction of the general theory of integration at the beginning of this century. (This viewpoint has also been expressed by Berberian in his review [Bb].) A possible starting point for a defense of this remark is found in section two of the present work, where I have collected some instances of historical "uniform thinking" in analysis. The latter portion of the second section sets the stage for the discussion of the modern material found in section three. This material summarizes certain formal connections between uniform and analytic ideas that have developed in the last ten years.

The formal results found in the third section and other new results in uniform spaces are due largely to the efforts of mathematicians in Czechoslovakia, France, Russia, and the United States. Specifically, in Czechoslovakia the research group consisting of Z. Frolík, M. Hušek, J. Pelant, J. Pachl, J. Vilímovský, P. Pták, P. Simon, V. Rödl, and J. Reiterman has contributed a number of new results, many of which can be found in the notes of Seminar Uniform Spaces 1973-1977 (Mathematics Institute of Czechoslovak Academy of Sciences, Prague, denoted in this paper by [SUS]). In France, a number of mathematicians associated with the mathematical school in Lyon, including H. Buchwalter, R. Pupier, and A. Deaibes have contributed results to the theories of uniform measures and algebras of functions (see the references found in [De]). In Russia, I.A. Berezanskii, V.P. Fedorova, and D.A. Raikov have contributed basic ideas to uniform measure theory, while the schools centering around V.A. Efremovič and Y.M. Smirnov have done basic work on proximity theory, uniform geometry, and dynamical systems (see the survey articles [EV] and [Sm]). Finally, in the United States pioneering work in uniform spaces and algebras of functions was done by J.R. Isbell (see $[I]_{1,2}$) and the uniform connections with the theory of vector lattices and approximation was greatly expanded in the work of A.W. Hager (see references $[Ha]_{4-10}$). The later work of his students M.D. Rice and G. Tashjian discusses further connections between uniform spaces, measurable functions, and the descriptive theory of sets (see $[Ri]_{1-5}$ and

$[\mathrm{Ta}]_{1,2}$).

I would like to thank the editors of the Real Analysis Exchange for this opportunity to present my views on the connections between uniform spaces and analysis. I hope that the patterns in uniform thinking will be congenial and easily recognized by an intended audience of real analysts. I would also like to thank my advisor A.W. Hager for introducing me to a large portion of the subject matter found in this paper and to thank my friends in Prague for many good conversations on uniform spaces.

IB. Notation

A uniform structure or uniformity on a set X can be described in terms of entourages, pseudometrics, or (uniform) covers which satisfy certain axioms. For the most part, the latter formulation will be the most convenient. Thus a <u>uniform space</u> will be a set X equipped with a family of uniform covers u which satisfies the following axioms: (i) if $\mathcal{U}$ and $\mathcal{V}$ are members of u, then $\mathcal{U}\wedge\mathcal{V}=\{U\cap V : U\in\mathcal{U} \text{ and } V\in\mathcal{V}\}$ is a member of u, (ii) if $\mathcal{U}$ is a member of u and $\mathcal{U}<\mathcal{V}$ (for each U in $\mathcal{U}$, there exists V in $\mathcal{V}$ such that $U \subseteq V$), then $\mathcal{V}$ is a member of u, and (iii) for each $\mathcal{V}$ in u, there exists $\mathcal{U}$ in u such that $\mathcal{U}^*=\{St(U,\mathcal{U}): U \in \mathcal{U}\}<\mathcal{V}$ (where $St(U,\mathcal{U}) = \bigcup\{U'\in\mathcal{U}: U'\cap U \neq \emptyset\}$. The uniform space will be denoted by uX. For example, if (M,d) is a pseudometric space, the uniformity generated by d consists of all covers which may be refined by covers of the form $\{B(x,r): x \in M\}$, $r>0$, where $B(x,r) = \{y: d(x,y)<r\}$. Each uniformity on a set X induces a topology on X which is completely regular; in general we will not be concerned with it except when considering uniformities on function spaces. We will use the fact that each completely regular space X admits a largest uniformity (called the fine uniformity) which induces the given topology. If a family of covers u satisfies (i) and (iii) (resp. (iii)), then u is a basis (resp. subbasis) for a uniformity on X. If u is a basis for a uniformity on X, a family $\mathcal{A}$ of subsets of X is uniformly discrete (with respect to u) if there exists $\mathcal{U}$ in u such that each member of $\mathcal{U}$ meets at most one member in $\mathcal{A}$.

If uX and vY are uniform spaces, a mapping $f: X \to Y$ is uniformly continuous if $f^{-1}(\mathcal{V})$ is a member of u for each $\mathcal{V}$ in v. The family of uniformly continuous mappings from uX to vY will be donoted by U(uX,vY)(or simply U(X,Y) if the uniformities are understood). If $\mathbb{R}$ is the real numbers with the usual absolute value metric, $U(uX, \mathbb{R})$ will be denoted by U(uX) (or simply U(X)). $U_b(X)$ will denote the set of bounded members of U(X). The sets of the form $Z(f) = \{x: f(x) = 0\}$ and $coz(f) = \{x: f(x) \neq 0\}$, f in U(X), are called (uniform) zero sets and cozero sets, respectively. The family of zero sets (resp. cozero sets) will be denoted by $Z(uX)$ (resp. $Coz(uX)$).

A family of mappings $\{f_s: uX \to vY\}$ is <u>equi-uniformly continuous</u> if for each $\mathcal{V}$ in v, there exists $\mathcal{U}$ in u such that $\mathcal{U} < f_s^{-1}(\mathcal{V})$ for every s.

A uniform space uX is <u>precompact</u> (<u>separable</u>) if each member of u has a finite (countable) subcover. uX is <u>complete</u> if each u-Cauchy filter $\mathcal{F}$ is convergent (where $\mathcal{F}$ is u-Cauchy if each cover in u contains some member of $\mathcal{F}$). The <u>completion</u> of a uniform space uX will usually be denoted by $\hat{X}$ if u is understood. For each uniformity u on a set X, there exists a largest uniformity pu contained in u which is precompact. The completion of X with respect to pu is called the <u>Samuel compactification</u> of uX and will be denoted by $\check{X}$.

We will also have occasion to use the theory of <u>proximity structures</u>, which axiomatizes the notion of nearness of sets and is essentially equivalent to the theory of precompact uniform spaces (see $[I]_2$ for details). Intuitively, the sets A and B are <u>near</u> (written $A\delta B$) with respect to a uniformity u if every member of u contains a set which intersects both A and B (if u is derived from a pseudometric d, this simply means that $d(A,B)=0$). If the sets A and B are not near, they are said to be <u>far</u>. A uniform space uX is said to be <u>finest in its proximity class</u> or <u>proximally fine</u> if u is the largest uniformity on X which induces the notion of nearness described above. It is a classical result of Efremovič (see[I]$_2$) that every metric space is proximally fine.

We will also need the following terminology from the theory of lattice-ordered algebras. A: $X \to \mathbb{R}$ will usually denote a vector lattice of real-valued functions on X which contains the constant mappings. A is uniformly closed if it is closed under the formation of uniformly convergent sequences. A is composition-closed if for each finite family $f_1, f_2, \ldots, f_n$ in A and every continuous mapping $g: \mathbb{R}^n \to \mathbb{R}$, the composition $g \circ (f_i): X \to \mathbb{R}$ is a member of A. A is inversion-closed if f in A and $f(x) \neq 0$ for every x implies $1/f$ is a member of A. Finally, A is a ring if it is closed under pointwise products and the ring is said to be Von-Neumann regular if for each f in A, there exists g in A such that $f^2 g = f$.

Finally, I will use the following terminology for analytic concepts. A sigma-field on the set X will usually be denoted by Σ; a measure on Σ will usually refer to a positive countably additive set function and be denoted by μ or ν. The family of measurable functions between two measurable spaces will be denoted by M(X,Y) if the sigma-fields are understood; $M(X, \mathbb{R})$ will refer to the real line with the usual Borel structure.

If μ is a measure on Σ, the notation $\Sigma(\mu)$ will refer to Σ equipped with the pseudometric d defined by

$d(A,B)=\mu(A\Delta B)$, where Δ is the operation of symmetric difference.

Given a uniform space uX, Baire(uX) (or Baire(X)) will denote the sigma-field generated by the uniform zero sets and Baire(X,M) will denote the family of real-valued mappings which are measurable with respect to Baire(X) and the usual Borel structure on the metric space M. The notation $\text{Baire}_\alpha(X)$ (resp. $\text{Baire}_\alpha(X,M)$) will be used to denote the Baire sets of (ambiguous) class α (resp. the Baire measurable mappings of class α - the pre-image of every open set is a Baire set of class α). Souslin(X) will denote the family of sets which are derived from the uniform zero sets using the (A) - or Souslin operation (see $[K]_1$ for the precise definition). bi-Souslin(X) will denote the family of sets A such that both A and its complement are Souslin sets. It follows from general results on the (A)-operation that Baire(X) $\subseteq$ bi-Souslin(X) is always valid.

IIA. Historical Ideas

The first part of this section will present several examples from the theories of topological groups, topological vector spaces, and dynamical systems which illustrate the historical pattern of establishing global results from local assumptions. These global results can frequently be expressed in the language of uniform spaces.

It is reasonable to assume that the early investigation of topological groups led to the definition of a uniform space by Weil in $[W]_1$. Whatever the motivation, it was clearly recognized that the right and left uniform structures associated with a topological group reflected global properties of the group, which in turn were derived from local assumptions. Thus a compact neighborhood of the identity guaranteed not only a uniform cover consisting of compact sets, but also an invariant Haar measure which assigned the same value to each member of the cover. This recognition of the role of uniformity led to the construction of invariant measures in certain uniform spaces [Lo]. Conversely, it was also shown that the presence of a left invariant measure in a measurable group forced the group to be locally bounded (see chapter twelve in [H1]) and hence by a result in $[Ri]_6$ the associated left uniform structure must have a uniform basis consisting of star-finite covers. Thus a necessary condition for the existence of an invariant measure can be expressed in terms of the uniform structure. Moreover, additional important properties of the enveloping algebra of the group are directly related to the coincidence of the right and left uniform structures (see the remarks in [Bb]).

Perhaps the most important early results in the theory of linear operators hinged on the use of the Baire category theorem to deduce uniform boundedness principles (see [Dn] for a complete discussion). These results essentially showed that the hypothesis of pointwise boundedness for a family of bounded linear operators on a Banach space guaranteed a uniform bound on the norms of the operators and hence, in our terminology, that the family is equi-uniformly continuous. Using this result, one can establish the standard closed graph theorem. In turn, the extension of the closed graph theorem to the more general setting of topological vector spaces has required the use of various completeness assumptions (for example, V. Ptak's B-completeness and J.L. Kelley's use of hypercompleteness) as well as new assumptions on the domain space (such as the notion of a barreled space - see [Ht] for further details). Finally, the work of the French school on topological vector spaces has required the introduction of uniform structures in many settings. Here I will only mention the work involving the vector space topologies on families of mappings induced by the uniformities of bounded or precompact convergence, the use of equi-uniformly continuous families of mappings in duality theory, and the accompanying characterizations of completeness. These examples once again illustrate the fundamental idea that analysts have often resorted to the language of uniformities to state results of a global or collective nature.

Before turning to real analysis proper, I would also like to remind the reader of the extensive modern usage of the language of uniformities in the theory of dynamical systems. For example, key ideas in stability theory (such as an almost periodic motion) depend on the underlying uniform structure and Lyapunov stability is based on equi-uniform continuity (see [Sa] and [Se] for further details). The work of the Russian school has illustrated the usefulness of considering uniform and proximity structures on vector fields and Riemannian spaces in connection with questions in differentiable dynamics. ([EV] contains a large number of references on these contributions.)

IIB. Measure and integration

This subsection is divided into an initial part involving measure and a second part motivated by the integral. The second part leads naturally into the modern theory of uniform spaces, while the first part has not been integrated into a modern uniform space context.

1. Measure

In the theory of outer measures one already encounters a concept of nearness for sets (essentially known to F. Riesz). The very definition of a metric outer measure on a metric space (X,d) requires that $\mu^*(A) + \mu^*(B) = \mu^*(A \cup B)$ when $d(A,B) > 0$, that is, if the sets A and B are far, the set function acts in an additive manner. There is no difficulty in substituting the phrase "A is far from B with respect to a proximity δ" and defining a δ-outer measure. If Σ is the sigma-field of μ^* - measurable sets, one can easily show that $\text{Baire}(uX) \subseteq \Sigma$ for every uniformity u which induces the proximity δ. In another direction, given an arbitrary outer measure μ^*, one can define a concept or nearness by saying that A and B are far in $\mu^*(A)+\mu^*(B) = \mu^*(A \cup B)$. To my knowledge, there has been no systematic examination of this interplay between proximity and measure.

The classical idea of absolute continuity provides another excellent illustration of uniform thinking. Let μ and ν be finite measures on (X,Σ). It is well known that μ is absolutely continuous with respect to ν if and only if the mapping $\mu\colon \Sigma(\nu) \to \mathbb{R}$ is uniformly continuous. Furthermore, the Vitali-Hahn-Saks theorem says more about this relationship: if $\mu_n \ll \nu$, $n = 1,2,\ldots$, and $\lim_{n\to\infty} \mu_n(E)$ exists for all E in Σ, then the family (μ_n) is equi-uniformly continuous on $\Sigma(\nu)$. In addition, the pointwise limit $\mu = \lim_{n\to\infty} \mu_n$ is countably additive and the countable additivity is uniform - if $E_1 \supset E_2 \supset \ldots$ with $\bigcap_{i=1}^{\infty} E_i = \emptyset$ and $r>0$, there exists m such that $\mu_n(E_m)<r$ for all $n=1,2,\ldots$ Moreover, Dubrovskii [Du] showed that the uniform countable additivity of a family of measures is equivalent to equi-uniform

continuity of the family with respect to a fixed $\Sigma(\nu)$. Hence one can see that the concepts mentioned above are only different expressions of the same uniform phenomena. Without citing further examples, I will only say that this uniform phenomena is the key to all known characterizations of compactness, not only in the setting of spaces of measures, but also in the setting of function spaces (Arzela-Ascoli theorems) (see [DS] for specific results in these directions).

2. Integration

Of course, the work on absolute continuity discussed above is related to convergence theorems for integrals. This fact has been known for a long time: in 1907 Vitali [V] proved that whenever the Lebesgue integrals $\int_0^1 f_n$ converge to the integral $\int_0^1 f$, the indefinite integrals $_E\int f_n$, n=1,2,... must be uniformly absolutely continuous with respect to Lebesgue measure. Therefore any condition on the pointwise convergent family (f_n) which guarantees the convergence of the integrals must, in some sense, be a "collective" condition (like equi-uniform continuity, although this condition does not insure the convergence of the integrals). The well known Monotone and Dominated Convergence theorems supply such sufficient conditions, which will be discussed below in connection with the inversion property. In turn, the Monotone Convergence theorem is incorporated in the definition of the Daniell integral as a linear functional defined on a space of real-valued functions, with the Riesz representation theorem representing the connection between the two approaches to integration theory (see the historical notes in chapters 3 and 4 of [DS]).

In the early 1940's A.D. Alexandrov [A] developed a theory of linear functionals on families of real-valued functions and a related theory of measures. This theory was based on what he called A-spaces and A-maps; in our terminology we can recognize his completely normal A-spaces as exactly the pairs $(X,Z(X))$, where $Z(X)$ is the family of all zero sets with respect to some uniformity on X, and his A-maps as the functions $f\colon X\to Y$ such that the pre-image of each zero set is a zero set (see $[Ha]_5$, pages 542-545 and $[Ha]_9$ for a detailed discussion). Given a uniformly closed vector lattice $A\colon X\to\mathbb{R}$, Alexandrov defined Z to be the family of zero sets of members in A and $C(X,Z)$ to be the family of A-maps (relative Z) from X to the real line. The properties of linear functionals on A and $C(X,Z)$ were discussed and related to certain measures on the field and sigma-field generated by Z. In particular, a representation theorem was established that related the continuous linear functionals on the bounded members of $C(X,Z)$ to the regular finitely additive measures on the field generated by Z. (This result is essentially found in [DS], IV. 6.2 for normal topological spaces.) Furthermore, if the family Z satisfied a compactness condition (#) every countable subfamily of Z with the finite intersection property has non-empty intersection, then the measures could be chosen to be regular and countably additive on the sigma-field generated by Z.

The preceding construction evidently raises a number of questions. What is the relationship between A and C(X,Z)? What are the properties of C(X,Z)? Why can one obtain a satisfactory representation theory? A complete answer to the first two questions is given in $[Ha]_9$. First, note that A is a subfamily of C(X,Z). It can be shown that C(X,Z) is the smallest uniformly closed vector lattice and ring which is inversion-closed and contains A. Hence A = C(X,Z) if and only if A is inversion-closed. The latter fact is also essentially found in Hausdorff's discussion of complete ordinary function systems, which are precisely the objects C(X,Z) (see [Hs]- 41.III and 41.VIII and $[M]_2$) and indicates that very early in the century these families were already being considered. The connection with uniform spaces is the following. Given a uniformly closed vector lattice A: $X \to \mathbb{R}$, let u_A denote the smallest uniformity on X such that every member of A is uniformly continuous. It is shown in $[Ha]_5$ that for each separable uniformity u there exists a special uniformity mu containing u which has the basis consisting of all countable cozero covers (with respect to u) and satisfies Coz(muX) = Coz(uX). Furthermore, U(muX) is the smallest uniformly closed inversion-closed vector lattice of functions which contains U(uX). In terms of the preceding work, we obtain $C(X,Z) = U(mu_A X)$ and $A = C(X,Z)$ if and only if $U(u_A X) = A = U(mu_A X)$. Hence in this setting, we see not only that the Alexandrov formulation is somewhat clarified, but also that the operation $u \rightsquigarrow mu$ induces a categorical isomorphism between a certain class of uniform spaces and the class of uniformly closed inversion-closed vector lattices or real-valued functions ($[Ha]_5$, section 6). The separable uniformities u such that u = mu are called <u>metric-fine</u>; the reason for the name comes from the fact that uX is metric-fine if and only if each uniformly continuous mapping $f\colon uX \to M$ to a metric space M is also uniformly continuous with respect to the fine uniformity on M (which has the basis consisting of all open covers of M). The basic theory of separable metric-fine uniform spaces is developed in $[Ha]_5$; the work of Gordon [Go] also contains results of a related nature.

The preceding discussion shows that the Alexandrov setting is essentially uniform space setting, but why does there exist a satisfactory representation theory? For example, if Z is the family of zero sets of a Tychonov space X, why can every positive linear functional on C(X,Z) (the family of all continuous real-valued functions) be represented by a countably additive measure on the Baire sets (the sigma-field generated by Z)? (This is a result of Hewitt [He].) One possible answer lies in the fact that the vector lattice C(X,Z) is inversion-closed. A recent result tends to support this assertion. M. Zahradník has shown ([SUS], 1973 - 1974) that U(uX) is inversion-closed if and only if u has the Daniell property: whenever $(f_n) \subseteq U(X)$ and $f_n \downarrow 0$, then (f_n) is equi-uniformly continuous. This result is related to our earlier discussion of the convergence of integrals - for a Monotone Convergence theorem to hold, monotone increasing families must exhibit some "collective" behavior and this behavior is provided by the inversion property.

Our discussion has indicated how the uniform concept of a metric-fine space has naturally arisen from fundamental analytic questions. Of course, our restriction to real-valued mappings means that we need only consider separable uniformities; for the discussion of mappings with infinite dimensional range the preceding work is not sufficient. During the period 1972-1978, a general theory of metric-fine spaces was developed by Z. Frolík, A.W. Hager, and M.D. Rice (see $[Fr]_{3,7,11,12}$, and $[Ha]_{5,7,11}$, and $[Ri]_{1,2,6}$). Briefly, to each uniform space uX one may associate a smallest metric-fine uniformity mu containing u; that is, each uniformly continuous mapping $f: muX \to M$ to a metric space is also uniformly continuous with respect to the fine uniformity on M, and this assignment is functorial: if $g: uX \to vY$ is uniformly continuous, then $g: muX \to mvY$ is also uniformly continuous. The uniformity mu has the basis of covers of the form $\{C_n \cap U_s^n\}$, where $\{C_n\}$ is a countable cozero cover and each cover $\mathcal{U}^n = \{U_s^n\}$, $n=1,2,\ldots$ is a member of u. It follows that each countable member of mu may be refined by a countable u-cozero cover; hence mu and u have the same cozero sets and in a certain sense m is the largest operator on uniform spaces which does not alter cozero structure. There are many other characterizations of mu - see for example [Fr]$_{13}$. Here we will only note that the idea of "piecing" together covers like the one given above seems to be central to a number of ideas connecting uniform spaces and the descriptive theory of sets. We will also see this idea used in our discussion of measurable spaces in section three.

There are several additional connections between uniform theory and representation theory which should be noted. Recall that when the compactness condition (#) of Alexandrov is satisfied, the measure representing a bounded functional can be assumed countably additive (this is essentially found in [DS], III.5.13). The family of zero sets of uX satisfies (#) (is a semi-compact paving in the sense of [Me]) if and only if uX is <u>precompact</u> and <u>metric-fine</u> (equivalently, precompact and U(X) is inversion-closed). Following Glicksberg

[Gl] (see (De), 4.5.15), one can also show that this condition is equivalent to each of the following conditions: (i) every positive linear functional on $U_b(X)$ satisfies the Monotone Convergence theorem, (ii) every finitely additive Baire measure (regular with respect to zero sets) is countably additive, (iii) U(X) satisfies Dini's theorem: if $(f_n) \subseteq U_b(X)$ and $f_n \downarrow 0$, then f_n converges uniformly to zero. Notice that condition (iii) follows naturally from the Zahradník characterization - if $f_n \downarrow 0$, (f_n) is an equi-uniform family on a precompact space and hence by the Arzela-Ascoli theorem precompact in the uniformity of uniform convergence, so f_n must converge uniformly to zero. Hence the notion of a precompact metric-fine space represents the amalgamation of several important analytical concepts.

I remarked earlier that the representation $A = U(u_A X)$ is valid for every inversion-closed vector lattice A of real-valued funcitons. Representations of this type have been studied by a number of authors including Isbell, Hager, Fenstad, Csaszar, and Nöbeling-Bauer in the context of generalized approximation theorems. Here I will cite some examples which are relevant to the present subject matter. The reader is referred to $[Ha]_{4, 8, 10}$ for further details and references.

If A is a vector lattice of <u>bounded</u> real-valued functions on X, Nöbeling and Bauer showed that A is always uniformly dense in $U(u_A X)$, a result which is essentially equivalent to the usual Stone-Weierstrass theorem (for real-valued functions). It follows from this result that when A is a subalgebra of $U_b(uX)$ and $u_A = pu$, then A is uniformly dense in $U_b(uX)$ (for X compact this condition is equivalent to the assumption that A separates the points of X). In another direction, Isbell $[I]_1$ established that each composition-closed vector lattice A of real-valued functions has the form $A = U(u_A X)$. It follows from either of the preceding results that the family R of (proper) Riemann integrable functions on a closed interval X can be represented as $U(u_R X)$. It should be noted that in general such a representation is not possible - the family A of Lebesgue integrable functions on $(0,\infty)$ provides a nice example (for $f(x) = (x+1)^{-2}$ is a member of A and the mapping $g(t) = |t|^{\frac{1}{2}}$ is a member of U(R), but $g \circ f$ is not a member of A). Finally, we note that the following approximation results are valid for any vector lattice A of real-valued functions (see $[Ha]_4$): (i) every non-negative member of $U(mu_A X)$ is the pointwise limit of an increasing sequence from A and (ii) A is sequentially dense in $U(mu_A X)$ equipped with the compact-open topology. The first result is clearly related to questions involving the extension of a positive functional L on A to a Daniell integral on C(X,Z) (or $U(mu_A X)$ in the present notation); for the extended functional $\check{L}$ to satisfy the Monotone Convergence theorem, $\check{L}$ must be defined from L in the natural way - if $(f_n) \subseteq A$ and $f_n \uparrow f$, we must have $L(f_n) \to \check{L}(f)$, so $\check{L}$ is a unique extension of L. It is also interesting to note the following consequence of (i). If the weak topology generated by a vector lattice A of real-valued functions is Lindelöf, every non-negative continuous real-valued function on X is the pointwise limit of an increasing sequence from A. (This follows from the fact that the only compatible metric-fine uniformity on a Lindelöf space is the fine uniformity.)

IIIA. Measurable funcitons and sigma-fields

The second half of the preceding section demonstrated how analytic concepts have suggested ideas in uniform spaces, which in turn have acted as unifying themes. The present section will present further modern uniform ideas that are connected with some central ideas in abstract analysis.

Everyone intuitively knows that measurability of a mapping is really a <u>global</u> concept. This idea may be expressed in a number of ways: (i) the topology generated by the measurable mappings is usually discrete, (ii) measurability (in the absence of a measure, density topology, etc.) can not be expressed locally, or

(iii) in the following manner. Let (X,Σ_1) and (Y,Σ_2) be measurable spaces. Given a sigma-field Σ, let $\underline{u(\Sigma)}$ denote the uniformity with the basis of countable Σ-partitions of the underlying set. Then one can easliy establish that $f\colon X\to Y$ is measurable (with respect to Σ_1 and Σ_2) if and only if $f\colon u(\Sigma_1)X\to u(\Sigma_2)Y$ is uniformly continuous. Moreover, if S is a separable metric space with the usual Borel structure, one obtains the equality $M(X,S) = U(u(\Sigma)X,S)$. Hence one can say that measurability is a global concept because the real-valued measurable mappings are precisely the uniformly continuous mappings with respect to an appropriate uniformity! This idea was introduced in the early 1970's by A.W. Hager and Z. Frolík. One may also take into account the almost everywhere setting. If (X,Σ,μ) is a measure space, the family of separable metric valued mappings on X which are measurable a.e. is the set of members of $U(\check{u}(\Sigma)X,M)$, where $\check{u}(\Sigma)$ has the basis of partitions of the form $\{A_n\colon n=1,2,\ldots\}\cup\{\{p\}\colon p\in A_0\}$, where each A_n is a member of Σ and $\mu(A_0) = 0$.

It is natural to inquire about the structure of $u(\Sigma)$. Using familiar facts about measurable mappings, one can easily show that $U(u(\Sigma)X)$ is a Von Neumann regular ring closed under the formation of sequential point-wise limits. In particular, the ring is inversion-closed, so one suspects some connection with the metric-fine spaces and representation theory discussed in section two. This is indeed the case. Following $[\text{Ha}]_6$, define a separable uniform space uX to be <u>measurable</u> if $u = u(\Sigma)$ for some sigma-field Σ on the set X (which must therefore be Baire(uX)). One may then establish the following representation theory and characterizations of measurable uniform spaces (see $[\text{Ha}]_6$). There is a categorical isomorphism between the Von Neumann regular rings of real-valued functions and the separable measurable uniform spaces - in particular, if A is such a ring, the uniformity u_A is measurable. Furthermore, the following statements are equivalent for a separable uniform space uX: (i) uX is measurable, (ii) uX is metric-fine and Coz(uX) is a sigma-field, (iii) U(uX) is closed under the formation of sequential pointwise limits, (iv) every uniform subspace of uX is metric-fine, and (v) U(uA) is inversion-closed for each subspace A of X. The last two conditions are particularly enlightening; in some sense the present discussion is just a hereditary (on every subspace) version of the discussion given in section two. There is also a measurable operator m_*, which assigns to each separable uniformity u the uniformity $m_* u = u(\Sigma)$, where Σ is the sigma-field Baire(uX). $m_* u$ is the smallest measurable uniformity containing u and the assignment $u \rightsquigarrow m_* u$ is functorial in nature.

In the discussion above we restricted our attention to separable uniform spaces. In view of the equivalent conditions (i)-(v), there are several possible ways to extend the preceding theory to arbitrary uniform spaces. Z. Frolík $[Fr]_5$ defined uX to be <u>measurable</u> if U(uX,M) is closed under the formation of pointwise sequential limits for every metric space; Rice $[Ri]_1$ essentially defined the same concept by requiring that every uniform subspace is metric-fine; in particular the definitions are equivalent ($[Fr]_5$) and each is equivalent to the condition that uX is metric-fine and Coz(uX) is a sigma-field. There is an associated functorial operation m_* which assigns to u the smallest measurable uniformity m_*u containing u, with m_*u having a basis of covers of the form $\{B_n \cap U_s^n\}$, where $\{B_n\}$ is a countable Baire partition of X and each $\mathcal{U}^n = \{U_s^n\}$, n=1,2,... is a member of u (hence once again a certain uniformly local operation on covers seems to have a special significance for analytic concepts). Furthermore, the uniformity generated by the countable members of m_*u is precisely u(Baire(X)), so u and m_*u have the same Baire sets. Many other results about the theory of measurable uniform spaces can be found in $[Fr]_{4-6}$ and $[Ri]_{1,2}$; we will have occasion to mention some of these in a moment. First, however, I would like to mention one problem with the non-separable theory described above.

The problem is simply that the metric-valued measurable mappings (with respect to Baire(uX)) need not coincide with the mappings which are uniformly continuous with respect to $m_* u$ (and the metric uniformity on the range space). In fact, a similar problem was present in the metric-fine theory: the cozero mappings from uX to an arbitrary metric space M (those for which the inverse of an open set is a cozero set) are not precisely the members of U(muX,M) in all cases; the latter family is usually smaller! In the present setting we have the same problem - $U(m_* uX,M)$ is generally a proper subset of the Baire measurable mappings from uX to M. To illustrate the problem, I will use a set-theoretic example which can be constructed using Martin's axiom and the negation of the Continuum Hypothesis. (There are other examples which do not require special set-theoretic assumptions.) Let X be an uncountable subset of the real line such that every subset of X is a relative G_δ set. Let u denote the subspace uniformity on X and let M denote the metric space based on X with the discrete 0-1 metric. Then every subset of X is a member of Baire(uX), so the identity mapping i: X→M is Baire measurable, but i is not uniformly continuous with respect to $m_* u$ since this uniformity has a basis consisting of countable covers.

Perhaps the problem stems from a wrong choice for the uniformity. Here is another possibility. Given a measurable space (X,Σ), let u_Σ denote the smallest uniformity on X such that every Σ-measurable mapping to a metric space is uniformly continuous (in the above example this uniformity consists of all covers of X). By definition, for every metric space M, Baire(uX,M) is contained in U(u_ΣX,M) (Σ= Baire (uX)). Hence when $m_* u = u_\Sigma$, the three classes Baire(uX,M), U(m_*uX,M), and U(u_ΣX,M) coincide. Conversely, one may prove (i) that if Baire(uX,M) = U(m_*uX,M) for every metric space M, then $m_* u = u_\Sigma$ and (ii) that this property holds exactly when the uniform space m_*uX is finest in its proximity class (see (Ri)$_{3,4}$), that is to say, when m_*u is the largest uniformity which induces the nearness relation δ, where $A \delta B$ if they cannot be separated by disjoint cozero sets. The measurable proximally fine spaces are called Baire-fine and have been studied in [Fr]$_{4-6}$, [Ha]$_{11}$, and [Ri]$_{2,4,5}$.

There is a secondary question implicit in the above discussion. Even though the families of mappings need not coincide, perhaps the sigma-fields do coincide, i.e. Baire (u_ΣX) = Σ for every measurable space (X,Σ). For example, this equality is implied by the equality $m_* u = u_\Sigma$. Unfortunately, Σ is generally only a proper subfamily of Baire(u_ΣX), as the following example illustrates. Let Y be a set with power greater than c and let Σ be the sigma-field on X = YxY generated by all rectangles A x B. Each of the covers $\mathcal{U}_1$ = {{y}x Y:y$\in$Y} and $\mathcal{U}_2$ ={Y {y}: y$\in$Y} is a member of u_Σ; since the meet of the two covers consists of all singleton subsets of X, it follows that u_Σ contains all covers of X, so Baire($u_\Sigma X$) contains all subsets of X. On the other hand, it is well known that for $|Y|>c$, the diagonal $\{(y,y): y\in Y\}$ is not a member of Σ.

The further analysis of sigma-fields requires a notion which lies at the heart of modern work in descriptive set theory - complete additivity.

Let (X,Σ) be a measurable space. A disjoint subfamily A of Σ is called completely additive if the union of every subfamily of A is a member of A. The following two examples will serve as orientation. First, every completely additive Baire family in a compact space or a complete separable metric space is countable. Second, there are clearly uncountable completely additive Lebesgue measurable families; consider the family of singleton sets in any uncountable set which has measure zero. One may prove that the family of completely additive Σ-partitions of X is a subbasis for the uniformity u_Σ; this means that every member of u can be refined by a finite meet of completely additive Σ-partitions. (Note that in the above example the family of such partitions is not a basis for u_Σ.) If the family of completely additive Σ-partitions is a basis for u_Σ, we say that Σ is proximally fine (see [Fr]$_1$ and [Ri]$_{3,4}$).

Each of the following statements is equivalent to the assertion that Σ is proximally fine. (i) Σ = Baire($u_\Sigma X$), (ii) u_Σ is the largest uniformity on X such that Σ = Baire($u_\Sigma X$), and if one of these conditions holds, the uniformity u_Σ must be measurable and proximally fine (but this latter statement is not equivalent to the above conditions as the preceding example shows). One may also establish that Σ is proximally fine if and only if for each countable family $\{f_n : X \to M_n\}$ of metric-valued Σ-measurable mappings, the mapping $(f_n): X \to \prod_{n=1}^{\infty} M_n$ is Σ-measurable with respect to the Borel field on the product space (see $[Ri]_5$). Examples of proximally fine sigma-fields include (i) all countably generated sigma-fields (assuming the Continuum Hypothesis), (ii) all sigma-fields on sets of power at most c (assuming Martin's axiom), (iii) the Lebesgue measurable sets (essentially communicated to me by W. Fleissner, but also a consequence of work in $[Fr]_{14}$), and (iv) the Borel field on every complete metric space. We will have more to say about (iv) below.

The following material will hopefully provide some justification for the introduction of the preceding concepts. Let uX be an arbitrary uniform space. I previously described a special basis for m_*u which, unfortunately, is not the most useful one for comparing m_*u and u_Σ(Σ= Baire(uX)). One may show, however, that m_*u also has the basis consisting of all σ-uniformly discrete (with respect to u) Baire partitions of <u>bounded class</u> $[Fr]_5$, that is, there exists a countable ordinal α such that each member of the cover belongs to Baire_α (X) and the cover has the form $\bigcup_{n=1}^{\infty} C^n$, where each family C^n is uniformly discrete with respect to u. Since m_*u is always contained in u_Σ, the question of equality hinges on whether every completely additive Baire partition has bounded Baire class and can be decomposed into a countable number of uniformly discrete subfamilies. In the early 1970's, the work by Hansell $[Hn]_{1,2}$ on σ-discrete decomposability almost achieved the required decomposition for a complete metric uniformity u, without proving that completely additive Baire families must be of bounded Baire class. This fact was established by Preiss in $[Pr]_1$. The following theorem could then be established: $[Ri]_{4,5}$ - for every complete metric space uX, m_*uX is proximally fine ($m_*u = u_{Borel}$) and Borel(X) is proximally fine. Other authors (see [F1] and $[Po]_{1,2}$) have also considered analogous decompositions for point-finite completely additive families in general metric spaces using additional set-theoretic assumptions. For example, it follows from work in [F1] that (i) assuming a set-theoretic axiom stronger than the Continuum Hypothesis ($\diamondsuit$ for stationary subsets of ω_1), m_*M is proximally fine for every metric space of power at most c, and (ii) in a model of set theory obtained by collapsing a supercompact cardinal to ω_2, m_*M is proximally fine for every metric space M.

One of the primary consequences of the preceding theorem is that for a complete metric domain (or a metric domain, depending on your set-theoretic inclinations) classical statements about measurable mappings can be exactly formulated as statements about uniformly continuous mappings. The following results illustrate this idea. Let M be a complete metric space. First, if $\{f_n: M \to M_n\}$ is a countable family of metric-valued Borel measurable mappings, then $(f_n): M \to \prod_{n=1}^{\infty} M_n$ is also a Borel mapping. Second, if (M_n) is a countable family of complete metric spaces, then Borel $(\prod_{n=1}^{\infty} M_n)$ is the smallest proximally fine sigma-field which contains the product sigma-field $\bigotimes_{n=1}^{\infty}$ Borel(M_n). Finally, if $f:M \to N$ is a Borel measurable mapping to the metric space N, the graph of f $(= \{(x,f(x)): x\ M\})$ is a Borel set in M x N. There are further consequences related to the descriptive theory of sets which will be explored in the next section. Before discussing these results, I would like to suggest another question related to proximally fine sigma-fields. First, to the best of my knowledge, there are no examples of metric spaces M such that Borel(M) is not proximally fine. Now consider the following difinition: a countable collection $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ of disjoint completely additive subfamilies of a sigma-field Σ <u>weakly generates</u> Σ if the collection $\mathcal{B}$ of all unions of subfamilies of $\mathcal{A}$ generates Σ in the usual sense - Σ is the smallest sigma-field containing

B. (The definition is motivated by the existence of a σ-discrete basis for any metric topology.) Then one may establish the following (unpublished) representation theorem. If Σ is a proximally fine sigma-field and contains a weakly generating subfamily, then Σ = Borel(M) for some zero-dimensional metric space M. This result can be viewed as a sigma-algebraic metrization theorem, which may possibly characterize the Borel families of metric spaces (in a manner analogous to the familiar characterization on the Borel families on separable metric spaces as precisely the countably generated sigma-fields).

IIIB. Descriptive set theory

The present section will consider a selection of topics in the non-separable theory of descriptive sets which are closely connected with uniform concepts (such as uniform discreteness and measurable uniformities).

Let uX be an arbitrary uniform space. Using the descriptions of m_*u previously given, one can show that every uniformly continuous metric-valued mapping $f: m_*uX \to M$ is a Baire mapping of class α, for some $\alpha < \omega_1$. Now assume that $g: X \to M$ is a metric-valued Borel measurable mapping on the complete metric space uX. By the preceding work, g is a member of $U(u_{Borel}X, M) = U(m_*uX, M)$; hence g is a Baire mapping of class α, for some $\alpha<\omega_1$ (see [Ri]$_4$ and [Fr]$_2$ - the proof in the second paper is incomplete, but it is essentially completed by the result in [Pr]$_1$). The fact that we just established answers an old question of Kuratowski in [K]$_2$.

The preceding result is valid for metric-valued measurable mappings defined on any uX such that $m_* uX$ is proximally fine. In particular, Tashjian [Ta]$_2$ has shown that $m_* uX$ is proximally fine for any arbitrary product of complete metric spaces by proving the following factorization theorem, which is interesting in its own right. Let $\{X_i: i \in I\}$ be an uncountable family of non-empty sets and let Σ be the sigma-field on $X = \prod_{i \in I} X_i$ consisting of all sets which depend on countably many co-ordinates (where $A \subset X$ depends on the coordinates $J \subset I$ if $a \in A$ and $y_j = a_j$ for $j \in J$ implies $y \in A$). Let $f: X \to M$ be a metric-valued mapping on the product space. Then g may be factored in the form $h \circ p$, where $p: X \to \prod_{j \in J} X_j$ ($|J| \leq \aleph_0$,) is a projection onto a countable subproduct, if and only if g is Σ-measurable. Now if uX is the product of the uniform spaces $u_i X_i$, one can show that every Baire(uX) set (in fact, every Souslin set) is a member of Σ; hence we obtain the following corollary: every metric-valued Baire measurable mapping on a product of uniform spaces may be factored through a countable subproduct. Using this result, one can make the following comprehensive statement about measurable mappings on special uniform spaces $[\mathrm{Ri}]_5$. Assume that $m_* uX$ is proximally fine. If $f: uX \to M$ is a metric-valued Baire measurable mapping, there exists a metric space M_2 which isometrically contains M and a complete metric space M_1 such that f may be factored as

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & M \subset M_2 \\ & {\scriptstyle e}\searrow \quad \nearrow{\scriptstyle g} & \\ & M_1 & \end{array}$$

$g \circ e$, where $e: uX \to M_1$ is uniformly continuous and $g: M_1 \to M_2$ is Borel measurable.

We will now turn to the discussion of separation theorems in the descriptive theory of sets. First, recall the classical results of Lusin (First Separation theorem) and Souslin: in a complete separable metric space disjoint Souslin sets can be separated by a Borel set and the bi-Souslin sets are precisely the Borel sets. This theorem was extended to the non-separable setting by Hansell in $[\mathrm{Hn}]_2$ - disjoint Souslin sets in a complete metric space can be separated by a hyperBaire set, where hyperBaire(uX) is the smallest sigma-field containing the zero sets that is closed under the formation of uniformly discrete unions (with respect to u). Hansell also established that every hyperBaire set in a metric space is a bi-Souslin set, so the equality hyperBaire(X) = bi-Souslin(X) is valid for complete metric spaces.

In [Ri]$_3$ I considered some generalizations of these results to uniform spaces and established the following two theorems. First, each Souslin (resp. bi-Souslin or Baire) set in a uniform space is the pre-image of a Souslin (resp. bi-Souslin or Baire) set in some separable metric space under a uniformly continuous mapping. Second, each pair of disjoint Souslin sets in a uniform space may be separated by a bi-Souslin set; moreover, essentially <u>any</u> result in the descriptive theory of sets which is valid for <u>all</u> separable metric spaces is valid for <u>all</u> uniform spaces. Unfortunately, the relationship between hyperBaire sets and bi-Souslin sets in a general setting is somewhat unclear. It follows from the first result mentioned above that every bi-Souslin set in a uniform space is a hyperBaire set, but the converse is generally false. Consider the following example. For each ordinal $\alpha<\omega_1$, let X_α be a copy of the countable ordinals with the uniformity consisting of all covers and let $Z = \Pi X_\alpha$. For each $\alpha<\omega_1$, define $S_\alpha = \{z\in Z: z_\alpha = \alpha\}$. Then each S_α is a member of Coz (Z) and $S = \bigcup S_\alpha$ is a hyperBaire set which is not a bi-Souslin set, since it does not depend on countably many co-ordinates. In fact, the example provides the key to the following (unpublished) result of Tashjian-Rice: Let X be an arbitrary product of complete metric spaces and let Σ be the sigma-field consisting of all sets which depend on countably many co-ordinates. Then bi-Souslin(X) = hyperBaire(X) $\cap \Sigma$.

We will conclude this section with some connections between the Lusin First Separation theorem and the uniformly local operation on covers that I have mentioned before. We will need the following definition. A uniform space uX is <u>locally fine</u> (see $[I]_2$, chapter 7 and [GI]) if each cover of the form $\{V_s \cap U_t^s\}$ is a member of u, where $\{V_s\}$ and each $\mathcal{U}^s = \{U_t^s\}$ is a member of u. This property forces nice behavior on the descriptive set-theoretic sets associated with uX; for example, if uX is locally fine, each of the classes Coz(X) and Souslin(X) is closed under the formation of uniformly discrete unions. In 1977, Jan Pelant (unpublished manuscript) characterized the locally fine spaces as precisely the uniform subspaces of fine spaces. One should also note that every separable metric-fine space is locally fine, but not every measurable space. The following theorem asserts the last statement in a forceful manner. Let uX be a complete metric space. Then the following statements are equivalent $[Ri]_5$: (i) bi-Souslin(X) = Borel(X), (ii) m_*uX is locally fine, and (iii) M is the union of a separable and a σ-discrete subspace. Hence the complete metric spaces which satisfy the Lusin theorem are essentially separable (the perfect kernel, which is the largest dense in itself subspace, is separable) and this condition can be expressed in uniform terms.

Further descriptive set-theoretic conditions on Baire-fine spaces which guarantee the locally fine property are found in $[Fr]_4$ and $[Ri]_2$.

IIIC. Uniform measures

The intuitive discussion of linear functionals on the space C(X,Z) given in section two can be formalized into a theory of <u>uniform measures</u>, which includes the theories of countably additive and cylindrical measures as special cases. This generalization has been considered by a number of the authors mentioned in the introduction. In particular, the reader is referred to Z. Frolík's articles $[Fr]_{10,15}$ and the thesis of A. Deaibes [De] for a further discussion of the present subject matter.

The following notation will be needed for the discussion. An equi-uniformly continuous uniformly bounded subfamily of $U_b(uX)$ will be called a UEB set; an equi-uniformly continuous pointwise bounded subfamily of U(uX) will be called a UE set. M(X) will denote the dual space of the Banach space $U_b(X)$. We say that μ in M(X) is a <u>uniform measure</u> if μ is continuous (in the pointwise topology) on every UEB set. Analogously, we say that a linear functional μ on U(X) is a <u>free uniform measure</u> if μ is continuous in the pointwise topology on every UE set. The family of uniform (resp.

free uniform) measures is denoted by $M_U(X)$ (resp. $M_F(X)$). We will also need the following familiar concepts. μ in $M(X)$ is σ-additive if $(f_n) \subseteq U_b(X)$ and $f_n \downarrow 0$ implies $\mu(f_n) \to 0$ and Radon if μ is continuous on the unit ball of $U_b(X)$ equipped with the compact-open topology. The family of σ-additive and Radon measures is denoted by $M_\sigma(X)$ and $M_t(X)$, respectively. One can establish that every Radon measure is a σ-additive uniform measure; further relationships between the classes will be discussed below.

Intuitively, the uniform measures represent the linear functionals on $U_b(X)$ which can be uniformly approximated on UEB sets by molecular measures (the finite linear combinations of point-mass (Dirac) measures). Formally, one defines the complete vector space topology on $M_U(X)$ to be the topology of uniform convergence on the UEB sets; by general results in duality theory, it follows that the continuous linear functionals on $M_U(X)$ are represented exactly by the members of $U_b(X)$. Furthermore, the canonical injection $\delta: X \to M_U(X)$ defined by sending x to the Dirac measure δ_x is a uniform embedding, with the molecular measures dense in $M_U(X)$. It follows from this fact that every uniformly continuous mapping $h: X \to E$ to a complete locally convex linear space with bounded range can be uniquely extended to a continuous linear mapping $\check{h}: M_U(X) \to E$ such that $\check{h} \circ \delta = h$. (The statements given above also have analogues for free uniform measures using the topology of uniform convergence on UE sets.)

The preceding ideas can be used to express basic ideas about linear functionals in terms of uniform measures on special uniform spaces. Recall from the earlier discussion of vector lattices A of real-valued functions on X that $U(mu_A X)$ is the smallest inversion-closed vector lattice containing A. In particular, if $A = U(uX)$, the theorem for extending Daniell integrals from A to $U(mu_A X)$ is formalized by the following result ([De],p.105) : $M_\sigma(X) = M_\sigma(meuX)$, where eu is the uniformity generated by all countable uniform covers in u. Furthermore, ([De],4.1.10) one obtains $M_\sigma(uX) = M_U(meuX)$, so the σ-additive measures are exactly the uniform measures with respect to the special metric-fine uniformity meu (which has the basis consisting of all countable u-cozero covers). Then the family $ca(\Sigma)$ consisting of all bounded countably additive set functions on the measurable space (X,Σ) may be represented as $ca(\Sigma) = M_\sigma(u(\Sigma)X) = M_U(u(\Sigma))X)$, where $u(\Sigma)$ is the measurable uniformity generated by the countable Σ-partitions. This result expresses the following approximation theorem. Let F be a UEB set with respect to $u(\Sigma)$ (that is F is a uniformly bounded family and for each $\delta>0$, there exists a countable Σ-partition $\{A_n\}$ such that every member of F varies by at most δ on each A_n). Then for each $\varepsilon>0$ and μ in $ca(\Sigma)$, there exists a molecular measure

$\sum_{i=1}^{n} c_i\delta_{x_i}$ such that $\sup_{f\in F} |\int f\, d\mu - \sum_{i=1}^{n} c_i f(x_i)| < \varepsilon$.

One may also show that the uniform measures associated with metric spaces provide an alternate description of the Radon measures. By ([De],4.4.6), if X is metric and μ is a positive linear functional on $U_b(X)$, then μ is a uniform measure if and only if for each $\varepsilon>0$, there exists a closed precompact subset P of X such that $|\mu(f)| < \varepsilon \, ||f||_\infty$ for each f in $U_b(X)$ which vanishes on P. Using this result, one can establish that for a (complete) metric space X, $M_U(X)$ is precisely the set of linear functionals on $U_b(X)$ which are continuous on the unit ball of $U_b(X)$ in the topology of uniform convergence on (compact) precompact sets. Hence for a complete metric space X, one has ([De],4.4.8,[Fr]$_{8-10}$): $M_t(X) = M_U(X) \subset M_\sigma(X)$. Pachl [Pa]$_3$,1.4) has also established the inclusion $M_U(X) \subset M_\sigma(X)$ for any uniform space X such that U(X) is inversion-closed and used this result to show that (under certain set-theorectic assumptions) on complete spaces of this type, the uniformity generated by U(X) must also be complete. Results of this form are called "generalized Katětov-Shirota" theorems. The preceding result and others have also been presented in [Ha]$_5$ and [Ri]$_6$; the reader is referred to [RR] for a discussion of results of this type.

In view of the above results, it is natural to inquire about the validity of the inclusion $M_\sigma(X) \subset M_U(X)$. We have already seen that $M_\sigma(uX) = M_U(meuX)$; using the fact that $eu \subset meu$ it follows that $M_\sigma(uX) \subset M_U(euX)$. Unfortunately, the general answer depends on set-theoretic assumptions. Define uX to be a D-space if no uniformly discrete subspace of X has real-measurable cardinality. Then ([Pa],2.1) establishes that X is a D-space if and only if $M_\sigma(X) \subset M_U(X)$. Since it is not known whether c is real-measurable and the statement that real-measurable cardinals exist is consistent with the usual axioms of set theory, the status of the inclusion $M_\sigma(X) \subset M_U(X)$ remains somewhat unclear.

For a final example, we will demonstrate (following [CS]) how the concept of a cylindrical measure can be expressed in terms of uniform measures. Let E and F be a pair of vector spaces in duality and let u be the smallest uniformity on E such that every member of F is uniformly continuous. Then the members of the corresponding space of uniform measures $M_U(uE)$ are exactly the projective limits of Radon measures taken over finite dimensional quotients of E, so $M_U(uE)$ is the space of cylindrical measures.

There are a number of results in the preceding theory which have not been mentioned. For example, there are results involving functorial questions, vector-valued uniform measures, compactness in spaces of uniform measures, and topics in free uniform measures which have also been investigated (see [CS], $[Fr]_{10}$, $[Pa]_{2-4}$,[R], [Fe], [Be],[To], and the bibliography in [De]).

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