TSUKUBA J. MATH.
Vol. 43 No. 2 (2019), 191–210

SHIMURA AND SHINTANI LIFTINGS OF CERTAIN CUSP FORMS OF HALF-INTEGRAL AND INTEGRAL WEIGHTS

By

Manish Kumar Pandey, B. Ramakrishnan and Anup Kumar Singh

Abstract. In [4], W. Kohnen constructed explicit Shintani lifts from the space of cusp forms of weight $2k$ on $\Gamma_0(N)$, which are mapped into the plus space $S^+_{k+1/2}(\Gamma_0(4N))$, where N is an odd integer. This lifting is adjoint to the (modified) Shimura map defined by him with respect to the Petersson scalar product. Using this construction along with the theory of newforms on $S^+_{k+1/2}(\Gamma_0(4N))$ (where N is an odd square-free natural number) developed in [3], he derived explicit Waldspurger theorem for the newforms belonging to the space $S^+_{k+1/2}(\Gamma_0(4N))$. Further in this work, Kohnen considered the ± 1 eigen subspaces corresponding to the Atkin-Lehner involution and showed that the intersection of this $\pm$ spaces with the corresponding newform spaces are isomorphic under the Shimura correspondence. A natural question is whether a similar result can be obtained for the intersection of this (± 1) subspaces with the space of oldforms. In this direction, S. Choi and C. H. Kim [2] considered the case where N is an odd prime p and constructed similar Shimura and Shintani maps between subspaces of forms of half-integral and integral weights. The subspaces considered by Choi and Kim are nothing but $+1$ eigenspace under the Fricke involution. In this paper, we generalise the work of Choi and Kim to the case where N is an odd square-free natural number.

2010 *Mathematics Subject Classification*: Primary 11F37, 11F67; Secondary 11F11.
Key words and phrases: Shimura and Shintani correspondences, Modular forms, Atkin-Lehner W-operators.
Received May 13, 2019.
Revised September 12, 2019.

1. Introduction

Let $\Gamma_0(N)$ denote the congruence subgroup of the full modular group $SL_2(\mathbf{Z})$. The work of G. Shimura [6] in 1973 gave the foundation of the theory of modular forms of half-integral weight and also a correspondence between the spaces of modular forms of half-integral weight and integral weight. Later in 1975, T. Shintani [7] used theta kernel to construct a modular form of half-integral weight from a given modular form of integral weight. These are referred to in the literature as the Shimura and Shintani correspondences. In [3], W. Kohnen constructed a canonical subspace of $S_{k+1/2}(\Gamma_0(4N))$ (the $\mathbf{C}$-vector space of all cusp forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4N)$), when N is an odd positive integer. This subspace is called the plus space, denoted by $S^+_{k+1/2}(\Gamma_0(4N))$, which consists of cusp forms $f \in S_{k+1/2}(\Gamma_0(4N))$, whose n-th Fourier coefficients $a_f(n)$ satisfy the condition that $a_f(n) \neq 0$ implies that $(-1)^k n \equiv 0, 1 \pmod 4$. Following Kohnen, we also denote the plus space $S^+_{k+1/2}(\Gamma_0(4N))$ simply by $S_{k+1/2}(N)$. When N is odd, Kohnen [3] defined a modified Shimura lift on $S_{k+1/2}(N)$, which is mapped into the space $S_{2k}(N)$, the vector space of cusp forms of weight $2k$ on $\Gamma_0(N)$. In that work he also developed an analogous theory of newforms on $S_{k+1/2}(N)$, when N is odd and square-free and showed that the subspaces spanned by newforms in the respective spaces $S_{k+1/2}(N)$ and $S_{2k}(N)$ are isomorphic under a linear combination of the modified Shimura lifts.

By constructing a kernel function, Kohnen [4] constructed Shintani liftings from the space $S_{2k}(N)$ into the plus space $S_{k+1/2}(N)$, when N is odd. Moreover, by construction, these Shintani liftings are adjoint (with respect to the Petersson scalar product) to the modified Shimura lifts defined by him. The W-operators (introduced by Atkin and Lehner in [1] in the case of integral weight modular forms) play a crucial role in the study of the theory of newforms in the space of modular forms (of integral and half-integral weights). In [3], Kohnen introduced analogous W-operators in the space $S_{k+1/2}(N)$, which we define below. Let p be a prime dividing N (since N is odd and square-free, p is an odd prime and $p^2 \nmid N$). Then the Atkin-Lehner W-operator in the space $S_{k+1/2}(N)$ is defined by

$$w_p := p^{-k/2+1/4} U(p) W(p), \tag{1}$$

where $U(p)$ is the usual U-operator defined by $U(p) : \sum_{n\geq 1} a(n) e^{2\pi i n z} \mapsto \sum_{n\geq 1} a(np) e^{2\pi i n z}$ and $W(p)$ is the operator defined in a similar manner as the Atkin-Lehner W-operator in the case of integral weight. Kohnen showed that the operator w_p is a hermitian involution and characterised its ± 1 eigenspaces in terms of certain properties of the Fourier coefficients of the forms. In fact, for a prime $p|N$, the ± 1 subspace corresponding to the W-operator w_p is defined as

$$S_{k+1/2}^{\pm,p}(N) := \{f \in S_{k+1/2}(N) \,|\, f|w_p = \pm f\}. \tag{2}$$

In [3], Kohnen showed that

$$S_{k+1/2}^{\pm,p}(N) = \left\{ f \in S_{k+1/2}(N) \,|\, a_f(n) = 0, \text{ if } \left(\frac{(-1)^k n}{p}\right) = \mp 1 \right\} \tag{3}$$

and proved that $S_{k+1/2}^{new}(N)$ and $S_{2k}^{new}(N)$ (respective subspaces generated by newforms) are isomorphic under a linear combination of the modified Shimura lifts. Further, he proved that the intersection of the newforms space with the above $\pm$, p subspace is also isomorphic via the (modified) Shimura correspondence to the subspace $S_{2k}^{new}(N) \cap S_{2k}^{\pm,p}(N)$, where $S_{2k}^{\pm,p}(N) = \{F \in S_{2k}(N) \,|\, F|W_p = \pm F\}$ (here W_p is the Atkin-Lehner operator in the space $S_{2k}(N)$ for $p|N$). A natural question is whether such a correspondence via the Shimura maps can be given for the intersection of the W-operator eigenspace with the old class. In 2017, S. Choi and C. H. Kim [2] considered this problem when N is an odd prime and showed that one can use the kernel function constructed by Kohnen [4] to define a new kernel function for the required mappings for the old class.

In this paper, we generalise the work of Choi and Kim to the case of odd and square-free level N. More precisely, we construct Shimura and Shintani maps corresponding to the subspaces of $S_{k+1/2}(N)$ and $S_{2k}(N)$, consisting of cusp forms which are eigenforms (with $+1$ eigenvalue) for the operator $\prod_{p|N} w_p$ and $\prod_{p|N} W_p$ respectively. We first decompose this $+1$ eigen subspace into a direct sum of component subspaces and we shall be deriving our mapping properties via these component subspaces. After presenting the necessary preliminary details we describe our main results in §2 and in section 3 we give a proof of our results.

2. Statement of Results

Throughout this paper $N > 1$ denotes an odd square-free natural number and we assume that N is a product of v prime divisors, i.e., $v(N) = v$.

As mentioned in the introduction, we use the notation $S_{k+1/2}(N)$ for the Kohnen plus space $S_{k+1/2}^{+}(\Gamma_0(4N))$. Let w_p be the Atkin-Lehner W-operator defined by (1) (due to Kohnen). The ± 1 eigen subspace (in $S_{k+1/2}(N)$) corresponding to each of the primes p dividing N is denoted as $S_{k+1/2}^{\pm,p}(N)$ (Eq. (2)). We now define the following subspaces of $S_{k+1/2}(N)$ and $S_{2k}(N)$, which are ± 1

eigen subspaces with respect to the product of the W-operators:

$$S_{k+1/2}^{\pm,N}(N) := \left\{ f \in S_{k+1/2}(N) \mid f|\prod_{p|N} w_p = \pm f \right\}, \tag{4}$$

$$S_{2k}^{\pm,N}(N) := \left\{ F \in S_{2k}(N) \mid F|\prod_{p|N} W_p = \pm F \right\}. \tag{5}$$

It is to be noted that in the case of integral weight, the Fricke involution $H_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ is equivalent to the product of all the W-operators. i.e., $H_N = \prod_{p|N} W_p$ on $S_{2k}(N)$.

The spaces $S_{k+1/2}(N)$ and $S_{2k}(N)$ are decomposed as orthogonal direct sums (with respect to the Petersson scalar product) of these $\pm$ subspaces as follows:

$$\begin{aligned} S_{k+1/2}(N) &= S_{k+1/2}^{+,N}(N) \oplus S_{k+1/2}^{-,N}(N), \\ S_{2k}(N) &= S_{2k}^{+,N}(N) \oplus S_{2k}^{-,N}(N). \end{aligned} \tag{6}$$

In this paper we shall be constructing the required correspondences between the spaces $S_{k+1/2}^{+,N}(N)$ and $S_{2k}^{+,N}(N)$. In order to get our construction of the maps, we need to further decompose these spaces as follows. As per our assumption, N is a product of v primes and so we consider the $(\pm, p)$ subspaces corresponding to each of the v primes and consider their intersection. Totally there are 2^v subspaces consisting of forms in $S_{k+1/2}(N)$ which are eigenfunctions with respect to all the w_p operators, $p|N$ with ± 1 eigenvalues. Let us assume that $f \in S_{k+1/2}(N)$ is an eigenfunction with respect to w_p with $+1$ eigenvalue for r number of primes (say $p_1, p_2, \ldots, p_r$) and with -1 eigenvalue for s number of primes (say $q_1, q_2, \ldots, q_s$), with $r + s = v$. The notation to keep the primes as p_i and q_j are only local and we do not assume these in the general situation. i.e., when we have a partition $r + s = v$, r, s are non-negative integers, we assume that the r primes are $p_1, p_2, \ldots, p_r$ and the s primes are $q_1, \ldots, q_s$. So, when we vary the numbers r, s in the partition of v, the number of primes in each group will also vary corresponding to the partition. Then it is clear that such an f belongs to $S_{k+1/2}^{+,N}(N)$ (resp. $S_{k+1/2}^{-,N}(N)$) only when s is even (resp. s is odd). Moreover, the number of such subspaces is equal to $\binom{v}{s}$. Let us denote the subspace as defined above for a given partition (r,s) of v as $S_{k+1/2}^{(r,s)}(N)$. To be precise, let r, s be non-negative integers such that $r + s = v$. Then we define

$$S_{k+1/2}^{(r,s)}(N) = \{ f \in S_{k+1/2}(N) \mid f|w_p = \pm f, p|N, s = \#\{p : f|w_p = -f\}\}. \tag{7}$$

It follows that when s is a non-negative integer, then

$$S_{k+1/2}^{(r,s)}(N) \subset \begin{cases} S_{k+1/2}^{+,N}(N) & \text{if } s \text{ is even}, \\ S_{k+1/2}^{-,N}(N) & \text{if } s \text{ is odd}. \end{cases}$$

We denote by $S_{k+1/2}^{(r,s)_e}(N)$ (resp. $S_{k+1/2}^{(r,s)_o}(N)$), the subspace $S_{k+1/2}^{(r,s)}(N)$ with s even (resp. s odd). From the construction of these spaces it is clear that the spaces $S_{k+1/2}^{\pm,N}(N)$ are decomposed as follows:

$$\begin{aligned} S_{k+1/2}^{+,N}(N) &= \bigoplus_{\substack{s=0,\,2|s\\ r+s=v}}^{v} \oplus_{\sigma} S_{k+1/2}^{(r,s)_e}(N), \\ S_{k+1/2}^{-,N}(N) &= \bigoplus_{\substack{s=1,\,s \text{ odd}\\ r+s=v}}^{v} \oplus_{\sigma'} S_{k+1/2}^{(r,s)_o}(N). \end{aligned} \tag{8}$$

In the second direct sum, σ runs over all the $\binom{v}{s}$ choices of s with $2|s$ and σ' runs over similar choices when s is odd. Correspondingly we also decompose the space $S_{2k}(N)$ in a similar fashion as follows:

$$\begin{aligned} S_{2k}^{+,N}(N) &= \bigoplus_{\substack{s=0,\,2|s\\ r+s=v}}^{v} \oplus_{\sigma} S_{2k}^{(r,s)_e}(N), \\ S_{2k}^{-,N}(N) &= \bigoplus_{\substack{s=1,\,s \text{ odd}\\ r+s=v}}^{v} \oplus_{\sigma'} S_{2k}^{(r,s)_o}(N), \end{aligned} \tag{9}$$

where $S_{2k}^{(r,s)_e}(N)$ (resp. $S_{2k}^{(r,s)_o}(N)$) is the subspace of $S_{2k}(N)$ consisting of forms F with the property the $F|W_p = \pm F$, $p|N$ and r number of primes p with $+1$ eigenvalue and s number of primes with -1 eigenvalue such the s is even (resp. odd) and $r+s=v$. In the above direct sums, σ and σ' have the same property as in (8).

Using the above decompositions, it is clear that in order to get the required correspondences between $S_{k+1/2}^{+,N}(N)$ and $S_{2k}^{+,N}(N)$, it is enough to construct the mappings between the spaces $S_{k+1/2}^{(r,s)}(N)$ and $S_{2k}^{(r,s)}(N)$ when s is even. For this reason, from now onwards we shall be assuming the following:

Assumption. Let r, s be non-negative integers such that $r+s=v$ with $2|s$.

Due to the above assumption that $2|s$, to simplify the notation, we shall remove the subscript 'e' and write the subspace as $S_{k+1/2}^{(r,s)}(N)$.

Kernel Function Constructed by Kohnen. We first describe the kernel function introduced by Kohnen, which will be used for our construction. Let $\mathcal{D}$ denote the set of all discriminants, i.e.,

$$\mathcal{D} = \{d \in \mathbf{Z} \mid d \equiv 0, 1 \pmod 4\}.$$

For $d \in \mathcal{D}$, let us denote by $\mathcal{Q}_{d,N}$, the set of all integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ with $N|a$ and $b^2 - 4ac = d$. Since N divides a, it is to be noted that $\mathcal{Q}_{d,N}$ is an empty set unless $d \equiv \square \ (4N)$. The meaning of this symbol is that d is a square modulo $4N$. We make use of this symbol from now onwards. For an integer $k \geq 2$ and $D, D' \in \mathcal{D}$ with $DD' > 0$, define a function denoted by $F_{k,N}(z; D, D')$ as follows.

$$F_{k,N}(z; D, D') = \sum_{Q \in \mathcal{Q}_{DD',N}} \frac{\omega_D(Q)}{Q(z, 1)^k}, \tag{10}$$

$z \in \mathcal{H}$, the upper half-plane. In the above definition, $\omega_D(Q)$ denotes the generalised genus character whose value is equal to $\left(\frac{D}{r}\right)$, if $(a, b, c, D) = 1$ and Q represents r with $(r, D) = 1$ and takes the value zero otherwise. The series converges absolutely and uniformly on compact sets and it defines a cusp form of weight $2k$ on $\Gamma_0(N)$. Moreover, it is non-zero only when $DD' \equiv \square \ (4N)$.

For $D \in \mathcal{D}$ with $(-1)^k D > 0$, let $P_{k,N;|D|}(\tau)$ denote the $|D|$-th Poincaré series in the space $S_{k+1/2}(N)$, characterised by

$$\langle f, P_{k,N;|D|} \rangle = c(|D|) a_f(|D|), \tag{11}$$

where $f \in S_{k+1/2}(N)$ and

$$c(|D|) = i_{4N}^{-1} \frac{\Gamma(k - 1/2)}{(4\pi|D|)^{k-1/2}}. \tag{12}$$

The kernel function constructed by Kohnen is defined as follows. For a fundamental discriminant $D \in \mathcal{D}$ with $(-1)^k D > 0$ and for $z, \tau \in \mathcal{H}$, let

$$\Omega_{k,N}(z, \tau; D) = i_N c_{k,D}^{-1} \sum_{\substack{m \geq 1 \\ (-1)^k m \equiv \square \ (4)}} m^{k-1/2}$$
$$\times \left(\sum_{t|N} \mu(t) \left(\frac{D}{t}\right) t^{k-1} F_{k,N/t}(tz; D, (-1)^k m) \right) e^{2\pi i m \tau}, \tag{13}$$

where

$$c_{k,D} = (-1)^{[k/2]}|D|^{-k+1/2}\pi\binom{2k-2}{k-1}2^{-3k+2}, \tag{14}$$

and i_N denotes the index of $\Gamma_0(N)$ in $SL_2(\mathbf{Z})$. Since $F_{k,N}(z;D,(-1)^k m)$ belongs to $S_{2k}(N)$, it follows that $\Omega_{k,N}(z,\tau;D)$ belongs to $S_{2k}(N)$ with respect to the z variable (for a fixed $\tau \in \mathcal{H}$). In Theorem 1 of [4], Kohnen showed that the omega function defined above can be expressed in terms of the Poincaré series in $S_{k+1/2}(N)$. We give below the theorem proved by Kohnen.

Theorem A ([4, Theorem 1]). *The function $\Omega_{k,N}(z,\tau;D)$ defined by (13) has the Fourier development*

$$\Omega_{k,N}(z,\tau;D) = i_N c_{k,D}^{-1}\delta_k \sum_{n\geq 1} n^{k-1}\left(\sum_{\substack{d\mid n\\(d,N)=1}}\left(\frac{D}{d}\right)(n/d)^k P_{k,N;n^2|D|/d^2}(\tau)\right)e^{2\pi i n z} \tag{15}$$

with respect to z, where

$$\delta_k = \frac{(-1)^{[k/2]}3(2\pi)^k}{(k-1)!}. \tag{16}$$

In particular, for a fixed $z \in \mathcal{H}$, $\Omega_{k,N}(z,\tau;D)$ is a cusp form in $S_{k+1/2}(N)$.

In the following, we make use of the above kernel function defined by Kohnen to construct the required kernel function for our liftings. For $D \in \mathcal{D}$ with $(-1)^k D > 0$, set

$$\varphi_{k,N}(z,\tau) := \sum_{\substack{m\geq 1\\(-1)^k m \equiv \square\ (4)}} m^{k-1/2}F_{k,N}(z;D,(-1)^k m)e^{2\pi i m\tau}. \tag{17}$$

As remarked earlier, the function $F_{k,N}(z;D,(-1)^k m)$ is non-zero only when $|D|m$ is a square modulo $4N$. Therefore, we have

$$\varphi_{k,N}(z,\tau) = \sum_{\substack{m\geq 1\\|D|m \equiv \square\ (4N)}} m^{k-1/2}F_{k,N}(z;D,(-1)^k m)e^{2\pi i m\tau}. \tag{18}$$

Our first result is to show (using Theorem A) that the function $\varphi_{k,N}(z,\tau)$ belongs to the Kohnen plus space with respect to the τ variable.

PROPOSITION 1. *For a fundamental discriminant $D \in \mathcal{D}$ with $(-1)^k D > 0$, the function $\varphi_{k,N}(z,\cdot)$ belongs to the space $S_{k+1/2}(N)$ for a fixed $z \in \mathcal{H}$.*

Next, we use the following definition to choose the fundamental discriminants for the purpose of Shimura and Shintani liftings.

DEFINITION 1. *For a fixed partition $r+s=v$ with r, s non-negative integers, and for an integer a, we say that $a \equiv \square_{r,s} \pmod{4N}$, if $a \equiv \square(4)$ and the number of primes $p|N$ for which a is a quadratic residue mod p is equal to r and the number of primes $p|N$ for which a is a quadratic non-residue mod p is equal to s. In other words, $a \in \mathcal{D}$ is such that $r = \#\left\{p|N : \left(\frac{a}{p}\right) = 1\right\}$, $s = \#\left\{p|N : \left(\frac{a}{p}\right) = -1\right\}$ and $r+s=v$.*

REMARK 1. Using the characterisation (3), obtained by Kohnen, we see that the space $S_{k+1/2}^{(r,s)}(N)$ can be defined in terms of the Fourier coefficients as follows.

$$S_{k+1/2}^{(r,s)}(N) = \{f \in S_{k+1/2}(N) \mid a_f(n) = 0 \text{ unless } (-1)^k n \equiv \square_{r,s} \pmod{4N}\}. \tag{19}$$

For a fixed pair (r,s) as chosen above with $2|s$ and $r+s=v$, let D be a fundamental discriminant with $(-1)^k D > 0$ satisfying the condition $D \equiv \square_{r,s}$ $\pmod{4N}$. The genus character has the following property for the action of the Atkin-Lehner operators W_t, $t|N$, where $W_t = \prod_{p|t} W_p$ (see [4, p. 243]):

$$\omega_D(Q \circ W_t) = \left(\frac{D}{t}\right)\omega_D(Q). \tag{20}$$

Further, $Q_{DD',N}$ (where $D, D' \in \mathcal{D}$, $DD' > 0$) is invariant under the action of the W-operators. Therefore, it follows that for $D \equiv \square_{r,s} \pmod{4N}$, the function $F_{k,N}(z; D, (-1)^k m)$ belongs to the space $S_{2k}^{(r,s)}(N)$. So, the function $\varphi_{k,N}(z,\tau)$ defined by (17) (which is equivalent to (18)), is indeed a function (with respect to the z variable) belonging to the space $S_{2k}^{(r,s)}(N)$. Hence, for a fundamental discriminant D with $(-1)^k D > 0$ and $D \equiv \square_{r,s} \pmod{4N}$, we consider the function $\varphi_{k,N}(z,\tau)$, defined by (18) which belongs to the space $S_{2k}^{(r,s)}(N)$ (due to our choice of D) as our kernel function with respect to the z variable.

By Proposition 1, $\varphi_{k,N}(z,\tau)$ belongs to $S_{k+1/2}(N)$ and the space is decomposed as follows:

$$S_{k+1/2}(N) = S_{k+1/2}^{(r,s)}(N) \oplus S_{k+1/2}^{(r,s)\perp}(N), \tag{21}$$

where the above is an orthogonal direct sum with respect to the Petersson scalar product. So, for a fixed $z \in \mathcal{H}$, as a function of τ, we project the function $\varphi_{k,N}(z,\tau)$ onto the subspace $S_{k+1/2}^{(r,s)}(N)$ and write it as $\varphi_{k,N}^{(r,s)}(z,\tau)$. So, we have

$$\varphi_{k,N}(z,\tau) = \varphi_{k,N}^{(r,s)}(z,\tau) + \varphi_{k,N}^{(r,s)\perp}(z,\tau). \tag{22}$$

To make it uniform, we denote our kernel function as $\varphi_{k,N}^{(r,s)}(z,\tau)$ and as a function of z, it is nothing but (18). In fact, as a function of z, $\varphi_{k,N}^{(r,s)\perp}(z,\tau) = 0$. Using this kernel function, we define the required Shimura and Shintani maps. Let $D \in \mathcal{D}$ be a fundamental discriminant with $(-1)^k D > 0$ and assume that $D \equiv \square_{r,s} \pmod{4N}$. Then the $|D|$-th Shimura map $\mathscr{S}_{D,(r,s)}$ on $S_{k+1/2}^{(r,s)}(N)$ and the $|D|$-th Shintani map on $S_{2k}^{(r,s)}(N)$ are defined as follows.

$$f|\mathscr{S}_{D,(r,s)}(z) := \frac{i_N}{c_{k,D}^*}\langle f, \varphi_{k,N}^{r,s}(-\bar{z},\cdot)\rangle_\tau, \tag{23}$$

$f \in S_{k+1/2}^{(r,s)}(N)$, and

$$F|\mathscr{S}_{D,(r,s)}^*(\tau) := \frac{i_N}{c_{k,D}^*}\langle F, \varphi_{k,N}^{r,s}(\cdot,-\bar{\tau})\rangle_z, \tag{24}$$

$F \in S_{2k}^{(r,s)}(N)$, where the constant $c_{k,D}^*$ is defined by

$$c_{k,D}^* = (-1)^{[k/2]} 2^k c_{k,D}, \tag{25}$$

with $c_{k,D}$ as in (14). In the above $\langle\cdot,\cdot\rangle_\tau$ and $\langle\cdot,\cdot\rangle_z$ denote the inner products with respect to τ and z respectively. The following mapping property follows from the fact that the function $\varphi_{k,N}^{(r,s)}(z,\tau)$ belongs to $S_{k+1/2}^{(r,s)}(N)$ as a function of τ and belongs to $S_{2k}^{(r,s)}(N)$ as a function of z:

$$\begin{aligned} \mathscr{S}_{D,(r,s)} &: S_{k+1/2}^{(r,s)}(N) \to S_{2k}^{(r,s)}(N), \\ \mathscr{S}_{D,(r,s)}^* &: S_{2k}^{(r,s)}(N) \to S_{k+1/2}^{(r,s)}(N). \end{aligned} \tag{26}$$

In the following theorem we give the properties of the Shimura and Shintani maps as defined above.

Theorem 1. *Let r and s be non-negative integers such that $r+s=v$ and $2|s$. Assume that $D \in \mathcal{D}$ is a fundamental discriminant with $(-1)^k D > 0$ and $D \equiv \square_{r,s} \pmod{4N}$. Then the Shimura and Shintani maps defined by* (23), (24) *satisfy the following properties.*

(a) *We have the following explicit description of the Shimura map in terms of the Fourier coefficients.*

$$f|\mathscr{S}_{D,(r,s)} = \delta_k \left[\sum_{\substack{n\geq 1\\(n,N)=1}} n^{k-1} \left(\sum_{\substack{d|n,\\(d,N)=1}} \left(\frac{D}{d}\right)(n/d)^k a_f(n^2|D|/d^2)c(n^2|D|/d^2) \right) e^{2\pi inz} \right.$$
$$+ \sum_{\substack{t|N\\t\neq 1}} \left(\frac{D}{t}\right) t^{k-1} \sum_{n\geq 1} n^{k-1}$$
$$\left. \times \left(\sum_{\substack{d|n\\(d,N/t)=1}} \left(\frac{D}{d}\right)(n/d)^k \langle f, P^{r,s}_{k,N/t;n^2|D|/d^2}(\tau)\rangle \right) e^{2\pi intz} \right], \tag{27}$$

where δ_k *is the constant defined by* (16) *and* $c(\alpha)$ *is defined by* (12). (*For the definition of* $P^{r,s}_{k,N;n}(\tau)$, *we refer to* §3.3.)

(b) *The Shimura and Shintani liftings defined above commute with the action of Hecke operators. Indeed, if* $(\ell, N) = 1$, *then for* $f \in S^{(r,s)}_{k+1/2}(N)$ *and* $F \in S^{(r,s)}_{2k}(N)$, *we have*

$$\begin{aligned} f|\mathscr{S}_{D,(r,s)}|T(\ell) &= f|T(\ell^2)|\mathscr{S}_{D,(r,s)}, \\ F|\mathscr{S}^*_{D,(r,s)}|T(\ell^2) &= F|T(\ell)|\mathscr{S}^*_{D,(r,s)}. \end{aligned} \tag{28}$$

The next theorem is concerned about the nature of the intersection of the space $S^{(r,s)}_{k+1/2}(N)$ with the old class. We show that exactly half of the oldforms (in terms of the dimension of the space) belong to the space $S^{+,N}_{k+1/2}(N)$. We prove this fact by producing explicit generators for the old class with respect to the component spaces $S^{(r,s)}_{k+1/2}(N)$.

THEOREM 2. *Let* $S^{+,N;old}_{k+1/2}(N) = S^{+,N}_{k+1/2}(N) \cap S^{old}_{k+1/2}(N)$, *where the space* $S^{old}_{k+1/2}(N)$ *is the orthogonal complement of the space of newforms* $S^{new}_{k+1/2}(N)$ *in* $S_{k+1/2}(N)$. *Then the dimension of the space* $S^{+,N;old}_{k+1/2}(N)$ *is given by*

$$\dim S^{+,N;old}_{k+1/2}(N) = \frac{1}{2} \dim S^{old}_{k+1/2}(N). \tag{29}$$

REMARK 2. In the proof of the above theorem (presented in §3.4), we have given explicit description of the generators for the space $S^{+,N;old}_{k+1/2}(N)$.

3. Proofs of Theorems

3.1. Proof of Proposition 1. Using the kernel function $\Omega_{k,N}(z,\tau;D)$ defined by (13), it follows that the function $\varphi_{k,N}(z,\tau)$ given by (18) can be expressed as follows.

$$\begin{aligned}\varphi_{k,N}(z,\tau) &= i_N^{-1} c_{k,D} \Omega_{k,N}(z,\tau;D) \\ &\quad - \sum_{\substack{t|N\\ t\neq 1}} \mu(t)\left(\frac{D}{t}\right) t^{k-1} \sum_{\substack{m\geq 1\\ |D|m\equiv\square \pmod{4N}}} m^{k-1/2} F_{k,N/t}(tz;D,(-1)^k m) e^{2\pi i m\tau} \\ &= i_N^{-1} c_{k,D} \Omega_{k,N}(z,\tau;D) - \sum_{\substack{t|N\\ t\neq 1}} \mu(t)\left(\frac{D}{t}\right) t^{k-1} \varphi_{k,N/t}(tz,\tau). \end{aligned} \tag{30}$$

Now we use induction on the number of prime factors of N. If $N = p$ is a prime, then the second part in the last line of the above equation has only one term which is $\left(\frac{D}{p}\right) p^{k-1} \varphi_{k,1}(pz,\tau)$. But $\varphi_{k,1}(z,\tau)$ is nothing but (upto a constant) $\Omega_{k,1}(z,\tau;D)$ which is an element of $S_{k+1/2}(1)$ and $\Omega_{k,p}(z,\tau;D) \in S_{k+1/2}(p)$. Therefore, $\varphi_{k,p}(z,\tau) \in S_{k+1/2}(p)$. Hence, by induction (on the number of prime factors) we see that $\varphi_{k,t}(z,\tau) \in S_{k+1/2}(t)$ for each $t|N$, $t < N$, from which the proof follows.

3.2. Proof of Theorem 1(a). In order to get the required property, we need the following properties of the Poincaré series in $S_{k+1/2}(N)$. For a natural number n with $(-1)^k n \equiv \square$ (4), we have the n-th Poincaré series $P_{k,N;n}(\tau)$ in $S_{k+1/2}(N)$. As per the assumptions of the theorem, let us take the pair (r,s) where r and s are non-negative integers such that $r + s = v$ and $2|s$. We consider the projection of $P_{k,N;n}(\tau)$ onto the space $S^{(r,s)}_{k+1/2}(N)$, denoted by $P^{r,s}_{k,N;n}(\tau)$. Further, one also has the n-th Poincaré series in $S^{(r,s)}_{k+1/2}(N)$, denoted by $P^{r,s}_n(\tau)$, characterised by the property $\langle f, P^{r,s}_n\rangle = c(n)a_f(n)$, for $f \in S^{(r,s)}_{k+1/2}(N)$. The following proposition gives some properties of these Poincaré series.

Proposition 2. *Let* $\alpha \in \mathbf{N}$, $(-1)^k\alpha \equiv \square$ (4). *Then for fixed non-negative integers* r, s *with* $r + s = v$, $2|s$, *we have the following properties.*

(i) *If* $(-1)^k\alpha \equiv \square_{r,s} \pmod{4N}$, *then* $P^{r,s}_{k,N;\alpha} = P^{r,s}_\alpha$. *As* $(\alpha, N) = 1$, *we also have* $P_{k,N;\alpha} = P^{r,s}_{k,N;\alpha}$.

(ii) *If* $(-1)^k\alpha \not\equiv \square_{r,s} \pmod{4N}$, *then* $P^{r,s}_{k,N;\alpha} = 0$.

PROOF OF PROPOSITION 2. We write the Poincaré series $P_{k,N;\alpha}(\tau)$ as

$$P_{k,N;\alpha}(\tau) = P^{r,s}_{k,N;\alpha}(\tau) + P^{r,s\perp}_{k,N;\alpha}(\tau),$$

where $P^{r,s\perp}_{k,N;\alpha}(\tau)$ belongs to the orthogonal complement (with respect to the Petersson scalar product) of $S^{(r,s)}_{k+1/2}(N)$ in $S_{k+1/2}(N)$. Using (19), we write the Fourier expansion of $P^{r,s}_{k,N;\alpha}(\tau)$ as

$$P^{r,s}_{k,N;\alpha}(\tau) = \sum_{\substack{\ell \geq 1,\, (-1)^k\ell \equiv \square(4), \\ (-1)^k\ell \equiv \square_{r,s} \pmod{4N}}} a_{P^{r,s}_{k,N;\alpha}}(\ell) e^{2\pi i \ell\tau}.$$

Then for each $\ell \in \mathbf{N}$ with $(-1)^k\ell \equiv \square(4)$, $(-1)^k\ell \equiv \square_{r,s} \pmod{4N}$, we have

$$\begin{aligned} a_{P^{r,s}_{\ell}}(\alpha)c(\alpha) &= \langle P^{r,s}_{\ell}, P_{k,N;\alpha}\rangle = \langle P^{r,s}_{\ell}, P^{r,s}_{k,N;\alpha}\rangle \\ &= \overline{\langle P^{r,s}_{k,N;\alpha}, P^{r,s}_{\ell}\rangle} = c(\ell)\overline{a_{P^{r,s}_{k,N;\alpha}}(\ell)}, \end{aligned} \tag{31}$$

where $c(\alpha)$ is defined by (12). This implies that

$$a_{P^{r,s}_{k,N;\alpha}}(\ell) = \frac{c(\alpha)}{c(\ell)}\overline{a_{P^{r,s}_{\ell}}(\alpha)} = 0, \quad \text{unless } (-1)^k\alpha \equiv \square_{r,s} \pmod{4N},$$

since $P^{r,s}_{\ell} \in S^{(r,s)}_{k+1/2}(N)$. This implies that if $(-1)^k\alpha \not\equiv \square_{r,s} \pmod{4N}$, then $P^{r,s}_{k,N;\alpha} = 0$, which proves (ii). To prove (i), consider

$$a_{P^{r,s}_{\ell}}(\alpha)c(\alpha) = \langle P^{r,s}_{\ell}, P^{r,s}_{\alpha}\rangle = \overline{\langle P^{r,s}_{\alpha}, P^{r,s}_{\ell}\rangle} = c(\ell)\overline{a_{P^{r,s}_{\alpha}}(\ell)}. \tag{32}$$

Combining (31) and (32), we get $P^{r,s}_{k,N;\alpha} = P^{r,s}_{\alpha}$, which proves the first part of (i). Since $(-1)^k\alpha \equiv \square_{r,s} \pmod{4N}$ it follows that $(\alpha, N) = 1$. For α, β with $(-1)^k\alpha \equiv \square_{r,s} \pmod{4N}$ and $(-1)^k\beta \equiv \square_{r,s} \pmod{4N}$, we have

$$\begin{aligned} \langle P^{r,s\perp}_{k,N;\alpha}, P^{r,s\perp}_{k,N;\beta}\rangle &= \langle P_{k,N;\alpha} - P^{r,s}_{\alpha}, P_{k,N;\beta} - P^{r,s}_{\beta}\rangle \\ &= c(\beta)a_{P_{k,N;\alpha}}(\beta) - \langle P_{k,N;\alpha}, P^{r,s}_{\beta}\rangle - \langle P^{r,s}_{\alpha}, P_{k,N;\beta}\rangle + \langle P^{r,s}_{\alpha}, P^{r,s}_{\beta}\rangle \\ &= c(\beta)a_{P_{k,N;\alpha}}(\beta) - c(\alpha)\overline{a_{P^{r,s}_{\beta}}(\alpha)} - c(\beta)a_{P^{r,s}_{\alpha}}(\beta) + c(\beta)a_{P^{r,s}_{\alpha}}(\beta) \\ &= c(\beta)(a_{P_{k,N;\alpha}}(\beta) - a_{P^{r,s}_{\alpha}}(\beta)) \quad (\text{using (32)}) \\ &= c(\beta)a_{P^{r,s\perp}_{k,N;\alpha}}(\beta). \end{aligned}$$

Therefore, we have the following:

$$c(\beta)a_{P^{r,s\perp}_{k,N;\alpha}}(\beta) = c(\alpha)\overline{a_{P^{r,s\perp}_{k,N;\beta}}(\alpha)}. \tag{33}$$

Now, consider

$$
\begin{aligned}
\langle P_{k,N;\alpha}, P^{r,s\perp}_{k,N;\beta}\rangle &= \langle P^{r,s}_{k,N;\alpha} + P^{r,s\perp}_{k,N;\alpha}, P^{r,s\perp}_{k,N;\beta}\rangle \\
&= \langle P^{r,s\perp}_{k,N;\alpha}, P^{r,s\perp}_{k,N;\beta}\rangle \\
&= c(\beta) a_{P^{r,s\perp}_{k,N;\alpha}}(\beta) \\
&= c(\alpha)\overline{a_{P^{r,s\perp}_{k,N;\beta}}(\alpha)} \quad \text{(using (33))} \\
&= 0 \quad \text{since} (-1)^k\alpha \equiv \square_{r,s} \pmod{4N} \text{ by our assumption.}
\end{aligned}
$$

(In the above, we have used the fact that $\langle P^{r,s}_{k,N;\alpha}, P^{r,s\perp}_{k,N;\beta}\rangle = 0$, which follows from (21).) This shows that $a_{P^{r,s\perp}_{k,N;\alpha}}(\beta) = 0$ for all β with $(-1)^k\beta \equiv \square_{r,s} \pmod{4N}$, i.e., whenever $(-1)^k\alpha \equiv \square_{r,s} \pmod{4N}$ and $(\alpha, N) = 1$, the Poincaré series $P_{k,N;\alpha}$ behaves like a Poincaré series in the (r,s) space. This implies that $P_{k,N;\alpha} = P^{r,s}_{\alpha}$ and completes the proof of Proposition 2. □

Now we return to the proof of 1(a). Using inductive arguments, it follows from (30) that the function $\varphi_{k,N}(z,\tau)$ can be expressed as follows.

$$
\varphi_{k,N}(z,\tau) = i_N^{-1} c_{k,D}\Omega_{k,N}(z,\tau;D) + \sum_{\substack{t|N\\ t\neq 1}} \left(\frac{D}{t}\right) t^{k-1} i_{N/t}^{-1} c_{k,D}\Omega_{k,N/t}(tz,\tau;D). \quad (34)
$$

Now using Theorem A, we can express the omega functions in terms of the half-integral weight Poincaré series. Explicitly, we have the following expression for $\varphi_{k,N}(z,\tau)$:

$$
\begin{aligned}
\varphi_{k,N}(z,\tau) = \delta_k \Bigg(&\sum_{n\geq 1} n^{k-1} \Bigg(\sum_{\substack{d|n,\\ (d,N)=1}} \left(\frac{D}{d}\right) (n/d)^k P_{k,N;n^2|D|/d^2}(\tau) \Bigg) e^{2\pi i n z} \\
&+ \sum_{\substack{t|N\\ t\neq 1}} \left(\frac{D}{t}\right) t^{k-1} \sum_{n\geq 1} n^{k-1} \\
&\times \Bigg(\sum_{\substack{d|n,\\ (d,N/t)=1}} \left(\frac{D}{d}\right) (n/d)^k P_{k,N/t;n^2|D|/d^2}(\tau) \Bigg) e^{2\pi i n t z} \Bigg). \quad (35)
\end{aligned}
$$

As a function of the τ variable, we project the above function onto the space $S^{(r,s)}_{k+1/2}(N)$ and obtain the following expression.

$$\varphi^{r,s}_{k,N}(z,\tau) = \delta_k \left(\sum_{n\geq 1} n^{k-1} \left(\sum_{\substack{d|n,\\(d,N)=1}} \left(\frac{D}{d}\right)(n/d)^k P^{r,s}_{k,N;n^2|D|/d^2}(\tau) \right) e^{2\pi i n z} \right.$$
$$+ \sum_{\substack{t|N\\t\neq 1}} \left(\frac{D}{t}\right) t^{k-1} \sum_{n\geq 1} n^{k-1}$$
$$\left. \times \left(\sum_{\substack{d|n,\\(d,N/t)=1}} \left(\frac{D}{d}\right)(n/d)^k P^{r,s}_{k,N/t;n^2|D|/d^2}(\tau) \right) e^{2\pi i n t z} \right). \tag{36}$$

Since $D \equiv \square_{r,s} \pmod{4N}$, by Proposition 2, the first part of the above expression is non-zero only when $(n,N)=1$. Moreover, in this case the projected Poincaré series coincides with the Poincaré series in the space (Proposition 2 (i)) and hence we get the following expression for the Fourier expansion of the Shimura map.

$$f|\mathscr{S}_{D,(r,s)} = \delta_k \left[\sum_{\substack{n\geq 1\\(n,N)=1}} n^{k-1} \left(\sum_{\substack{d|n,\\(d,N)=1}} \left(\frac{D}{d}\right)(n/d)^k a_f(n^2|D|/d^2) c(n^2|D|/d^2) \right) e^{2\pi i n z} \right.$$
$$+ \sum_{\substack{t|N\\t\neq 1}} \left(\frac{D}{t}\right) t^{k-1} \sum_{n\geq 1} n^{k-1}$$
$$\left. \times \left(\sum_{\substack{d|n\\(d,N/t)=1}} \left(\frac{D}{d}\right)(n/d)^k \langle f, P^{r,s}_{k,N/t;n^2|D|/d^2}(\tau)\rangle \right) e^{2\pi i n t z} \right], \tag{37}$$

where δ_k and $c(\alpha)$ are constants as defined before. This completes the proof of 1(a).

3.3. Proof of Theorem 1(b). In this section, we prove the required commutative properties with respect to the Hecke operators. For a positive integer $\ell \geq 1$ with $(\ell, N) = 1$, the Hecke operators $T(\ell^2)$ and $T(\ell)$ on the spaces $S_{k+1/2}(N)$ and $S_{2k}(N)$ are hermitian with respect to the Petersson scalar product.

Further, we have the following orthogonal decomposition (with respect to the Petersson scalar product) of the spaces:

$$S_{k+1/2}(N) = S_{k+1/2}^{(r,s)}(N) \oplus S_{k+1/2}^{(r,s)\perp}(N),$$

$$S_{2k}(N) = S_{2k}^{(r,s)}(N) \oplus S_{2k}^{(r,s)\perp}(N).$$

The first decomposition is already given by (21) and the second decomposition is the analogous one for the integral weight case. Therefore, the Hecke operators $T(\ell^2)$ and $T(\ell)$ preserve the respective subspaces. Thus, we have the following lemma.

Lemma 3. *Let* $(\ell, N) = 1$. *Then one has*

$$(\varphi_{k,N}|T(\ell^2))^{r,s} = \varphi_{k,N}^{r,s}|T(\ell^2),$$

and

$$(\varphi_{k,N}|T(\ell))^{r,s} = \varphi_{k,N}^{r,s}|T(\ell).$$

In the following we shall be using the inner products in both the spaces of half-integral and integral weights. To distinguish this, we use the notation $\langle \cdot, \cdot \rangle_\tau$ for the inner product in $S_{k+1/2}(N)$ and the notation $\langle \cdot, \cdot \rangle_z$ for the inner product in $S_{2k}(N)$. Also, we denote the D-th Shimura map on $S_{k+1/2}(N)$ defined by Kohnen [3] as $\mathscr{S}_{D,N}^{+}$.

For a positive integer $t|N$, let $f \in S_{k+1/2}(t)$. Then

$$\begin{aligned}\langle f, \Omega_{k,t}(-\bar{z}, \tau; D)|T(\ell^2)\rangle_\tau &= \langle f|T(\ell^2), \Omega_{k,t}(-\bar{z}, \tau; D)\rangle_\tau \\ &= f|T(\ell^2)\mathscr{S}_{D,t}^{+} \\ &= f|\mathscr{S}_{D,t}^{+}T(\ell) \\ &= \langle f, \Omega_{k,t}(-\bar{z}, \tau; D)\rangle_\tau|T(\ell) \\ &= \langle f, \Omega_{k,t}(-\bar{z}, \tau; D)|T(\ell)\rangle_\tau.\end{aligned}$$

In the above we used the fact that the Shimura map $\mathscr{S}_{D,t}^{+}$ commutes with Hecke operators. Therefore, the above computation shows that for each $t|N$,

$$\Omega_{k,t}(-\bar{z}, \tau; D)|T(\ell^2) = \Omega_{k,t}(-\bar{z}, \tau; D)|T(\ell). \tag{38}$$

In particular,

$$\begin{aligned}\Omega_{k,N/t}(-t\bar{z},\tau;D)|T(\ell^2) &= \Omega_{k,N/t}(-\bar{z},\tau;D)|T(\ell^2)|B(t)_z \\ &= \Omega_{k,N/t}(-\bar{z},\tau;D)|T(\ell)|B(t)_z \quad (\text{by } (38)) \\ &= \Omega_{k,N/t}(-t\bar{z},\tau;D)|T(\ell) \quad (\text{since } (\ell,N)=1),\end{aligned}$$

where $B(t)_z$ is the operator which maps a function $h(z)$ into $h(tz)$. Now, proceeding as done in [2, p. 312], it is easy to see that the following commuting rule holds.

$$f|\mathscr{S}_{D,(r,s)}|T(\ell) = f|T(\ell^2)|\mathscr{S}_{D,(r,s)}, \tag{39}$$

where $f \in S_{k+1/2}^{(r,s)}(N)$. In a similar way one can prove that for $F \in S_{2k}(t)$, $t|N$, one has

$$\langle F, \Omega_{k,t}(\cdot,-\bar{\tau};D)|T(\ell)\rangle_z = \langle F, \Omega_{k,t}(\cdot,-\bar{\tau};D)|T(\ell^2)\rangle_z, \tag{40}$$

from which it follows that

$$F|\mathscr{S}^*_{D,(r,s)}|T(\ell^2) = F|T(\ell)|\mathscr{S}^*_{D,(r,s)}. \tag{41}$$

3.4. Proof of Theorem 2. The case $N = p$ was already considered by Choi and Kim in [2, Lemma 4.1]. So, we assume that N has at least two prime factors. We first consider the case $N = p_1p_2$, $p_1 \neq p_2$ and then discuss about the general case. When $N = p_1p_2$, the oldforms in $S_{k+1/2}(p_1p_2)$ consists of 3 eigenclasses, namely, $S_{k+1/2}(1)$, $S_{k+1/2}^{new}(p_i)$, $i = 1,2$. By the theory of newforms developed by Kohnen, it is enough to determine forms belonging to each eigenclass in the projected space $S_{k+1/2}^{+,N}(N)$. In this case the pairs (r,s) are given by $(2,0)$, $(0,2)$ and $(1,1)$, of which we need to consider the first two cases where s is even. Note that the pair $(1,1)$ corresponds to the two subspaces $(+,-)$, $(-,+)$, the ± 1 subspaces corresponding to the primes p_1, p_2. First we assume that $(r,s) = (2,0)$. i.e., the subspace consisting of forms which are eigenfunctions with respect to w_{p_i}, $i = 1,2$ with eigenvalue $+1$. Let g be an oldform in the eigenclass generated by forms in $S_{k+1/2}(1)$ such that $g|w_{p_i} = g$, $i = 1,2$. The function g can be written as $g = f_1 + f_2|w_{p_1} + f_3|w_{p_2} + f_4|w_{p_1p_2}$, where f_i, $1 \le i \le 4$ are cusp forms in $S_{k+1/2}(1)$. We claim that $f_1 = f_2 = f_3 = f_4$. First, we use the fact that $g|w_{p_1} = g$. Since $w_{p_1}^2$ is identity, this implies the following.

$$\begin{aligned} f_1|w_{p_1} + f_2 + f_3|w_{p_1p_2} + f_4|w_{p_2} &= f_1 + f_2|w_{p_1} + f_3|w_{p_2} + f_4|w_{p_1p_2}, \\ (f_2 - f_1) + (f_1 - f_2)|w_{p_1} &= (f_3 - f_4)|w_{p_2} + (f_4 - f_3)|w_{p_1p_2} \\ &= [(f_3 - f_4) + (f_4 - f_3)|w_{p_1}]|w_{p_2}. \end{aligned}$$

In the last equation, the left-hand side is a function in the space $S_{k+1/2}(p_1)$, whereas the right-hand side is a function in $S_{k+1/2}(p_1p_2)$, which is of the form $h|w_{p_2}$, where $h = (f_3 - f_4) + (f_4 - f_3)|w_{p_1} \in S_{k+1/2}(p_1)$. This implies that the functions on both the sides are zero. To get this, we use Eq. (44) of [3] to get $h|w_{p_2} = \left(\frac{p_1}{p_2}\right) p_2^{-k+1}(-h|U(p_2^2) + h|T(p_2^2))$. Using the Shimura correspondence and denoting the Shimura image of h as $H \in S_{2k}(p_1)$, we see that both H and $H|U(p_2)$ belong to $S_{2k}(p_1)$, from which it follows that $H = 0$ (this follows from [1, Lemma 16]). In other words $h = 0$. Therefore, we have

$$f_1 - f_2 = (f_1 - f_2)|w_{p_1},$$

$$f_3 - f_4 = (f_3 - f_4)|w_{p_1}.$$

Using the same argument as above (note that the LHS functions are in $S_{k+1/2}(1)$ and the RHS functions are of the form $h_1|w_{p_1}$, where $h_1 \in S_{k+1/2}(1)$), we conclude that $f_1 = f_2$ and $f_3 = f_4$. Repeating the above arguments by assuming that $g|w_{p_2} = g$, we obtain $f_1 = f_3$ and $f_2 = f_4$. Thus, $f_1 = f_2 = f_3 = f_4$ and g is written as $g = f + f|w_{p_1} + f|w_{p_2} + f|w_{p_1p_2}$, where $f \in S_{k+1/2}(1)$. Similarly, for the case $(r,s) = (0,2)$, it follows that g takes the form $g = f - f|w_{p_1} - f|w_{p_2} + f|w_{p_1p_2}$, where f is Hecke eigenform in $S_{k+1/2}(1)$. Therefore, the contribution from $S_{k+1/2}(1)$ to the intersection $S^{+,N}_{k+1/2}(N) \cap S^{old}_{k+1/2}(N)$ is given by

$$\langle f + f|w_{p_1} + f|w_{p_2} + f|w_{p_1}w_{p_2}, f - f|w_{p_1} - f|w_{p_2} + f|w_{p_1}w_{p_2} : f \in S_{k+1/2}(1)\rangle.$$

This shows that the (dimension) contribution of the space $S_{k+1/2}(1)$ in the subspace $S^{+,N}_{k+1/2}(N)$ is exactly $2 \dim S_{k+1/2}(1)$. Next, we consider the contributions from $S^{new}_{k+1/2}(p_i)$, $i = 1,2$. An element in the old class will be of the form $g = f_1 + f_2|w_{N/p_i}$, $f_1, f_2 \in S^{new}_{k+1/2}(p_i)$ are newforms. We need the properties that $g|w_{p_i} = g$, $i = 1,2$. Arguing as before, this would lead to $f_1 = f_2 = f$ (say) and moreover, as f is a newform, we must have $f|w_{p_i} = f$, in other words, $g = f + f|w_{N/p_i}$, $f \in S^{(+,p_i),new}_{k+1/2}(p_i)$. The above discussion is for the case $(2,0)$. Now, similar arguments imply that the contribution from $S^{new}_{k+1/2}(p_i)$ (for the case $(0,2)$) is given by $g = f - f|w_{N/p_i}$, $f \in S^{(-,p_i),new}_{k+1/2}(p_i)$. Therefore, the contribution of $S^{new}_{k+1/2}(p_i)$ in the subspace $S^{+,N}_{k+1/2}(N)$ is given by

$$\langle f \pm f|w_{N/p_i}, f \in S^{(\pm,p_i),new}_{k+1/2}(N)\rangle.$$

Since $S^{new}_{k+1/2}(p_i) = S^{(+,p_i),new}_{k+1/2}(p_i) \oplus S^{(-,p_i),new}_{k+1/2}(p_i)$, we see that the (dimension) contribution of the space $S^{new}_{k+1/2}(p_i)$ in $S^{+,N}_{k+1/2}(N)$ is exactly $\dim S^{new}_{k+1/2}(p_i)$. Thus, we have

$$\dim S^{+,N}_{k+1/2}(N) \cap S^{old}_{k+1/2}(N) = 2 \dim S_{k+1/2}(1) + \sum_{i=1}^{2} \dim S^{new}_{k+1/2}(p_i)$$
$$= \frac{1}{2} \dim S^{old}_{k+1/2}(N).$$

Now we consider the general case. Let us consider the old class generated by the space $S^{new}_{k+1/2}(t)$, where $t|N$, $t < N$. From the newform theory, it is known that this old class has dimension equal to $d(N/t) \dim S^{new}_{k+1/2}(t)$, where $d(n)$ is the number of divisors of n. Note that as N is square-free, $d(N/t) = 2^{\alpha}$, where α is the number of prime factors of N/t. We require that an old form g in this class to be an eigenform with respect to the W-operators w_p, where $p|N/t$. Note that as the base functions are newforms in $S^{new}_{k+1/2}(t)$, they are already eigenforms with respect to w_p, for $p|t$. For each prime $p|N/t$, if we require the condition that g is an eigenfunction with respect to w_p, then the number of components in the old class reduces by a factor of 2. We shall be repeating this process for each prime $p|N/t$ and so we will be repeating α times. So, finally in order that g is an eigenfunction under each W-operator for $p|N/t$, the number of components becomes $2^{\alpha}/2^{\alpha} = 1$. Therefore, for each Hecke eigenform (newform) in $S^{new}_{k+1/2}(t)$, there is a unique form which belongs to the space $S^{old}_{k+1/2}(N)$, which is an eigenform w.r.t all w_p, $p|N/t$. Since we have to consider the (r,s) case, we shall get into the precise contribution of a newform in the required old class. If p_i, $1 \le i \le r$ is one of the primes dividing the level t, then the newform $f \in S^{new}_{k+1/2}(t)$ is an eigenfunction with respect to w_{p_i}, for $p_i|t$ with eigenvalue $+1$. Similarly, if q_j divides t, then $f|w_{q_j} = -f$. For the rest of the primes p_i and q_j not dividing t, we need to assume the respective eigenvalue ($+1$ or -1) and finally we will end up with only one linear combination generated by a newform in $S^{new}_{k+1/2}(t)$, which is described below.

$$g = \pm^{r,s} \sum_{l|N/t} f|w_l, \tag{42}$$

where $w_l = \prod_{p|l} w_p$ and the sign in the linear combination is -1, if the number of primes q_j dividing l is odd. Now, we compute the dimension of the space $S^{+,N;old}_{k+1/2}(N)$.

Case (i): Let $f \in S^{+,t;new}_{k+1/2}(t)$. This implies that $\#\{q_j : 1 \le j \le s, q_j|N/t\}$ is even. As the number of prime divisors of N/t is α, the total contribution (in

$S_{k+1/2}^{+,N;old}(N))$ from $S_{k+1/2}^{+,t;new}(t)$ is given by

$$\sum_{s=0,s\ \text{even}}^{\alpha} \binom{\alpha}{s} \dim S_{k+1/2}^{+,t;new}(t). \tag{43}$$

Case (ii): Let $f \in S_{k+1/2}^{-,t;new}(t)$. In this case, $\#\{q_j : 1 \le j \le s, q_j | N/t\}$ is odd. The total contribution of $S_{k+1/2}^{+,t;new}(t)$ in $S_{k+1/2}^{+,N;old}(N)$ is then given by

$$\sum_{s=0,s\ \text{odd}}^{\alpha} \binom{\alpha}{s} \dim S_{k+1/2}^{-,t;new}(t). \tag{44}$$

Since $\sum_{s=0,s\ \text{even}}^{\alpha} \binom{\alpha}{s} = \sum_{s=0,s\ \text{odd}}^{\alpha} \binom{\alpha}{s} = 2^{\alpha-1}$, combining these two cases, we see that the total contribution from $S_{k+1/2}^{new}(t)$ in the old class $S_{k+1/2}^{+,N;old}(N)$ is

$$2^{\alpha-1}(\dim S_{k+1/2}^{+,t;new}(t) + \dim S_{k+1/2}^{-,t;new}(t)) = 2^{\alpha-1} \dim S_{k+1/2}^{new}(t).$$

Therefore, we have

$$\dim S_{k+1/2}^{+,N;old}(N) = \sum_{t|N,\,t<N} 2^{\alpha-1} \dim S_{k+1/2}^{new}(t) = \frac{1}{2} \dim S_{k+1/2}^{old}(N). \tag{45}$$

Note that explicit description of the space $S_{k+1/2}^{+,N;old}(N)$ follows from (42). This completes the proof of Theorem 2.

We remark that in [5, Theorem 5.11] a decomposition of the old class in a general set up was given. In our proof we have not used this and carried out the characterisation by only using the old class decomposition in terms of the W-operators.

Acknowledgements

The work was started when all the authors were visiting IMSc, Chennai during May 2018 and was completed during our visit to KSoM, Kozhikode. We wish to thank Sanoli Gun and Manickam for providing excellent facilities during our visits. Manish Kumar Pandey and Anup Kumar Singh thank the Infosys foundation for the scholarship provided to them through HRI, Prayagraj (Allahabad). Finally, we thank the referee for making suggestions that improved the exposition of this article.

References

[1] Atkin, A. O. L. and Lehner, J., Hecke operators on $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160.

[2] Choi, S. and Kim, C. H., Shintani and Shimura lifts of cusp forms on certain arithmetic groups and their applications, Open Math. **15** (2017), 304–316.
[3] Kohnen, W., Newforms of half-integral weight, J. reine angew Math. **333** (1982), 32–72.
[4] Kohnen, W., Fourier coefficients of modular forms of half-integral weight, Math. Ann. **271** (1985), 237–268.
[5] Manickam, M. and Ramakrishnan, B., On Shimura, Shintani and Eichler-Zagier correspondences, Trans. Amer. Math. Soc. **352** (2000), 2601–2617.
[6] Shimura, G., On modular forms of half integral weight, Ann. Math (2) **97** (1973), 440–481.
[7] Shintani, T., On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J. **58** (1975), 83–126.

Harish-Chandra Research Institute, HBNI
Chhatnag Road, Jhunsi
Prayagraj (Allahabad)-211 019
India
E-mail: manishpandey@hri.res.in
ramki@hri.res.in
anupsingh@hri.res.in