TSUKUBA J. MATH.
Vol. 43 No. 2 (2019), 145–189

# 48 DIMENSIONAL EVEN UNIMODULAR NEARLY EXTREMAL LATTICES.

By

Michio OZEKI

**Abstract.** In this paper we introduce the notion of an even unimodular nearly extremal lattice in the case of 48 dimension. We prove two basic properties of such lattices $L$. First we prove that any nearly extremal lattice $L$ is generated by the vectors of norm 4 and norm 6 in $L$. Next we prove that the Siegel theta series of degree 2 associated with an even unimodular nearly extremal lattice is determined by the Fourier coefficients which are connected with the vectors of norm 4.

## 1. Introduction

An even unimodular 48 dimensional lattice with the least non-zero minimumal norm 6 is called an even unimodular extremal lattice of dimension 48. Only four such lattices are known (c.f. [22]).

In the present paper we consider a class of the even unimodular lattices of dimension 48 with the least non-zero minimal norm 4. We call such a lattice as a nearly extremal lattice. Later we will discuss some basic arithmetical properties of such lattices. Then we study some examples of such lattices.

Our first result is that any 48-dimensional even unimodular nearly extremal lattice $L$ is generated by the norm 4 vectors and the norm 6 vectors in $L$. (Theorem 3.7). This result is showed by using the extension of our previous method [27], [28].

Our second result is that the Siegel theta series of degree 2 associated with an nearly extremal 48-dimensional lattice is determined by the Fourier

---

2000 *Mathematics Subject Classification*: Primary 11E12, Secondary 11F11, 11F46.
*Key words and phrases*: Even Unimodular Lattice, Nearly Extremal, theta series with spherical functions, Siegel theta series.
Received February 22, 2019.
Revised October 10, 2019.

coefficients that are connected with the norm 4 vectors in the lattice (Theorem 5.10).

The present author has a strong feeling that the generating subsets with the minimal norms of an even unimodular lattice would determine the Siegel theta series completely for past ten years. As evidences for such an expectation we collect some results.

In [15] Leech constructed the even unimodular 24-dimensional lattice with the minimal vectors of norm 4. From his paper the fact that the norm 4 vectors generate the full lattice can be read off. By the works of Niemeier [23] and Venkov [40] we observe that any 24-dimensional even unimodular lattice is generated by the vectors of norm 2, 4 and 6. In degrees 1, 2 and 3 we computed some Fourier coefficiens of the Siegel theta series that are associated with the even unimodular overlattices of the root lattices of type $E_8^3$, $D_{24}$, $A_{24}$, $E_7 \oplus A_{17}$, $D_6^4$, $E_6 \oplus D_7 \oplus A_{11}$. The set $M(g,k)$ of all Siegel modular forms of degree $g$ and even weight $k$ forms a linear space of certain dimension. The set of Siegel cusp forms of degree $g$ and weight $k$ forms a linear subspace $S(g,K)$ of $M(g,k)$. In the Appendix we briefly give a description of $M(g,k)$, $S(g,k)$ with $g \geq 3$, $k = 12, 16$. In these cases we only need to use the Fourier coefficients that are connected with the vectors of norm 2 and norm 4. In [31] the present author has showed that the Fourier coefficients of the Siegel theta series of degrees up to 5 associated with the Leech lattice can be computed in principle. The computations are restricted to the case when the index $T$ of the Fourier coefficients $a(T,L)$ come from the norm 4 vectors in the Leech lattice $L$. But the method is easily extendable to more general types of indices $T$.

In [41] Venkov has proved that any even unimodular 32-dimensional extremal lattices is generated by the minimal vectors (norm 4). In [24] Oura-Ozeki have showed that any even unimodular 32-dimensional extremal lattice has the identical Siegel theta series of degrees up to 3.

In [27] the present author has showed that any even unimodular 40-dimensional extremal lattice is generated by the minimal norm vectors (of norm 4) and the next minimal vectors of norm 6. In [28] the present author has showed that any even unimodular 48-dimensional extremal lattice is generated by the minimal vectors of norm 6. Later other people take interests in this line of research. In [13] Kominers and Abel has showed the same kind of result in 40, 80, 120 dimensions. See also [5]. In [14] Kominers extends the same kind of results in dimensions 56, 72 and 96.

In [33] Salvati Manni showed that the Siegel theta series of degree up to 3 associated with the even unimodular 32 dimensional extremal lattices is unique and that the difference of the two Siegel theta series of degree 4 associated with the two non-isometric extremal lattices is a constant multiple of the square $J^2$ of the Shottky modular form $J$ of the Siegel cusp form of degree 4 and weight 16. In [24] Oura and the present author have showed that the Fourier coefficients of such series are in principle computable (some of them are given there). In the same article we showed that the Fourier coefficients of the Siegel theta series of degree 4 associated with such lattices $L$ are computable if we could determine one peculiar Fourier coefficient $a(T, L)$, where the index $T$ is connected with the vectors of norm 4 in $L$. In [25] we developed a method to compute $a(T, L)$ for the peculiar $T$ in the cases that the lattices come from the binary extremal self-dual codes, which are classified by [2], [3]. The same kind of results are given in [30] for the 40 dimensional case and in [32] for the 48 dimensional case. In the treated cases we confined ourselves to the Siegel theta series of low degrees. Our conviction would be that the generating subsets of small norms determine the Siegel theta series of higher degrees also.

The present article is a trial to show that the two trends of researches described above may collaborate.

## 2. Some Preliminaries

**2.1. Some Definitions from Lattice Theory.** Let $\mathbf{Z}$ be the ring of rational integers and $\mathbf{Q}$ the field of rational numbers. A finitely generated $\mathbf{Z}$-module $L$ in $\mathbf{Q}^g$ with a positive definite metric is called a positive definite quadratic lattice. Since we treat only the positive definite quadratic lattices, we shall omit the adjectives "positive definite quadratic". A lattice $L$ is integral if $L$ satisfies $(\mathbf{x}, \mathbf{y}) \in \mathbf{Z}$ for any $\mathbf{x}, \mathbf{y} \in L$ where $(\,,\,)$ is the bilinear form associated to the metric. Two integral lattices $L_1$ and $L_2$ are said to be isometric if and only if there exists a bijective linear mapping from $L_1$ to $L_2$ preserving the metric. The dual lattice $L^\#$ of $L$ is defined by

$$L^\# = \{\mathbf{y} \in L \otimes_{\mathbf{Z}} \mathbf{Q} \mid (\mathbf{x}, \mathbf{y}) \in \mathbf{Z},\ \forall \mathbf{x} \in L\}.$$

A lattice $L$ is called even if it holds that $(\mathbf{x}, \mathbf{x}) \equiv 0 \pmod 2$ for all $\mathbf{x} \in L$, and $L$ is unimodular if $L = L^\#$. The maximal number of linearly independent vectors over $\mathbf{Q}$ in $L$ is called the rank of $L$. It is known that the rank of an even unimodular lattice is divisible by 8. A lattice $L$ is even if any element $\mathbf{x}$ of $L$ has even norm $(\mathbf{x}, \mathbf{x})$.

Even unimodular lattices exist only when $n \equiv 0 \pmod 8$. The minimal norm of a lattice is $\mathrm{Min}(L) = \min_{\mathbf{x} \in L \setminus \{0\}} (\mathbf{x}, \mathbf{x})$.

When $L$ is even unimodular of rank $n$ it holds that (conf. [18])

$$\mathrm{Min}(L) \le 2\left[\frac{n}{24}\right] + 2.$$

A lattice which attains the above maximum is called an extremal lattice.

In the case that the dimension is 48 the extremal even unimodular lattice satisfies $Min(L) = 6$. We say that an even unimodular lattice $L$ is called nearly extremal if it satisfies the condition $Min(L) = 4$.

In an even lattice $L$, for the non-zero vector $\mathbf{x} \in L$ the inner product $(\mathbf{x}, \mathbf{x})$ is an even integer, and we say that $\mathbf{x}$ is a $2m$-vector if $(\mathbf{x}, \mathbf{x}) = 2m$ holds for some natural number $m$. Let $\Lambda_{2m}(L)$ be the set defined by

$$\Lambda_{2m}(L) = \{\mathbf{x} \in L \mid (\mathbf{x}, \mathbf{x}) = 2m\}. \tag{2.1}$$

**2.2. Some Definitions from Coding Theory.** We confine ourselves to the binary codes. For the codes over other finite fields or rings one may refer to [17].

Let $\mathbf{F}_2 = GF(2)$ be the field of 2 elements. Let $V = \mathbf{F}_2^n$ be the vector space of dimension $n$ over $\mathbf{F}_2$. A linear $[n, k]$ code $\mathbf{C}$ is a vector subspace of $V$ of dimension $k$. An element $\mathbf{u}$ in $\mathbf{C}$ is called a code word of $\mathbf{C}$. In $V$, the inner product, which is denoted by $\mathbf{u} \cdot \mathbf{v}$ for $\mathbf{u}$, $\mathbf{v}$ in $V$, is defined as usual. Two codes are said to be equivalent if after a suitable change of coordinate positions the code words in the two codes coincide. The dual code $\mathbf{C}^\perp$ of $\mathbf{C}$ is defined by

$$\mathbf{C}^\perp = \{\mathbf{u} \in V \mid \mathbf{u} \cdot \mathbf{v} = \mathbf{0},\ \forall \mathbf{v} \in \mathbf{C}\}.$$

The code $\mathbf{C}$ is called self-orthogonal if it satisfies $\mathbf{C} \subseteq \mathbf{C}^\perp$, and self-dual if it satisfies $\mathbf{C} = \mathbf{C}^\perp$. Self-dual $[n, k]$ codes exist only if $n \equiv 0 \pmod 2$ and $k = \frac{n}{2}$.

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ be a vector in $V$. The Hamming weight $wt(\mathbf{u})$ of the vector $\mathbf{u}$ is defined to be the number of $i$'s such that $u_i \neq 0$. The Hamming distance $d(\mathbf{u}, \mathbf{v})$ on $V$ is defined by $d(\mathbf{u}, \mathbf{v}) = wt(\mathbf{u} - \mathbf{v})$. The minimal distance $d(\mathbf{C})$ of a code $\mathbf{C}$ is defined by

$$\begin{aligned} d(\mathbf{C}) &= \mathrm{Min}_{\mathbf{u}, \mathbf{v} \in \mathbf{C}, \mathbf{u} \neq \mathbf{v}}\ d(\mathbf{u}, \mathbf{v}) \\ &= \mathrm{Min}_{\mathbf{u} \in \mathbf{C}, \mathbf{u} \neq \mathbf{0}}\ wt(\mathbf{u}). \end{aligned}$$

A binary $[n, k]$ code with the minimal distance $d$ is called a binary $[n, k, d]$ code.

**2.3. Code Construction of Lattices.** We confine ourselves to the binary code construction of even unimodular latties. For the construction of even unimodular lattices by the codes over ring one may refer to [4], [38] or [29].

Let $\mathbf{C}$ be a binary self-orthogonal $[n, k]$ code. We recall the construction $B_2$ for the lattices (c.f. Conway-Sloane [4], Chap. 5). Let

$$\rho : \mathbf{Z}^n \to \mathbf{F}_2^n$$

denote the reduction modulo 2. Then

$$M(\mathbf{C}) = \frac{1}{\sqrt{2}}\left\{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \rho^{-1}(\mathbf{C}) \,\middle|\, \sum_{i=1}^{n} x_i \equiv 0 \pmod 4\right\}$$

defines an even lattice. Suppose that $\mathbf{C}$ is a doubly even self-dual binary (not necessarily extremal) $[n, n/2]$ code. The so-called doubling process is as follows. Put

$$\gamma = \begin{cases} \frac{1}{\sqrt{8}}(1, \dots, 1, -3) & \text{if } n \equiv 8 \pmod{16}, \\ \frac{1}{\sqrt{8}}(1, \dots, 1, 1) & \text{if } n \equiv 0 \pmod{16}. \end{cases}$$

Then

$$\mathscr{N}(\mathbf{C}) = \mathscr{M}(\mathbf{C}) \cup (\gamma + \mathscr{M}(\mathbf{C}))$$

is an even unimodular lattice of rank $n$ for $n \geq 8$, $n \equiv 0 \pmod 8$. Minimal norm of the obtained lattice depends on the minimal distance of the code.

**2.4. Theta Series Associated with the Even Unimodular Lattice.** We collect two important facts about theta-series associated with the lattice. Before doing so, we need some preliminaries.

Let $L$ be an even unimodular lattice of rank $8k$, then the (ordinary) theta series for $L$ is defined by

$$\vartheta(z, L) = \sum_{\mathbf{x} \in L} \exp(\pi i(\mathbf{x}, \mathbf{x})z), \tag{2.2}$$

where $z$ is a complex variable with positive imaginary part. This series is rewritten as

$$\vartheta(z, L) = \sum_{m=0}^{\infty} a(2m, L) \exp(2\pi i m z), \tag{2.3}$$

where $a(2m, L) = |\Lambda_{2m}(L)|$ and $|X|$ is the cardinality of a set $X$.

**2.5. Modular Forms of Degree 1.** Let $\mathfrak{H}_1 = \{z = x + yi \in \mathbf{C} \mid y > 0\}$ be the complex upper-half plane. A complex valued function $f(z)$ on $\mathfrak{H}_1$ is called a modular form of (even) weight $2h$ belonging to the full modular group $SL_2(\mathbf{Z})$ if it is holomorphic on $\mathfrak{H}_1$ satisfying

(i) $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^{2h} f(z), \quad \textit{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}),$

(ii) $f(z)$ is holomorphic at infinity.

The condition (ii) means that when $f(z)$ is expanded as a Fourier series (this expansion is guaranteed by the condition (i)):

$$f(z) = \sum_{n=-\infty}^{n=+\infty} a_n(f) \exp(2\pi i n z), \tag{2.4}$$

then the terms $\sum_{n<0} a_n(f) \exp(2\pi i n z)$ vanish.

The set $M(1,k)$ $(k \equiv 0 \pmod 2)$ of the modular forms of weight $k$ and degree 1 forms a finite dimensional linear space over the field $\mathbf{C}$ of complex numbers and the dimension of $M(1,k)$ is also well-known. For the precise development of the theory we may refer the books [35] and [36], and for the theta series with spherical function we refer to [34] and [9].

A modular form $f(z) \in M(1,k)$ whose Fourier series expansion such as (2.4) satisfies $a_0 = 0$ is called a cusp form of weight $k$. The set $S(1,k)$ of the cusp forms of weight $k$ forms a vector subspace of $M(1,k)$.

Theta series with the spherical function is defined by

$$\vartheta(z, P_v, L) = \sum_{\mathbf{x} \in L} P_v(\mathbf{x}; \boldsymbol{\alpha}) \exp(\pi i (\mathbf{x}, \mathbf{x}) z), \tag{2.5}$$

where $\alpha$ is any vector in $\mathbf{R}^{8k}$ with $8k = \mathrm{rank}(L)$. This series is rewritten as

$$\vartheta(z, P_v, L) = \sum_{m=1}^{\infty} \sum_{\mathbf{x} \in \Lambda_{2m}(L)} P_v(\mathbf{x}; \boldsymbol{\alpha}) \exp(2\pi i m z). \tag{2.6}$$

We quote a well-known result without giving the proof as a proposition.

PROPOSITION 2.1. *Let $L$ be an even unimodular lattice of rank $8k$. Then it holds that*

(i) $\vartheta(z, L) \in M(1, 4k)$, *and*
(ii) $\vartheta(z, P_v, L) \in S(1, 4k + v)$.

We give some well-known modular forms:

$$E_4(z) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \cdots \in M(1,4),$$

$$E_6(z) = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \cdots \in M(1,6),$$

$$\Delta_{12}(z) = q - 24q^2 + 252q^3 - 1472q^4 + \cdots \in S(1,12).$$

Here we use the convention $q = \exp(2\pi i z)$.

**2.6. Siegel Theta Series.** A Siegel theta series of degree $g$ ($g \geq 2$) attached to the even unimodular lattice $L$ is defined by

$$\Theta_g(Z, L) = \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_g \in L} \exp(\pi i \sigma([\mathbf{x}_1, \ldots, \mathbf{x}_g]Z)),$$

where $Z$ is the variable on the Siegel upper-half space of degree $g$, $[\mathbf{x}_1, \ldots, \mathbf{x}_g]$ is a $g$ by $g$ square matrix whose $(i,j)$ entry is $(\mathbf{x}_i, \mathbf{x}_j)$ and $\sigma$ is the trace of the matrix.

The Siegel theta series is a class of Siegel modular forms of degree $g$. We do not go into the full detail of Siegel modular forms of degree $g$. Concerning this the readers may refer the book by Freitag [8]. Later we will need the linear space of the Siegel modular forms of degree 2 and the weight $k$, which is denoted by $M(2,k)$.

The Siegel theta series of degree $g$ can be expanded to

$$\Theta_g(Z, L) = \sum_{T \in \hat{\mathscr{P}}_g^s(\mathbf{Z})} a(T, L) e^{2\pi i \sigma(TZ)}.$$

Here $\hat{\mathscr{P}}_g^s(\mathbf{Z})$ is the set of positive semi-definite semi-integral symmetric square matrices of degree $g$, and $a(T,L) = |\{\langle \mathbf{x}_1, \ldots, \mathbf{x}_g \rangle \in L^g \mid [\mathbf{x}_1, \ldots, \mathbf{x}_g] = 2T\}|$. An element $T$ of $\hat{\mathscr{P}}_g^s(\mathbf{Z})$ is called an index for the Fourier coefficient $a(T, L)$.

We shall say that $2T$ is represented by the lattice $L$ if $a(T, L) \neq 0$.

We quote one importatn property of $a(T, L)$ above as a proposition:

Proposition 2.2. *Let $a(T, L)$ be a Fourier coefficient of the Siegel theta series of degree g. Then we have*

$$a(UTU^t, L) = a(T, L),$$

*where $U$ is a unimodular matrix of size $g$ and $U^t$ is the transpose of $U$.*

For the proof of this we refer to [37], formula (48).

### 3. Some Basic Properties of Nearly Extremal Lattices

**3.1. Relations among the Fourier Coefficients.** Let $\mathscr{L}_{ne}$ be any one of 48 dimensional even unimodular nearly extremal lattices. By Proposition 2.1 we have $\vartheta(z, \mathscr{L}_{ne}) \in M(1, 24)$. Since $\dim_{\mathbf{C}} M(1, 24) = 3$ we take $E_4^6(z)$, $E_4^3(z)\Delta_{12}(z)$, $\Delta_{12}^2(z)$ as a basis of $M(1, 24)$. The Fourier expansions of $E_4^6(z)$, $E_4^3(z)\Delta_{12}(z)$, $\Delta_{12}^2(z)$ are given by

$$E_4^6(z) = 1 + 1440q + 876960q^2 + 292072320q^3 + 57349833120q^4 + 6660135541440q^5 + \cdots,$$

$$E_4^3(z)\Delta_{12}(z) = q + 696q^2 + 162252q^3 + 12831808q^4 + 34188270q^5 + \cdots,$$

$$\Delta_{12}^2(z) = q^2 - 48q^3 + 1080q^4 - 15040q^5 + \cdots.$$

For the lattice $\mathscr{L}_{ne}$ by noting that $a(2, \mathscr{L}_{ne}) = 0$ the theta series is written as

$$\begin{aligned}\vartheta(z, \mathscr{L}_{ne}) &= 1 + a(4, \mathscr{L}_{ne})q^2 + a(6, \mathscr{L}_{ne})q^3 + a(8, \mathscr{L}_{ne})q^4 + a(10, \mathscr{L}_{ne})q^5 + \cdots \\ &= E_4^6(z) + c_1 E_4^3(z)\Delta_{12}(z) + c_2 \Delta_{12}^2(z),\end{aligned}$$

with some constants $c_1$ and $c_2$. By comparing the coefficients we have

$$c_1 = -1440, \quad a(4, \mathscr{L}_{ne}) = c_2 - 125280,$$

$$a(6, \mathscr{L}_{ne}) = 292072320 - 1440 \cdot 162252 - 48 \cdot c_2.$$

By rewriting the above relations we have

Proposition 3.1. *The Fourier expansion of the theta series $\vartheta(z, \mathscr{L}_{ne})$ is completely determined by the value $a(4, \mathscr{L}_{ne})$ and other values $a(2n, \mathscr{L}_{ne})$ $n \geq 3$ are explicitly expressed by $a(4, \mathscr{L}_{ne})$.*

For instance we have

$$a(6, \mathscr{L}_{ne}) = 52416000 - 48a(4, \mathscr{L}_{ne}), \tag{3.1}$$

$$a(8, \mathscr{L}_{ne}) = 39007332000 + 1048 \cdot a(4, \mathscr{L}_{ne}), \tag{3.2}$$

$$a(10, \mathscr{L}_{ne}) = 6609020221440 - 15040 \cdot a(4, \mathscr{L}_{ne}). \tag{3.3}$$

**3.2. Inner Product Relations.** By Proposition 2.1 the series $\vartheta(z, P_2, \mathscr{L}_{ne}) \in S(1, 26)$. Since $\dim_{\mathbf{C}} S(1, 26) = 1$ and it is spanned by $E_4^2(z)E_6(z)\Delta_{12}(z)$, we have the equality

$$\begin{aligned}\vartheta(z, P_2, \mathscr{L}_{ne}) &= \sum_{n=1}^{\infty} \sum_{\mathbf{x} \in \Lambda_{2n}} P_2(\mathbf{x}; \boldsymbol{\alpha}) q^n \\ &= c_3 E_4^2(z) E_6(z) \Delta_{12}(z) \\ &= c_3(q - 48q^2 - 195804q^3 - \cdots).\end{aligned}$$

Since $\Lambda_2 = \varnothing$, we have $c_3 = 0$. From this it follows that

Proposition 3.2.

$$\sum_{\mathbf{x} \in \Lambda_{2n}} P_2(\mathbf{x}; \boldsymbol{\alpha}) = 0 \quad \textit{for } n \geq 2.$$

By rewriting these equations we have

Proposition 3.3. *Let the notations be as above. Then we have*

$$\sum_{\mathbf{x} \in \Lambda_{2n}} (\mathbf{x}, \boldsymbol{\alpha})^2 = \frac{2n}{48} a(2n, \mathscr{L}_{ne})(\boldsymbol{\alpha}, \boldsymbol{\alpha}), \quad n \geq 2. \tag{3.4}$$

Similarly we have

$$\begin{aligned}\vartheta(z, P_4, \mathscr{L}_{ne}) &= \sum_{n=1}^{\infty} \sum_{\mathbf{x} \in \Lambda_{2n}} P_4(\mathbf{x}; \boldsymbol{\alpha}) q^n \\ &= c_4 E_4^4(z) \Delta_{12}(z) + c_5 E_4(z) \Delta_{12}^2(z) \\ &= c_4(q + 936q^2 + 331452q^3 + 53282368q^4 + 3468981150q^5 + \cdots) \\ &\quad + c_5(q^2 + 192q^3 - 8280q^4 + 147200q^5 - \cdots),\end{aligned}$$

$$\begin{aligned}\vartheta(z, P_6, \mathscr{L}_{ne}) &= \sum_{n=1}^{\infty} \sum_{\mathbf{x} \in \Lambda_{2n}} P_6(\mathbf{x}; \boldsymbol{\alpha}) q^n \\ &= c_6 E_4^3(z) E_6(z) \Delta_{12}(z) + c_7 E_6(z) \Delta_{12}^2(z) \\ &= c_6(q + 192q^2 - 205164q^3 - 80642048q^4 - 9217742250q^5 - \cdots) \\ &\quad + c_7(q^2 - 552q^3 + 8640q^4 + 116000q^5 - \cdots),\end{aligned}$$

$$\vartheta(z, P_8, \mathscr{L}_{ne}) = \sum_{n=1}^{\infty} \sum_{\mathbf{x} \in \Lambda_{2n}} P_8(\mathbf{x}; \boldsymbol{\alpha}) q^n$$
$$= c_8 E_4^5(z)\Delta_{12}(z) + c_9 E_4^2(z)\Delta_{12}^2(z)$$
$$= c_8(q + 1176q^2 + 558252q^3 + 134859328q^4 + 16978993230q^5 + \cdots)$$
$$+ c_9(q^2 + 432q^3 + 39960q^4 - 1418560q^5 + \cdots),$$

$$\vartheta(z, P_{10}, \mathscr{L}_{ne}) = \sum_{n=1}^{\infty} \sum_{\mathbf{x} \in \Lambda_{2n}} P_{10}(\mathbf{x}; \boldsymbol{\alpha}) q^n$$
$$= c_{10} E_4^4(z) E_6(z) \Delta_{12}(z) + c_{11} E_4(z) E_6(z) \Delta_{12}^2(z)$$
$$= c_{10}(q + 432q^2 - 156924q^3 - 129459968q^4 - 29013680250q^5 - \cdots)$$
$$+ c_{11}(q^2 - 312q^3 - 121680q^4 + 1004000q^5 + \cdots),$$

with some constants $c_4, c_5, \ldots, c_{11}$. From these equations we get (infinitely) many identities among $\sum_{\mathbf{x} \in \Lambda_k} P_m(\mathbf{x}; \boldsymbol{\alpha})$'s. We will use the following identities.

Proposition 3.4. *Let $\mathscr{L}_{ne}$ be any one of even unimodular 48 dimensional nearly extremal lattices, and $\Lambda_k$, $k = 4, 6, \ldots$ be the set of norm $k$ vectors in $\mathscr{L}_{ne}$. Then we have*

(i) $$\sum_{\mathbf{x} \in \Lambda_6} P_4(\mathbf{x}; \boldsymbol{\alpha}) = 192 \sum_{\mathbf{x} \in \Lambda_4} P_4(\mathbf{x}; \boldsymbol{\alpha}),$$

(ii) $$\sum_{\mathbf{x} \in \Lambda_6} P_6(\mathbf{x}; \boldsymbol{\alpha}) = -552 \sum_{\mathbf{x} \in \Lambda_4} P_6(\mathbf{x}; \boldsymbol{\alpha}),$$

(iii) $$\sum_{\mathbf{x} \in \Lambda_6} P_8(\mathbf{x}; \boldsymbol{\alpha}) = 432 \sum_{\mathbf{x} \in \Lambda_4} P_8(\mathbf{x}; \boldsymbol{\alpha}),$$

(iv) $$\sum_{\mathbf{x} \in \Lambda_6} P_{10}(\mathbf{x}; \boldsymbol{\alpha}) = -312 \sum_{\mathbf{x} \in \Lambda_4} P_{10}(\mathbf{x}; \boldsymbol{\alpha}),$$

(v) $$\sum_{\mathbf{x} \in \Lambda_8} P_4(\mathbf{x}; \boldsymbol{\alpha}) = -8280 \sum_{\mathbf{x} \in \Lambda_4} P_4(\mathbf{x}; \boldsymbol{\alpha}),$$

(vi) $$\sum_{\mathbf{x} \in \Lambda_8} P_6(\mathbf{x}; \boldsymbol{\alpha}) = 8640 \sum_{\mathbf{x} \in \Lambda_4} P_6(\mathbf{x}; \boldsymbol{\alpha}),$$

(vii) $$\sum_{\mathbf{x} \in \Lambda_8} P_8(\mathbf{x}; \boldsymbol{\alpha}) = 39960 \sum_{\mathbf{x} \in \Lambda_4} P_8(\mathbf{x}; \boldsymbol{\alpha}),$$

(viii) $\displaystyle\sum_{\mathbf{x}\in\Lambda_8} P_{10}(\mathbf{x};\boldsymbol{a}) = -121680 \sum_{\mathbf{x}\in\Lambda_4} P_{10}(\mathbf{x};\boldsymbol{a}),$

(ix) $\displaystyle\sum_{\mathbf{x}\in\Lambda_{10}} P_4(\mathbf{x};\boldsymbol{a}) = 147200 \sum_{\mathbf{x}\in\Lambda_4} P_4(\mathbf{x};\boldsymbol{a}),$

(x) $\displaystyle\sum_{\mathbf{x}\in\Lambda_{10}} P_6(\mathbf{x};\boldsymbol{a}) = 116000 \sum_{\mathbf{x}\in\Lambda_4} P_6(\mathbf{x};\boldsymbol{a}),$

(xi) $\displaystyle\sum_{\mathbf{x}\in\Lambda_{10}} P_8(\mathbf{x};\boldsymbol{a}) = -1418560 \sum_{\mathbf{x}\in\Lambda_4} P_8(\mathbf{x};\boldsymbol{a}),$

(xii) $\displaystyle\sum_{\mathbf{x}\in\Lambda_{10}} P_{10}(\mathbf{x};\boldsymbol{a}) = 1004000 \sum_{\mathbf{x}\in\Lambda_4} P_{10}(\mathbf{x};\boldsymbol{a}).$

Proof. From $\Lambda_2 = \emptyset$ and the $q$ expansion of $\vartheta(z, P_4, \mathscr{L}_{ne})$ we have

$$\begin{aligned}
\vartheta(z, P_4, \mathscr{L}_{ne}) &= \sum_{\mathbf{x}\in\Lambda_2} P_4(\mathbf{x};\boldsymbol{a})q + \sum_{\mathbf{x}\in\Lambda_4} P_4(\mathbf{x};\boldsymbol{a})q^2 + \sum_{\mathbf{x}\in\Lambda_6} P_4(\mathbf{x};\boldsymbol{a})q^3 + \cdots \\
&= \sum_{\mathbf{x}\in\Lambda_4} P_4(\mathbf{x};\boldsymbol{a})q^2 + \sum_{\mathbf{x}\in\Lambda_6} P_4(\mathbf{x};\boldsymbol{a})q^3 + \cdots \\
&= c_4(q + 936q^2 + 331452q^3 + 53282368q^4 + 53282368q^5 + \cdots) \\
&\quad + c_5(q^2 + 192q^3 - 8280q^4 + 147200q^5 - \cdots).
\end{aligned}$$

Thus we have $c_4 = 0$ and

$$\sum_{\mathbf{x}\in\Lambda_6} P_4(\mathbf{x};\boldsymbol{a}) = 192 \sum_{\mathbf{x}\in\Lambda_4} P_4(\mathbf{x};\boldsymbol{a}).$$

The equations (ii), ..., (xii) are proved in the same way. □

In the next subsection we will use the explicit form of the spherical functions $P_v(\mathbf{x};\boldsymbol{a})$. Concerning this functions the reader may refer to [9], Formulas (79) and (80) and [28]. We give here some.

$$P_2(\mathbf{x};\boldsymbol{a}) = (\mathbf{x},\boldsymbol{a})^2 - \frac{(\mathbf{x},\mathbf{x})(\boldsymbol{a},\boldsymbol{a})}{48}$$

$$P_4(\mathbf{x};\boldsymbol{a}) = (\mathbf{x},\boldsymbol{a})^4 - \frac{6}{52}(\mathbf{x},\boldsymbol{a})^2 \cdot (\mathbf{x},\mathbf{x})(\boldsymbol{a},\boldsymbol{a}) + \frac{3(\mathbf{x},\mathbf{x})^2(\boldsymbol{a},\boldsymbol{a})^2}{52 \cdot 50}$$

$\vdots \quad \vdots \quad \vdots$

$$P_{10}(\mathbf{x};\boldsymbol{\alpha}) = (\mathbf{x},\boldsymbol{\alpha})^{10} - \frac{45(\mathbf{x},\boldsymbol{\alpha})^8\cdot(\mathbf{x},\mathbf{x})\cdot(\boldsymbol{\alpha},\boldsymbol{\alpha})}{64} + \frac{630(\mathbf{x},\boldsymbol{\alpha})^6\cdot(\mathbf{x},\mathbf{x})^2(\boldsymbol{\alpha},\boldsymbol{\alpha})^2}{64\cdot 62}$$
$$- \frac{3150(\mathbf{x},\boldsymbol{\alpha})^4\cdot(\mathbf{x},\mathbf{x})^3(\boldsymbol{\alpha},\boldsymbol{\alpha})^3}{64\cdot 62\cdot 60} + \frac{4725(\mathbf{x},\boldsymbol{\alpha})^2\cdot(\mathbf{x},\mathbf{x})^4(\boldsymbol{\alpha},\boldsymbol{\alpha})^4}{64\cdot 62\cdot 60\cdot 58}$$
$$- \frac{945\cdot(\mathbf{x},\mathbf{x})^5(\boldsymbol{\alpha},\boldsymbol{\alpha})^5}{64\cdot 62\cdot 60\cdot 58\cdot 56}.$$

**3.3. First Results.** Let $\mathscr{L}_{ne}$ be a 48 dimensional even unimodular nearly extremal lattice, and $\mathscr{L}_{4,6}$ be the sublattice of $\mathscr{L}_{ne}$ generated by the subsets $\Lambda_4$ and $\Lambda_6$. We are going to prove that

$$\mathscr{L}_{4,6} = \mathscr{L}_{ne}. \tag{3.5}$$

To attain this goal we need some preliminary steps. We assume that the condition (3.5) does not hold. Then there should be a vector $\mathbf{x}_0 \in \mathscr{L}_{ne}\backslash\mathscr{L}_{4,6}$. Among such vectors we consider a vector $\mathbf{x}_0$ with the least norm $(\mathbf{x}_0,\mathbf{x}_0)$. We call this vector as a least obstruction vector for the condition (3.5). We remark that by the definition of the least obstruction vector we know that $(\mathbf{x}_0,\mathbf{x}_0) \geq 8$. We prove

LEMMA 3.5. (i) *If the least obstruction vector* $\mathbf{x}_0$ *has the norm* $(\mathbf{x}_0,\mathbf{x}_0) = 8$, *then it holds that*

$$|(\mathbf{x}_0,\mathbf{x})| \leq 2 \quad \textit{for any } \mathbf{x}\in\Lambda_4, \tag{3.6}$$

*and*

$$|(\mathbf{x}_0,\mathbf{y})| \leq 3 \quad \textit{for any } \mathbf{y}\in\Lambda_6. \tag{3.7}$$

(ii) *If the least obstruction vector* $\mathbf{x}_0$ *has the norm* $(\mathbf{x}_0,\mathbf{x}_0) \geq 10$, *then in addition to* (3.6) *and* (3.7) *it holds that*

$$|(\mathbf{x}_0,\mathbf{w})| \leq 4 \quad \textit{for any } \mathbf{w}\in\Lambda_8. \tag{3.8}$$

PROOF. Proof of (i). Without loss of generality we may restrict to the case when $(\mathbf{x}_0,\mathbf{x}) > 0$. Suppose that $(\mathbf{x}_0,\mathbf{x}) > 3$ holds for an $\mathbf{x}\in\Lambda_4$. Then we have $(\mathbf{x}-\mathbf{x}_0,\mathbf{x}-\mathbf{x}_0) = (\mathbf{x}_0,\mathbf{x}_0)+4-2(\mathbf{x}_0,\mathbf{x}) < (\mathbf{x}_0,\mathbf{x}_0)$. And $\mathbf{x}-\mathbf{x}_0 \equiv -\mathbf{x}_0 \bmod \mathscr{L}_{4,6}$. This contradicts to the minimality of the obstruction vector $\mathbf{x}_0$. This proves (3.6).

Other inequalities are proved likewise. □

When the least obstruction vector $\mathbf{x}_0$ has the norm $(\mathbf{x}_0, \mathbf{x}_0) \geq 8$ by Lemma 3.5,(i) we put

$$N_i = \#\{\mathbf{x} \in \Lambda_4 \mid (\mathbf{x}_0, \mathbf{x}) = i\}, \quad i = 0, \pm 1, \pm 2,$$

and

$$M_j = \#\{\mathbf{y} \in \Lambda_6 \mid (\mathbf{x}_0, \mathbf{y}) = j\}, \quad j = 0, \pm 1, \pm 2, \pm 3.$$

It is easy to show that $N_{-i} = N_i$ and $M_{-j} = M_j$.

We write the sum $\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}_0, \mathbf{x})^t$ $(t = 2, 4, \ldots)$ by using $N_i$'s.

$$\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}_0, \mathbf{x})^t = 2 \cdot N_1 + 2 \cdot 2^t N_2. \tag{3.9}$$

We have also

$$\sum_{\mathbf{y} \in \Lambda_6} (\mathbf{x}_0, \mathbf{y})^t = 2 \cdot M_1 + 2 \cdot 2^t M_2 + 2 \cdot 3^t M_3. \tag{3.10}$$

We now introduce the shortened notations of the right-hand sides of (3.9) and (3.10) respectively to save the space later:

$$\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}_0, \mathbf{x})^t = \sum_{i=1}^{2} 2 \cdot i^t N_i,$$

$$\sum_{\mathbf{y} \in \Lambda_6} (\mathbf{x}_0, \mathbf{y})^t = \sum_{j=1}^{3} 2 \cdot j^t M_j.$$

Then as the special cases of (3.4) we have (under the condition that $\alpha = \mathbf{x}_0$)

$$\sum_{i=1}^{2} 2 \cdot i^t N_i = \frac{4}{48} a(4, \mathscr{L}_{ne})(\mathbf{x}_0, \mathbf{x}_0) \tag{3.11}$$

and

$$\sum_{j=1}^{3} 2 \cdot j^2 M_j = \frac{6}{48} a(6, \mathscr{L}_{ne})(\mathbf{x}_0, \mathbf{x}_0). \tag{3.12}$$

Since we use $(\mathbf{x}_0, \mathbf{x}_0)$ frequently we set $m = (\mathbf{x}_0, \mathbf{x}_0)$. The lefthand-side of Proposition 3.4,(i) is

$$\sum_{\mathbf{y}\in\Lambda_6} P_4(\mathbf{y};\mathbf{x}_0) = \sum_{\mathbf{y}\in\Lambda_6}\left[(\mathbf{y},\mathbf{x}_0)^4 - \frac{6}{52}(\mathbf{y},\mathbf{y})m(\mathbf{y},\mathbf{x}_0)^2 + \frac{3}{52\cdot 50}(\mathbf{y},\mathbf{y})^2m^2\right]$$
$$= \sum_{j=1}^{3} 2\cdot j^4 M_j - \frac{6\cdot 6}{52}m\sum_{j=1}^{3} 2\cdot j^2 M_j + \frac{3\cdot 6^2}{52\cdot 50}m^2 a(6,\mathscr{L}_{ne}).$$

The right-hand side of Proposition 3.4,(i) is

$$192\sum_{\mathbf{x}\in\Lambda_4} P_4(\mathbf{x};\mathbf{x}_0) = 192\sum_{\mathbf{x}\in\Lambda_4}\left[(\mathbf{x},\mathbf{x}_0)^4 - \frac{6}{52}m(\mathbf{x},\mathbf{x})(\mathbf{x},\mathbf{x}_0)^2 + \frac{3}{52\cdot 50}(\mathbf{x},\mathbf{x})^2m^2\right]$$
$$= 192\left[\sum_{i=1}^{2} 2\cdot i^4 N_i - \frac{6\cdot 4}{52}m\sum_{i=1}^{2} 2\cdot i^2 N_i + \frac{3\cdot 4^2}{52\cdot 50}m^2 a(4,\mathscr{L}_{ne})\right].$$

Thus we have a linear condition on $N_1$, $N_2$, $M_1$, $M_2$, $M_3$ and $a(4,\mathscr{L}_{ne})$

$$\sum_{j=1}^{3} 2\cdot j^4 M_j - \frac{6\cdot 6\cdot m}{52}\sum_{j=1}^{3} 2\cdot j^2 M_j + \frac{3\cdot 6^2\cdot m^2}{52\cdot 50}a(6,\mathscr{L}_{ne}) \tag{3.13}$$
$$= 192\left[\sum_{i=1}^{2} 2\cdot i^4 N_i - \frac{6\cdot 4\cdot m}{52}\sum_{i=1}^{2} 2\cdot i^2 N_i + \frac{3\cdot 4^2\cdot m^2}{52\cdot 50}a(4,\mathscr{L}_{ne})\right].$$

In the same way from Proposition 3.4,(ii) we have

$$\sum_{j=1}^{3} 2\cdot j^6 M_j - \frac{15\cdot 6\cdot m}{56}\sum_{j=1}^{3} 2\cdot j^4 M_j + \frac{45\cdot 6^2\cdot m^2}{56\cdot 54}\sum_{j=1}^{3} 2\cdot j^2 M_j \tag{3.14}$$
$$- \frac{15\cdot 6^3\cdot m^3}{56\cdot 54\cdot 52}a(6,\mathscr{L}_{ne})$$
$$= -552\left[\sum_{i=1}^{2} 2\cdot i^6 N_i - \frac{15\cdot 4\cdot m}{56}\sum_{i=1}^{2} 2\cdot i^4 N_i\right.$$
$$\left. + \frac{45\cdot 4^2\cdot m^2}{56\cdot 54}\sum_{i=1}^{2} 2\cdot i^2 N_i - \frac{15\cdot 4^3\cdot m^3}{56\cdot 54\cdot 52}a(4,\mathscr{L}_{ne})\right].$$

From Proposition 3.4,(iii) we have

$$\sum_{j=1}^{3} 2\cdot j^8 M_j - \frac{28\cdot 6\cdot m}{60}\sum_{j=1}^{3} 2\cdot j^6 M_j + \frac{210\cdot 6^2\cdot m^2}{60\cdot 58}\sum_{j=1}^{3} 2\cdot j^4 M_j \tag{3.15}$$
$$- \frac{420\cdot 6^3\cdot m^3}{60\cdot 58\cdot 56}\sum_{j=1}^{3} 2\cdot j^2 M_j + \frac{105\cdot 6^4\cdot m^4}{60\cdot 58\cdot 56\cdot 54}a(6,\mathscr{L}_{ne})$$

$$
\begin{aligned}
&= 432\Big[\sum\nolimits_{i=1}^{2} 2\cdot i^8 N_i - \frac{28\cdot 4\cdot m}{60}\sum\nolimits_{i=1}^{2} 2\cdot i^6 N_i \\
&\qquad - \frac{210\cdot 4^2\cdot m^2}{60\cdot 58}\sum\nolimits_{i=1}^{2} 2\cdot i^4 N_i - \frac{420\cdot 4^3\cdot m^3}{60\cdot 58\cdot 56}\sum\nolimits_{i=1}^{2} 2\cdot i^2 N_i \\
&\qquad + \frac{105\cdot 4^4\cdot m^4}{60\cdot 58\cdot 56\cdot 54} a(4,\mathscr{L}_{ne})\Big].
\end{aligned}
$$

From Proposition 3.4,(iv) we have

$$
\begin{aligned}
(3.16)\quad &\sum\nolimits_{j=1}^{3} 2\cdot j^{10} M_j - \frac{45\cdot 6\cdot m}{64}\sum\nolimits_{j=1}^{3} 2\cdot j^8 M_j + \frac{630\cdot 6^2\cdot m^2}{64\cdot 62}\sum\nolimits_{j=1}^{3} 2\cdot j^6 M_j \\
&\quad - \frac{3150\cdot 6^3\cdot m^3}{64\cdot 62\cdot 60}\sum\nolimits_{j=1}^{3} 2\cdot j^4 M_j + \frac{4725\cdot 6^4\cdot m^4}{64\cdot 62\cdot 60\cdot 58}\sum\nolimits_{j=1}^{3} 2\cdot j^2 M_j \\
&\quad - \frac{945\cdot 6^5\cdot m^5}{64\cdot 62\cdot 60\cdot 58\cdot 56} a(6,\mathscr{L}_{ne}) \\
&= -312\Big[\sum\nolimits_{i=1}^{2} 2\cdot i^{10} N_i - \frac{45\cdot 4\cdot m}{64}\sum\nolimits_{i=1}^{2} 2\cdot i^8 N_i \\
&\qquad + \frac{630\cdot 4^2\cdot m^2}{64\cdot 62}\sum\nolimits_{i=1}^{2} 2\cdot i^6 N_i - \frac{3150\cdot 4^3\cdot m^3}{64\cdot 62\cdot 60}\sum\nolimits_{i=1}^{2} 2\cdot i^4 N_i \\
&\qquad + \frac{4725\cdot 4^4\cdot m^4}{64\cdot 62\cdot 60\cdot 58}\sum\nolimits_{i=1}^{2} 2\cdot i^2 N_i \\
&\qquad - \frac{945\cdot 4^5\cdot m^5}{64\cdot 62\cdot 60\cdot 58\cdot 56} a(4,\mathscr{L}_{ne})\Big].
\end{aligned}
$$

The equations $(3.1), (3.11), \dots, (3.16)$ allow us to express $a(4,\mathscr{L}_{ne})$, $N_1$, $N_2$, $M_1$, $M_2$, $M_3$ in terms of rational functions with integer coefficients of $m$. We simply give the results as a proposition.

Proposition 3.6. *Let $\mathscr{L}_{ne}$ be an even unimodular nearly extremal lattice of dimension* 48. *Suppose there exists an obstruction vector $\mathbf{x}_0$ with the norm $m = (\mathbf{x}_0, \mathbf{x}_0) \geq 8$ for* (3.5). *Let $a(4,\mathscr{L}_{ne})$, $N_1$, $N_2$, $M_1$, $M_2$, $M_3$ be the quantities discussed above. Then we have the following expressions.*

$$
a(4,\mathscr{L}_{ne}) = \frac{4320(43m^4 - 903m^3 + 6832m^2 - 21580m + 22880)}{(m-2)(m-4)(m-6)(m-9)},
$$

$$
N_1 = \frac{-24m\cdot(15m^5 - 655m^4 + 9504m^3 - 61252m^2 + 177268m - 179400)}{(m-2)(m-4)(m-6)(m-9)},
$$

$$N_2 = \frac{3m(30m^3 - 425m^2 + 1703m - 1300)}{(m-4)(m-9)},$$

$$M_1 = \frac{num}{(m-2)(m-4)(m-6)(m-9)},$$

$$M_2 = \frac{-288m(15m^6 - 630m^5 + 9305m^4 - 62763m^3 + 203339m^2 - 294866m + 145600)}{(m-2)(m-4)(m-6)(m-9)},$$

$$M_3 = \frac{16m(45m^5 - 1080m^4 + 9355m^3 - 35581m^2 + 56511m - 29250)}{(m-4)(m-6)(m-9)}.$$

*Here num is given by*

$$num = 144m(75m^6 - 3870m^5 + 81805m^4 - 844293m^3 + 4342159m^2 - 10375456m + 8875100).$$

We now prove

Theorem 3.7. *Let* $\mathscr{L}_{ne}$ *be an even unimodular nearly extremal lattice of dimension* 48. *Then* (3.5) *holds.*

Proof. The proof is not straightforward. We divide the proof into three steps.

**Step 1**. We assume that the condition (3.5) does not hold. Let $\mathbf{x}_0$ be the least obstruction vector against the condition (3.5). We suppose that $m = (\mathbf{x}_0, \mathbf{x}_0) \geq 8$. Let $N_1, N_2, \ldots, M_3$ the five quantities introduced above. Then by Proposition 3.6 these quantities are expressed as a rational functions of $m$. By our setting $a(4, \mathscr{L}_{ne})$ and other five quantities should be all non-negative integers. By a simple estimation we can show that the rational expression for $M_2$ is negative if $m \geq 50$. For even $m$ which are between 8 and 48 at least one of the rational expressions for $a(4, \mathscr{L}_{ne}), N_1, N_2, \ldots, M_3$ has the value that is not integer or negative except for two cases.

The parameters of the two exceptions are

Case 1. $m = 10$, $a(4, \mathscr{L}_{ne}) = 388800$, $N_1 = 97400$, $N_2 = 16150$, $M_1 = 8200800$, $M_2 = 2732400$, $M_3 = 218400$.

Case 2. $m = 12$, $a(4, \mathscr{L}_{ne}) = 236976$, $N_1 = 59832$, $N_2 = 14664$, $M_1 = 9575592$, $M_2 = 3821664$, $M_3 = 657624$.

**Step 2**. We will briefly show that the parameters of the case 1 can not pass a further test.

Since the possibility of the least obstruction vector $\mathbf{x}_0$ with $m = (\mathbf{x}_0, \mathbf{x}_0) = 8$ against the condition (3.5) is denied. We suppose that the norm of $\mathbf{x}_0$ is $m = (\mathbf{x}_0, \mathbf{x}_0) = 10$. Then we have $a(4, \mathscr{L}_{ne}) = 388800$, $N_1 = 97400$, $N_2 = 16150$, $M_1 = 8200800$, $M_2 = 2732400$, $M_3 = 218400$.

We note that

$$N_1 = \#\{\mathbf{x} \in \Lambda_4 \mid (\mathbf{x}, \mathbf{x}_0) = 1\} = 97400, \quad \text{and}$$

$$N_2 = \#\{\mathbf{x} \in \Lambda_4 \mid (\mathbf{x}, \mathbf{x}_0) = 1\} = 16150.$$

By (3.9) we can compute the values

$$\sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}_0, \mathbf{x})^t = 2 \cdot 97400 + 2 \cdot 2^t 16150. \tag{3.17}$$

By reckoning in Lemma 3.5,(ii) we put

$$R_j = |\{\mathbf{z} \in \Lambda_8 \mid (\mathbf{x}_0, \mathbf{z}) = j\}|, \quad j = 0, \pm 1, \pm 2, \pm 3, \pm 4.$$

Similarly to Eqns. (3.9) and (3.10) we have

$$\sum_{\mathbf{z} \in \Lambda_8} (\mathbf{x}_0, \mathbf{z})^t = 2 \cdot R_1 + 2 \cdot 2^t R_2 + 2 \cdot 3^t R_3 + 2 \cdot 4^t R_4. \tag{3.18}$$

As before we employ a short hand notation:

$$\sum_{\mathbf{z} \in \Lambda_8} (\mathbf{x}_0, \mathbf{z})^t = \sum_{i=1}^{4} 2 \cdot i^t R_i.$$

First from (3.4) we obtain

$$\sum_{i=1}^{4} 2 \cdot i^2 R_i = \frac{8}{48} a(8, \mathscr{L}_{ne})(\mathbf{x}_0, \mathbf{x}_0) = \frac{5}{3} a(8, \mathscr{L}_{ne}). \tag{3.19}$$

We use Proposition 3.4,(v), …, (viii) to obtain further linear equations on $R_1, \ldots, R_4$. By Proposition 3.4,(v) we know

$$\sum_{\mathbf{x} \in \Lambda_8} P_4(\mathbf{x}; \boldsymbol{\alpha}) = -8280 \sum_{\mathbf{x} \in \Lambda_4} P_4(\mathbf{x}; \boldsymbol{\alpha}).$$

We take $\mathbf{x}_0 = \alpha$ in the above equation and also in the equations (vi), (vii) and (viii) in Proposition 3.4. We note that the right-hand sides of the equations (v), ..., (viii) can be computable numerically by Eqn. (3.17). For instance, the summation at the right-hand side the equation (v) can be transformed to

$$\sum_{\mathbf{x}\in\Lambda_4} P_4(\mathbf{x};\mathbf{x}_0) = \sum_{\mathbf{x}\in\Lambda_4}\left[(\mathbf{x},\mathbf{x}_0)^4 - \frac{6}{52}(\mathbf{x},\mathbf{x})(\mathbf{x}_0,\mathbf{x}_0)(\mathbf{z},\mathbf{x}_0)^2 + \frac{3\cdot 10^2}{52\cdot 50}(\mathbf{x},\mathbf{x})^2(\mathbf{x}_0,\mathbf{x}_0)^2\right]$$

$$= 2\cdot 97400 + 2\cdot 2^4 16150 - \frac{6\cdot 4\cdot 10}{52}(2\cdot 97400 + 2\cdot 2^2 16150)$$

$$+ \frac{3\cdot 4^2\cdot 10^2}{52\cdot 50}a(4,\mathscr{L}_{ne})$$

$$= -66000.$$

In the same way $\sum_{\mathbf{x}\in\Lambda_4} P_k(\mathbf{x};\mathbf{x}_0)$, $k = 6, 8, 10$ can be computed numerically, and we only give the resulting values:

$$\sum_{\mathbf{x}\in\Lambda_4} P_6(\mathbf{x};\mathbf{x}_0) = -1968000/91,$$

$$\sum_{\mathbf{x}\in\Lambda_4} P_8(\mathbf{x};\mathbf{x}_0) = 5428400/29,$$

$$\sum_{\mathbf{x}\in\Lambda_4} P_{10}(\mathbf{x};\mathbf{x}_0) = 75918750/31.$$

As to the left-hand sides of (v) in Proposition 3.4 we have

$$\sum_{\mathbf{z}\in\Lambda_8} P_4(\mathbf{z};\mathbf{x}_0) = \sum_{\mathbf{z}\in\Lambda_8}\left[(\mathbf{z},\mathbf{x}_0)^4 - \frac{6}{52}(\mathbf{z},\mathbf{z})(\mathbf{x}_0,\mathbf{x}_0)(\mathbf{z},\mathbf{x}_0)^2 + \frac{3\cdot 10^2}{52\cdot 50}(\mathbf{z},\mathbf{z})^2(\mathbf{x}_0,\mathbf{x}_0)^2\right] \tag{3.20}$$

$$= \sum_{i=1}^4 2\cdot i^4 R_i - \frac{6\cdot 8\cdot 10}{52}\sum_{i=1}^4 2\cdot i^2 R_i$$

$$+ \frac{3\cdot 8^2\cdot 10^2}{52\cdot 50}a(8,\mathscr{L}_{ne})$$

$$= -8280\cdot(-66000).$$

By Eqns. (3.2), (3.19) and (3.20) we have

$$\sum_{i=1}^4 2\cdot i^4 R_i = 315864835200. \tag{3.21}$$

From Eqn. (vi) we have

$$(3.22)\qquad \sum\nolimits_{\mathbf{z}\in\Lambda_8} P_6(\mathbf{z};\mathbf{x}_0) = \sum\nolimits_{i=1}^4 2\cdot i^6 R_i - \frac{15\cdot 8\cdot 10}{56}\sum\nolimits_{i=1}^4 2\cdot i^4 R_i$$

$$+\frac{45\cdot 8^2\cdot 10^2}{56\cdot 54}\sum\nolimits_{i=1}^4 2\cdot i^2 R_i - \frac{15\cdot 8^3\cdot 10^3}{56\cdot 54\cdot 52}a(8,\mathscr{L}_{ne})$$

$$= 8640\cdot(-1968000/91).$$

By Eqns. (3.19), (3.21) and (3.22) we have

$$(3.23)\qquad \sum_{i=1}^4 2\cdot i^6 R_i = 31681640160000/13.$$

Since $R_1,\dots,R_4$ are all non-negative integers Eqn. (3.23) implies that Case 1 is impossible.

**Step 3**. It remains to show that $m = (\mathbf{x}_0,\mathbf{x}_0) = 12$ is impossible.

Suppose for an obstruction vector $\mathbf{x}_0$ against the condition (3.5) with $m = (\mathbf{x}_0,\mathbf{x}_0) = 12$. Then some parameters are detined by those in Case 2. Namely $a(4,\mathscr{L}_{ne}) = 236976$, $N_1 = 59832$, $N_2 = 14664$. This time we may note that

$$N_1 = \#\{\mathbf{x}\in\Lambda_4 \mid (\mathbf{x},\mathbf{x}_0) = 1\} = 59832,\quad \text{and}$$

$$N_2 = \#\{\mathbf{x}\in\Lambda_4 \mid (\mathbf{x},\mathbf{x}_0) = 1\} = 14664.$$

By (3.9) we can compute the values

$$(3.24)\qquad \sum_{\mathbf{x}\in\Lambda_4}(\mathbf{x}_0,\mathbf{x})^t = 2\cdot 59832 + 2\cdot 2^t 14664.$$

By Lemma 3.5,(ii) we may set

$$U_j = \#\{\mathbf{w}\in\Lambda_{10} \mid (\mathbf{x}_0,\mathbf{w}) = j\},\quad j = 0,\pm1,\pm2,\pm3,\pm4.$$

We easily see that $U_{-j} = U_j$, $j = 1,\dots,4$. We put

$$(3.25)\qquad \sum_{\mathbf{w}\in\Lambda_{10}}(\mathbf{x}_0,\mathbf{w})^t = 2\cdot U_1 + 2\cdot 2^t U_2 + 2\cdot 3^t U_3 + 2\cdot 4^t U_4 = \sum_{i=1}^4 2\cdot i^t U_i.$$

From (3.4) we obtain

$$(3.26)\qquad \sum_{i=1}^4 2\cdot i^2 U_i = \frac{10}{48}a(10,\mathscr{L}_{ne})(\mathbf{x}_0,\mathbf{x}_0) = \frac{5}{2}a(10,\mathscr{L}_{ne}).$$

We put $\mathbf{x}_0 = \alpha$ and use Proposition 3.4,(ix), ... (xi) to obtain further linear equations on $U_1, \dots, U_4$. From Eqn. (ix) we have

$$\sum_{\mathbf{y} \in \Lambda_{10}} P_4(\mathbf{y}; \mathbf{x}_0) = 147200 \sum_{\mathbf{x} \in \Lambda_4} P_4(\mathbf{x}; \mathbf{x}_0).$$

The right-hand sides of the equations (ix), ..., (xii) can be computable numerically by Eqn. (3.24). For instance we see that

$$\begin{aligned}\sum_{\mathbf{x} \in \Lambda_4} P_4(\mathbf{x}; \mathbf{x}_0) &= \sum_{\mathbf{x} \in \Lambda_4} \left[ (\mathbf{x}, \mathbf{x}_0)^4 - \frac{6}{52} (\mathbf{x}, \mathbf{x})(\mathbf{x}_0, \mathbf{x}_0)(\mathbf{x}, \mathbf{x}_0)^2 + \frac{3}{52 \cdot 50} (\mathbf{x}, \mathbf{x})^2 (\mathbf{x}_0, \mathbf{x}_0)^2 \right] \\ &= 2 \cdot 59832 + 2 \cdot 2^4 14664 - \frac{6 \cdot 4 \cdot 12}{52} (2 \cdot 59832 + 2 \cdot 2^2 14664) \\ &\quad + \frac{3 \cdot 4^2 \cdot 12^2}{52 \cdot 50} a(4, \mathscr{L}_{ne}) \\ &= -2339472/25.\end{aligned}$$

In the same way $\sum_{\mathbf{x} \in \Lambda_4} P_k(\mathbf{x}; \mathbf{x}_0)$, $k = 6, 8$ can be computed numerically, and we only give the resulting vlues:

$$\sum_{\mathbf{x} \in \Lambda_4} P_6(\mathbf{x}; \mathbf{x}_0) = 4536816/91,$$

$$\sum_{\mathbf{x} \in \Lambda_4} P_8(\mathbf{x}; \mathbf{x}_0) = 123418992/145.$$

$$\begin{aligned}\text{(3.27)} \qquad \textstyle\sum_{\mathbf{y} \in \Lambda_{10}} P_4(\mathbf{y}; \mathbf{x}_0) &= \textstyle\sum_{\mathbf{y} \in \Lambda_{10}} \Big[ (\mathbf{y}, \mathbf{x}_0)^4 - \frac{6}{52} (\mathbf{y}, \mathbf{y})(\mathbf{x}_0, \mathbf{x}_0)(\mathbf{y}, \mathbf{x}_0)^2 \\ &\qquad + \frac{3}{52 \cdot 50} (\mathbf{y}, \mathbf{y})^2 (\mathbf{x}_0, \mathbf{x}_0)^2 \Big] \\ &= \textstyle\sum_{i=1}^4 2 \cdot i^4 U_i - \frac{6 \cdot 10 \cdot 12}{52} \sum_{i=1}^4 2 \cdot i^2 U_i \\ &\quad + \frac{3 \cdot 10^2 \cdot 12^2}{52 \cdot 50} a(10, \mathscr{L}_{ne}) \\ &= 147200 \cdot (-2339472/25).\end{aligned}$$

By Eqns. (3.26) and (3.27) we have

$$\sum_{i=1}^4 2 \cdot i^4 U_i = 118884435032064. \tag{3.28}$$

From Proposition 3.4 Eqn. (x) we have

$$\begin{aligned}(3.29)\quad \sum\nolimits_{\mathbf{y}\in\Lambda_{10}} P_6(\mathbf{y};\mathbf{x}_0) &= \sum\nolimits_{i=1}^{4} 2\cdot i^6 U_i - \frac{15\cdot 10\cdot 12}{56}\sum\nolimits_{i=1}^{4} 2\cdot i^4 U_i \\ &\quad + \frac{45\cdot 10^2\cdot 12^2}{56\cdot 54}\sum\nolimits_{i=1}^{4} 2\cdot i^2 U_i - \frac{15\cdot 10^3\cdot 12^3}{56\cdot 54\cdot 52} a(10,\mathscr{L}_{ne}) \\ &= 116000\cdot(4536816/91).\end{aligned}$$

By Eqns. (3.27), (3.28) and (3.29) we have

$$(3.30)\qquad \sum_{i=1}^{4} 2\cdot i^6 U_i = 1371465442675200.$$

From Proposition 3.4, Eqn. (xi) we have

$$\begin{aligned}(3.31)\quad \sum\nolimits_{\mathbf{y}\in\Lambda_{10}} P_8(\mathbf{y};\mathbf{x}_0) &= \sum\nolimits_{i=1}^{4} 2\cdot i^8 U_i - \frac{28\cdot 10\cdot 12}{60}\sum\nolimits_{i=1}^{4} 2\cdot i^6 U_i \\ &\quad + \frac{210\cdot 10^2\cdot 12^2}{60\cdot 58}\sum\nolimits_{i=1}^{4} 2\cdot i^4 U_i \\ &\quad - \frac{420\cdot 10^3\cdot 12^3}{60\cdot 58\cdot 56}\sum\nolimits_{i=1}^{4} 2\cdot i^2 U_i \\ &\quad + \frac{105\cdot 10^4\cdot 12^4}{60\cdot 58\cdot 56\cdot 54} a(10,\mathscr{L}_{ne}) \\ &= -1418560\cdot(123418992/145).\end{aligned}$$

By Eqns. (3.26), (3.28), (3.30) and (3.31) we have

$$(3.32)\qquad \sum_{i=1}^{4} 2\cdot i^8 U_i = 21326995936257024.$$

The linear conditions (3.26), (3.28), (3.30) and (3.32) on $U_1,\dots,U_4$ are enough to solve them. We solve

$$U_1 = -1458949248672,\quad U_2 = 2348888715456,$$
$$U_3 = -288817699680,\quad U_4 = 182473363248.$$

These values show that $m = (\mathbf{x}_0,\mathbf{x}_0) = 12$ is impossible. We have proved the theorem. □

REMARK 3.8. In the proofs of Steps 2 and 3 of Theorem 3.7 there are more than one way to show the impossibilitity of the parameters $m, N_1, \ldots, M_3$. The point is that there are a lot of conditions for $R_1, \ldots, R_4$ (or $U_1, \ldots, U_4$) that will lead to the contradiction.

## 4. Examples of Nearly Extremal Lattices

**4.1. First Example.** An orthogonal sum of two copies of the Leech lattice. Let *Leech* be the 24-dimensional even unimodular extremal lattice. We form an orthogonal sum of the two copies of *Leech*: $\mathscr{L}_1 = Leech \oplus Leech$. Since the minimal non-zero norm of *Leech* is four, $\mathscr{L}_1$ is a nealy extremal 48-dimensional even unimodular lattice. *Leech* has a minimal vector basis, and so does $\mathscr{L}_1$. The lattice $\mathscr{L}_1$ has thus a finer basis than Theorem 3.7. Since $|\Lambda_4(Leech)| = 196560$, we compute that $|\Lambda_4(\mathscr{L}_1)| = 2 \cdot 196560 = 393120$. By Eqns. (3.1) and (3.2) we have

$$|\Lambda_6(\mathscr{L}_1)| = 33546240, \quad |\Lambda_8(\mathscr{L}_1)| = 39419321760.$$

**4.2. Second Example.** Lattice constructed from an orthogonal sum of the binary Golay code $\mathscr{G}_{24}$ of length 24. Let $\mathbf{C}_1 = \mathscr{G}_{24} \oplus \mathscr{G}_{24}$ be an orthogonal sum of the two copies of $\mathscr{G}_{24}$. $\mathbf{C}_1$ is a doubly even binary self-dual $[48, 24, 8]$ code of length 48. Since the homogeneous weight enumerator $W_{\mathscr{G}_{24}}(x, y)$ of the Golay code is given by $W_{\mathscr{G}_{24}}(x, y) = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}$, the homogeneous weight enumerator of $\mathscr{G}_{24} \oplus \mathscr{G}_{24}$ is computed to be

$$W_{\mathscr{G}_{24} \oplus \mathscr{G}_{24}}(x, y) = x^{48} + 1518x^{40}y^8 + 5152x^{36}y^{12} + 577599x^{32}y^{16} + 3910368x^{28}y^{20} + 7787940x^{24}y^{24} + \cdots. \tag{4.1}$$

Let

$$\rho : \mathbf{Z}^{48} \to \mathbf{F}_2^{48}$$

be the reduction modulo 2 map. We set

$$\mathscr{M}(\mathbf{C}_1) = \frac{1}{\sqrt{2}}\left\{\mathbf{x} = (x_1, x_2, \ldots, x_{48}) \in \rho^{-1}(\mathbf{C}_1) \,\middle|\, \sum_{i=1}^{48} x_i \equiv 0 \bmod 4\right\},$$

$\gamma = \frac{1}{\sqrt{8}}(1, \ldots, 1) \in \mathbf{R}^{48}$, and

$$\mathscr{L}_2 = \mathscr{M}(\mathbf{C}_1) \cup (\gamma + \mathscr{M}(\mathbf{C}_1)).$$

$\mathscr{L}_2$ is an even unimodular nearly extremal lattice. We examine $\Lambda_4(\mathscr{L}_2)$ and $\Lambda_6(\mathscr{L}_2)$ precisely.

$\Lambda_4(\mathscr{L}_2)$ consists of the two subsets $\Lambda_{4,1}(\mathscr{L}_2)$ and $\Lambda_{4,2}(\mathscr{L}_2)$ that are given by

$$\Lambda_{4,1}(\mathscr{L}_2) = \{((\pm\sqrt{2})^2, 0^{46})\},$$

$$\Lambda_{4,2}(\mathscr{L}_2) = \left\{\frac{1}{\sqrt{2}}(x_1, x_2, \ldots, x_{48}) \in \Lambda_4(\mathscr{L}_2) \mid i's\ (x_i \neq 0)\ come\ from\ \mathbf{C}_1(8), \right.$$
$$\left. \sum_{i=1}^{48} x_i \equiv 0 \bmod 4 \right\}.$$

Here non-zero coordinates of an element in $\Lambda_{4,1}(\mathscr{L}_2)$ place at the two coordiates out of the 48 coordinates, and $\mathbf{C}_1(8)$ is the subset of $\mathbf{C}_1$ consisting of the codewords of weight 8.

We observe that $\Lambda_4(\mathscr{L}_2) = \Lambda_{4,1}(\mathscr{L}_2) \cup \Lambda_{4,2}(\mathscr{L}_2)$ and

$$|\Lambda_4(\mathscr{L}_2)| = |\Lambda_{4,1}(\mathscr{L}_2)| + |\Lambda_{4,2}(\mathscr{L}_2)| = 4512 + 1518 \cdot 2^7 = 198816. \tag{4.2}$$

As to the set $\Lambda_6(\mathscr{L}_2)$ there are some efforts to consider.

Let $\mathbf{0} \neq \mathbf{u} \in \mathbf{C}_1$, then there are vectors $\mathbf{x} \in \left(\frac{1}{\sqrt{2}}\rho^{-1}(\mathbf{u})\right) \cap \mathscr{L}_2$ with various norms. Among such $\mathbf{x}$'s there are the vectors with the least norm $\frac{1}{2}wt(\mathbf{u})$. We write

$$Min(\mathbf{u}, \mathscr{L}_2) = \left\{\mathbf{x} \in \left(\frac{1}{\sqrt{2}}\rho^{-1}(\mathbf{u})\right) \cap \mathscr{L}_2 \mid (\mathbf{x}, \mathbf{x}) = \frac{1}{2}wt(\mathbf{u})\right\}.$$

In the set $Min(\mathbf{u}, \mathscr{L}_2)$ there is a unique vector $\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u})$, where non-zero coordinates of $\rho^{\#}(\mathbf{u})$ are all 1. Other vector $\mathbf{y} \in Min(\mathbf{u}, \mathscr{L}_2)$ is described by the difference vector $\sqrt{2}\delta = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \mathbf{y}$. Here the coordinate values of $\delta$ are determined as follows. Let $i(\sqrt{2}\mathbf{y})$ (resp. $i(\delta)$ be the $i$-th coordinate of $\sqrt{2}\mathbf{y}$ (resp. $\delta$) for $1 \leq i \leq 48$, then we see that

$$i(\delta) = \begin{cases} 1 & if\ \ i(\sqrt{2}\mathbf{y}) = -1, \\ 0 & if\ \ i(\sqrt{2}\mathbf{y}) = 1\ \ or\ \ 0. \end{cases}$$

We will use two notations. When a vecor $\mathbf{v}$ has integer coordinates whose entries consist of 1's and 0's, then we use $supp(\mathbf{v})$ to denote the set of coordinate positions at which $\mathbf{v}$ has 1 as the entry. This type of vector $\mathbf{v}$ will be called a $(0,1)$ vector. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two $(0,1)$ vectors of the same size. We write $\mathbf{v}_1 \subseteq \mathbf{v}_2$ if the inclusion relation $supp(\mathbf{v}_1) \subseteq supp(\mathbf{v}_2)$ holds. In this sense we may write $\delta \subseteq \rho^{\#}(\mathbf{u})$.

REMARK 4.1. We remark that $|Min(\mathbf{u}, \mathscr{L}_2)| = 2^{wt(\mathbf{u})-1}$ for $wt(\mathbf{u}) \leq 24$.

LEMMA 4.2. *Let* $\mathbf{y} = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta \in Min(\mathbf{u}, \mathscr{L}_2)$. *Then a neccesary and sufficient condition for* $\mathbf{y} - \gamma$ *belongs to* $\Lambda_6(\mathscr{L}_2)$ *is that* $\delta = \mathbf{0}$.

PROOF. We see that

$$\begin{aligned}(\mathbf{y} - \gamma, \mathbf{y} - \gamma) &= \left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta, \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta\right) \\ &\quad - 2\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta, \gamma\right) + (\gamma, \gamma) \\ &= \frac{(\delta, \delta)}{2} + 6.\end{aligned}$$

Therefore we conclude that

$$(\mathbf{y} - \gamma, \mathbf{y} - \gamma) = 6 \Leftrightarrow \delta = 0. \qquad \square$$

By Lemma 4.2 we obtain some subsets of $\Lambda_6(\mathscr{L}_2)$.

$$\Lambda_{6,0}(\mathscr{L}_2) = \left\{\pm\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right) \,\middle|\, \mathbf{u} \in \mathbf{C}_1(0)\right\},$$

$$\Lambda_{6,1}(\mathscr{L}_2) = \left\{\pm\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right) \,\middle|\, \mathbf{u} \in \mathbf{C}_1(8)\right\},$$

$$\Lambda_{6,2}(\mathscr{L}_2) = \left\{\pm\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right) \,\middle|\, \mathbf{u} \in \mathbf{C}_1(12)\right\},$$

$$\Lambda_{6,3}(\mathscr{L}_2) = \left\{\pm\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right) \,\middle|\, \mathbf{u} \in \mathbf{C}_1(16)\right\},$$

$$\Lambda_{6,4}(\mathscr{L}_2) = \left\{\pm\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right) \,\middle|\, \mathbf{u} \in \mathbf{C}_1(20)\right\},$$

$$\Lambda_{6,5}(\mathscr{L}_2) = \left\{\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right) \,\middle|\, \mathbf{u} \in \mathbf{C}_1(24)\right\}.$$

REMARK 4.3. It is easy to see that the set $\left\{-\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right) \,\middle|\, \mathbf{u} \in \mathbf{C}_1(8)\right\}$ equals the set $\left\{\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}^c) - \gamma\right) \,\middle|\, \mathbf{u}^c \in \mathbf{C}_2(40)\right\}$. The same arguement also applies to the pair $(\mathbf{C}_1(12), \mathbf{C}_1(36))$, the pair $(\mathbf{C}_1(16), \mathbf{C}_1(32))$ and the pair

$(\mathbf{C}_1(20), \mathbf{C}_1(28))$. On the other hand $\left\{-\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right) \,\middle|\, \mathbf{u} \in \mathbf{C}_1(24)\right\}$ is identical with $\left\{\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right) \,\middle|\, \mathbf{u} \in \mathbf{C}_1(24)\right\}$.

Besides the above six subsets of $\Lambda_6(\mathscr{L}_2)$ there are at least two subsets. One is

$$\Lambda_{6,6}(\mathscr{L}_2) = \bigcup_{\mathbf{u} \in \mathbf{C}_1(12)} Min(\mathbf{u}, \mathscr{L}_2).$$

Another subset of $\Lambda_6(\mathscr{L}_2)$ is

$$\Lambda_{6,7}(\mathscr{L}_2) = \bigcup_{\mathbf{u} \in \mathbf{C}_1(8)} \{\mathbf{x} - \mathbf{y} \mid \mathbf{x} \in Min(\mathbf{u}, \mathscr{L}_2), \mathbf{y} \in \Lambda_{4,1}(\mathscr{L}_2), (\mathbf{x}, \mathbf{y}) = 1\}.$$

As to the cardinality of the set $\Lambda_{6,7}(\mathscr{L}_2)$ we treat it as a lemma.

Lemma 4.4. *Let $\Lambda_{6,7}(\mathscr{L}_2)$ be as above. Then it holds that*

$$|\Lambda_{6,7}(\mathscr{L}_2)| = 1518 \cdot 2^7 \cdot 80 = 15544320.$$

Proof. Let $\mathbf{u} \in \mathbf{C}_1(8)$, $\mathbf{x} = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta \in Min(\mathbf{u}, \mathscr{L}_2)$, and $\mathbf{y} \in \Lambda_{4,1}(\mathscr{L}_2)$ such that $(\mathbf{x}, \mathbf{y}) = 1$. Then $\mathbf{x} - \mathbf{y}$ can be written as

$$\mathbf{x} - \mathbf{y} = \frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \sqrt{2}\delta' \pm \sqrt{2}\sigma,$$

where $\delta'$, $\sigma$ are both $(0,1)$ vectors that satisfy the relations

$$\delta' \subseteq \rho^{\#}(\mathbf{u}), \quad |supp(\delta')| = |supp(\delta)| \pm 1, \quad \sigma \not\subseteq \rho^{\#}(\mathbf{u}), \quad |supp(\sigma)| = 1.$$

For each fixed $\mathbf{u} \in \mathbf{C}_1(8)$ there are $2^7 \cdot 80$ such $\mathbf{x} - \mathbf{y}$'s in number. From this the lemma follows. □

Lemma 4.5. *Let the notations be as above. Then we have*

$$\Lambda_6(\mathscr{L}_2) = \bigcup_{0 \le i \le 7} \Lambda_{6,i}(\mathscr{L}_2).$$

Proof. It is obvious that

$$\Lambda_6(\mathscr{L}_2) \supseteq \bigcup_{0 \le i \le 7} \Lambda_{6,i}(\mathscr{L}_2),$$

and the righthand side is a disjoint union. Therefore we have the inequality

$$|\Lambda_6(\mathscr{L}_2)| \ge \sum_{i=0}^{7} |\Lambda_{6,i}(\mathscr{L}_2)|.$$

By the shape of homogeneous weight enumerator $\mathscr{M}(\mathbf{C}_1)$ we can count $|\Lambda_{6,i}(\mathscr{L}_2)|$, $0 \le i \le 7$. Indeed we have

$$|\Lambda_{6,0}(\mathscr{L}_2)| = 2, \quad |\Lambda_{6,1}(\mathscr{L}_2)| = 2 \cdot 1518, \quad |\Lambda_{6,2}(\mathscr{L}_2)| = 2 \cdot 5152,$$

$$|\Lambda_{6,3}(\mathscr{L}_2)| = 2 \cdot 577599, \quad |\Lambda_{6,4}(\mathscr{L}_2)| = 2 \cdot 3910368, \quad |\Lambda_{6,5}(\mathscr{L}_2)| = 7787940.$$

As to $|\Lambda_{6,6}(\mathscr{L}_2)|$ by reckoning in Remark 1 we compute that $|\Lambda_{6,6}(\mathscr{L}_2)| = 5152 \cdot 2^{11}$.

By Lemma 4.4 we know that $|\Lambda_{6,7}(\mathscr{L}_2)| = 15544320$. In all we have

$$\begin{aligned} |\Lambda_6(\mathscr{L}_2)| &\ge \sum_{i=0}^{7} |\Lambda_{6,i}(\mathscr{L}_2)| \\ &= 42872832. \end{aligned}$$

On the other hand from Eqns. (3.1) and (4.2) we have $|\Lambda_6(\mathscr{L}_2)| = 42872832$. This implies that the lemma should hold. □

REMARK 4.6. The present lattice is connected with the Golay code of length 24. But it is irreducible and is not generated by $\Lambda_4(\mathscr{L}_2)$ only. Actually $\gamma$ can not be expressed as a linear combination of the elements in $\Lambda_4(\mathscr{L}_2)$ with integer coefficients.

**4.3. Third Example.** The lattice constructed from the doubly even binary self-dual code $\mathbf{C}_2 = [48, 24, 12]$. The homogeneous weight enumerator of $\mathbf{C}_2$ is given by

$$\begin{aligned} W_{\mathbf{C}_2}(x, y) = x^{48} &+ 17296x^{36}y^{12} + 535095x^{32}y^{16} + 3995376x^{28}y^{20} \\ &+ 7681680x^{24}y^{24} + \cdots \end{aligned}$$

We employ several notations from the second example. Let

$$\rho : \mathbf{Z}^{48} \to \mathbf{F}_2^{48}$$

be the reduction modulo 2 map. We set

$$\mathscr{M}(\mathbf{C}_2) = \frac{1}{\sqrt{2}} \left\{ \mathbf{x} = (x_1, x_2, \dots, x_{48}) \in \rho^{-1}(\mathbf{C}_2) \,\middle|\, \sum_{i=1}^{48} x_i \equiv 0 \bmod 4 \right\},$$

$\gamma = \frac{1}{\sqrt{8}}(1, \dots, 1) \in \mathbf{R}^{48}$, and

$$\mathscr{L}_3 = \mathscr{M}(\mathbf{C}_2) \cup (\mathbf{Z}\gamma + \mathscr{M}(\mathbf{C}_2)).$$

We desribe $\Lambda_4(\mathscr{L}_3)$ and $\Lambda_6(\mathscr{L}_3)$ briefly.

Since minimal weight of $\mathbf{C}_2$ is 12, there is no norm 4 vector coming from the codeword. We see that

$$\Lambda_4(\mathscr{L}_3) = \{((\pm\sqrt{2})^2, 0^{46})\}.$$

Lemma 4.2 is valid if we replace $\mathscr{L}_3$ instead of $\mathscr{L}_2$. By Lemma 4.2 we obtain

$$\Lambda_{6,1}(\mathscr{L}_3) = \left\{\pm\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right)\middle|\, \mathbf{u} \in \mathbf{C}_2(0)\right\},$$

$$\Lambda_{6,2}(\mathscr{L}_3) = \left\{\pm\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right)\middle|\, \mathbf{u} \in \mathbf{C}_2(12)\right\},$$

$$\Lambda_{6,3}(\mathscr{L}_3) = \left\{\pm\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right)\middle|\, \mathbf{u} \in \mathbf{C}_2(16)\right\},$$

$$\Lambda_{6,4}(\mathscr{L}_3) = \left\{\pm\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right)\middle|\, \mathbf{u} \in \mathbf{C}_2(20)\right\},$$

$$\Lambda_{6,5}(\mathscr{L}_3) = \left\{\left(\frac{1}{\sqrt{2}}\rho^{\#}(\mathbf{u}) - \gamma\right)\middle|\, \mathbf{u} \in \mathbf{C}_2(24)\right\}.$$

Besides the above subsets we have

$$\Lambda_{6,6}(\mathscr{L}_3) = \bigcup_{\mathbf{u} \in \mathbf{C}_2(12)} Min(\mathbf{u}, \mathscr{L}_3).$$

By the shape of homogeneous weight enumerator $\mathscr{M}(\mathbf{C}_1)$ we can count $|\Lambda_{6,i}(\mathscr{L}_2)|$, $0 \le i \le 7$. Indeed we have

$$|\Lambda_{6,1}(\mathscr{L}_3)| = 2, \quad |\Lambda_{6,2}(\mathscr{L}_3)| = 2 \cdot 17296, \quad |\Lambda_{6,3}(\mathscr{L}_3)| = 2 \cdot 535095,$$

$$|\Lambda_{6,4}(\mathscr{L}_3)| = 2 \cdot 3995376, \quad |\Lambda_{6,5}(\mathscr{L}_3)| = 7681680, \quad |\Lambda_{6,6}(\mathscr{L}_3)| = 17296 \cdot 2^{11}.$$

We see that

$$\Lambda_6(\mathscr{L}_3) \supseteq \bigcup_{i=1}^{6} \Lambda_{6,i}(\mathscr{L}_3),$$

and

$$|\Lambda_6(\mathscr{L}_3)| \ge \sum_{i=1}^{6} |\Lambda_{6,i}(\mathscr{L}_3)| = 52199424.$$

Since $a(4, \mathscr{L}_3) = |\Lambda_4(\mathscr{L}_3)| = 4512$ and $a(6, \mathscr{L}_3)$ is obtained from Eqn. (3.1), we have $a(6, \mathscr{L}_3) = 52199424$. Thus we conclude that

$$\Lambda_6(\mathscr{L}_3) = \bigcup_{i=1}^{6} \Lambda_{6,i}(\mathscr{L}_3).$$

**4.4. Fourth Example.** By [16] there are exactly two non-equivalent ternary self-dual codes $Q_{24}$ and $P_{24}$ with minimum distance 9. Both codes lead to the unique even unimodular 24 dimensional extremal lattice, namely Leech lattice. From these two codes we may form three non-equivalent ternary self-dual $[48, 24, 9]$ codes: $Q_{24} \oplus Q_{24}$, $Q_{24} \oplus P_{24}$, $P_{24} \oplus P_{24}$. Since the complete weight enumerator $\mathscr{W}_{Q_{24}}(x, y, z) = \mathscr{W}_{P_{24}}(x, y, z)$ is known ([19]) to be

$$\begin{aligned}\mathscr{W}_{Q_{24}}(x, y, z) = x^{24} + y^{24} + z^{24} &+ 2024(x^{15}y^6z^3 + x^{15}y^3z^6 + x^3y^{15}z^6 \\ &+ x^6y^3z^{15} + x^6y^{15}z^3 + x^3y^6z^{15}) \\ &+ 46(x^{12}y^{12} + x^{12}z^{12} + y^{12}z^{12}) \\ &+ 10120(x^{12}y^9z^3 + x^{12}y^3z^9 + x^9y^{12}z^3 \\ &+ x^9y^3z^{12} + x^3y^{12}z^9 + x^3y^9z^{12}) \\ &+ 111320(x^9y^9z^6 + x^9y^6z^9 + x^6y^9z^9) \\ &+ 41492(x^{12}y^6z^6 + x^6y^{12}z^6 + x^6y^6z^{12}),\end{aligned}$$

we obtain $\mathscr{W}_{Q_{24} \oplus Q_{24}}(x, y, z)$. Here we give some beginning terms.

$$\begin{aligned}\mathscr{W}_{Q_{24} \oplus Q_{24}} = x^{48} &+ 4048(x^{39}y^3z^6 + x^{39}y^6z^3) + 82984x^{36}y^6z^6 \\ &+ 20240(x^{36}y^3z^9 + x^{36}y^9z^3) + 92(x^{36}y^{12} + x^{36}z^{12}) + \cdots\end{aligned}$$

Let $\mathbf{C}_3 = Q_{24} \oplus Q_{24}$ and

$$\phi : \mathbf{Z}^{48} \to \mathbf{F}_3^{48}$$

be the reduction modulo 3 map. We set

$$\mathscr{K}(\mathbf{C}_3) = \frac{1}{\sqrt{3}}\left\{\mathbf{x} = (x_1, x_2, \dots, x_{48}) \in \phi^{-1}(\mathbf{C}_3) \,\middle|\, \sum_{i=1}^{48} x_i \equiv 0 \bmod 6\right\},$$

$v = \frac{1}{\sqrt{12}}(1^{47}, (-5)) \in \mathbf{R}^{48}$, and

$$\mathscr{L}_4 = \mathscr{K}(\mathbf{C}_3) \cup (v + \mathscr{K}(\mathbf{C}_3)).$$

$\mathscr{L}_4$ is an even unimodular nearly extremal lattice. According to [29] from each codeword of weight 9 there arise 9 vectors in $\Lambda_4(\mathscr{L}_4)$ and from each codeword of weight 12 there arises one vector in $\Lambda_4(\mathscr{L}_4)$. Therefore we have

$$|\Lambda_4(\mathscr{L}_4)| = 4048(9 + 9) + 82984 + 20240 \cdot 2 + 92 \cdot 2 = 196512.$$

By Eq. (3.1) we have $a(6, \mathscr{L}_4) = 42983424$. At present it is hard to compute the arithmetical properties of the lattices constructed from the ternary codes.

## 5. Siegel Theta Series of Degree 2

**5.1. Relations among the Fourier Coefficients.** One strong motivation for conceiving the concept of the nearly extremal lattices is to construct explicit lattices which are comparatively tractable in computing the Fourier coefficients of the Siegel theta series.

For $\boldsymbol{a} \in \Lambda_4(\mathscr{L}_{ne})$ we put

$$v_i = |\{\mathbf{x} \in \Lambda_4(\mathscr{L}_{ne}) \mid (\mathbf{x}, \boldsymbol{a}) = i\}|,$$
$$\mu_j = |\{\mathbf{x} \in \Lambda_6(\mathscr{L}_{ne}) \mid (\mathbf{x}, \boldsymbol{a}) = j\}|,$$

where we remark that $i$ can only take the values $i = 0, \pm 1, \pm 2, \pm 4$ and that $j$ can only take the values $j = 0, \pm 1, \pm 2, \pm 3$. This fact can be shown in a similar way to that of Eqns. (3.6) and (3.7). It is easy to see that $v_{-i} = v_i$, $v_{-4} = v_4 = 1$.

For $\boldsymbol{a} \in \Lambda_6(\mathscr{L}_{ne})$ we put

$$\lambda_k = |\{\mathbf{y} \in \Lambda_6(\mathscr{L}_{ne}) \mid (\mathbf{y}, \boldsymbol{a}) = k\}|.$$

We note that $k$ can only take the values $k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6$. We note that $\lambda_{-k} = \lambda_k$, $\lambda_{-6} = \lambda_6 = 1$. We may use the symbol $v_i(\boldsymbol{a})$ instead of $v_i$ to emphasize the role of $\boldsymbol{a}$ at certain occasions.

We specify the Siegel theta series, which is introduced in Section 2.6, to $g = 2$:

$$\Theta_2(Z, L) = \sum_{T \in \hat{\mathscr{P}}_2^s(\mathbf{Z})} a(T, L) e^{2\pi i \sigma(TZ)}.$$

We pick up some peculiar indices $T \in \hat{\mathscr{P}}_2^s$. In Tables 1, 2 and 3

$$T = \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix} \in \hat{\mathscr{P}}_2^s$$

is expressed by $T = (a, b, c)$. We are going to discuss the Fourier coefficients $a(T, \mathscr{L}_{ne})$ of the Siegel theta series $\Theta_2(Z, \mathscr{L}_{ne})$ of degree 2 associated with a lattice

$\mathscr{L}_{ne}$. In the following proposition $T_j$ denotes the numbered binary matrix of size 2 given in Tables 1, 2, 3. We prove

Proposition 5.1. *Let* $\boldsymbol{\alpha} \in \Lambda_4(\mathscr{L}_{ne})$ *and* $t$ *be a positive even integer. Then it holds that*

$$\sum_{\boldsymbol{\alpha} \in \Lambda_4(\mathscr{L}_{ne})} \sum_{\mathbf{x} \in \Lambda_4(\mathscr{L}_{ne})} (\mathbf{x}, \boldsymbol{\alpha})^t = 2 \cdot 4^t a(4, \mathscr{L}_{ne}) + 2 \cdot 2^t a(T_1, \mathscr{L}_{ne}) + 2a(T_2, \mathscr{L}_{ne}). \tag{5.1}$$

*Let* $\boldsymbol{\alpha} \in \Lambda_6(\mathscr{L}_{ne})$ *and* $t$ *be a positive even integer. Then it holds that*

$$\sum_{\boldsymbol{\alpha} \in \Lambda_6(\mathscr{L}_{ne})} \sum_{\mathbf{x} \in \Lambda_4(\mathscr{L}_{ne})} (\mathbf{x}, \boldsymbol{\alpha})^t = 2 \cdot 3^t a(T_2, \mathscr{L}_{ne}) + 2 \cdot 2^t a(T_4, \mathscr{L}_{ne}) + 2a(T_5, \mathscr{L}_{ne}), \tag{5.2}$$

*and*

$$\begin{aligned} \sum_{\boldsymbol{\alpha} \in \Lambda_6(\mathscr{L}_{ne})} \sum_{\mathbf{y} \in \Lambda_6(\mathscr{L}_{ne})} (\mathbf{y}, \boldsymbol{\alpha})^t &= 2 \cdot 6^t a(6, \mathscr{L}_{ne}) + 2 \cdot 4^t a(T_4, \mathscr{L}_{ne}) + 2 \cdot 3^t a(T_7, \mathscr{L}_{ne}) \\ &\quad + 2 \cdot 2^t a(T_8, \mathscr{L}_{ne}) + 2a(T_9, \mathscr{L}_{ne}). \end{aligned} \tag{5.3}$$

Proof. Proof of Eqn. (5.1). Let $\boldsymbol{\alpha} \in \Lambda_4(\mathscr{L}_{ne})$. Then we have

$$\sum_{\mathbf{x} \in \Lambda_4(\mathscr{L}_{ne})} (\mathbf{x}, \boldsymbol{\alpha})^t = 2 \cdot 4^t + 2 \cdot 2^t v_2 + 2v_1.$$

From this we obtain

$$\begin{aligned} \sum_{\boldsymbol{\alpha} \in \Lambda_4(\mathscr{L}_{ne})} \sum_{\mathbf{x} \in \Lambda_4(\mathscr{L}_{ne})} (\mathbf{x}, \boldsymbol{\alpha})^t &= \sum_{\boldsymbol{\alpha} \in \Lambda_4(\mathscr{L}_{ne})} (2 \cdot 4^t + 2 \cdot 2^t v_2 + 2v_1) \\ &= 2 \cdot 4^t a(4, \mathscr{L}_{ne}) + 2 \cdot 2^t \sum_{\boldsymbol{\alpha} \in \Lambda_4(\mathscr{L}_{ne})} \sum_{\mathbf{x} \in \Lambda_4(\mathscr{L}_{ne}),\, (\mathbf{x}, \boldsymbol{\alpha}) = 2} 1 \\ &\quad + 2 \sum_{\boldsymbol{\alpha} \in \Lambda_4(\mathscr{L}_{ne})} \sum_{\mathbf{x} \in \Lambda_4(\mathscr{L}_{ne}),\, (\mathbf{x}, \boldsymbol{\alpha}) = 1} 1 \\ &= 2 \cdot 4^t a(4, \mathscr{L}_{ne}) + 2 \cdot 2^t a(T_1, \mathscr{L}_{ne}) + 2a(T_2, \mathscr{L}_{ne}). \end{aligned}$$

Eqns. (5.2) and (5.3) are similarly proved, but in the course of transformation we must use Proposition 2.2. $\square$

The following propositions give the relations among the values of $a(T_j, \mathscr{L}_{ne})$, $1 \le j \le 10$. To save the space we use $r_j$ instead of $a(T_j, \mathscr{L}_{ne})$. Further we use $\Sigma_1(t) = 2 \cdot 4^t a(4, \mathscr{L}_{ne}) + 2^{t+1} r_1 + 2r_2$, $\Sigma_2(t) = 2 \cdot 3^t r_2 + 2 \cdot 2^t r_4 + 2r_5$ and $\Sigma_3(t) = 2 \cdot 6^t a(6, \mathscr{L}_{ne}) + 2 \cdot 4^t r_4 + 2 \cdot 3^t r_7 + 2 \cdot 2^t r_8 + 2r_9$ for the short notations.

Proposition 5.2. *We have*

(i) $\Sigma_1(2) = \dfrac{1}{3} a(4, \mathscr{L}_{ne})^2$,

(ii) $\Sigma_2(2) = \dfrac{1}{2} a(4, \mathscr{L}_{ne}) a(6, \mathscr{L}_{ne})$,

(iii)
$$\Sigma_2(4) - \frac{6 \cdot 4 \cdot 6}{52} \Sigma_2(2) + \frac{3 \cdot 4^2 \cdot 6^2}{52 \cdot 50} a(4, \mathscr{L}_{ne}) a(6, \mathscr{L}_{ne}) = 192 \left\{ \Sigma_1(4) - \frac{6 \cdot 4 \cdot 4}{52} \Sigma_1(2) + \frac{3 \cdot 4^4}{52 \cdot 50} a(4, \mathscr{L}_{ne})^2 \right\},$$

(iv)
$$\Sigma_2(6) - \frac{15 \cdot 6 \cdot 4}{56} \Sigma_2(4) + \frac{45 \cdot 4^2 \cdot 6^2}{56 \cdot 54} \Sigma_2(2) - \frac{15 \cdot 6^3 \cdot 4^3}{56 \cdot 54 \cdot 52} a(4, \mathscr{L}_{ne}) a(6, \mathscr{L}_{ne})$$
$$= -552 \left\{ \Sigma_1(6) - \frac{15 \cdot 4^2}{56} \Sigma_1(4) + \frac{45 \cdot 4^4}{56 \cdot 54} \Sigma_1(2) - \frac{15 \cdot 4^6}{56 \cdot 54 \cdot 52} a(4, \mathscr{L}_{ne})^2 \right\},$$

(v)
$$\Sigma_2(8) - \frac{28 \cdot 6 \cdot 4}{60} \Sigma_2(6) + \frac{210 \cdot 6^2 \cdot 4^2}{60 \cdot 58} \Sigma_2(4) - \frac{420 \cdot 6^3 \cdot 4^3}{60 \cdot 58 \cdot 56} \Sigma_2(2) + \frac{105 \cdot 6^4 \cdot 4^4}{60 \cdot 58 \cdot 56 \cdot 54} a(4, \mathscr{L}_{ne}) a(6, \mathscr{L}_{ne})$$
$$= 432 \left\{ \Sigma_1(8) - \frac{28 \cdot 4^2}{56} \Sigma_1(6) + \frac{210 \cdot 4^4}{60 \cdot 58} \Sigma_1(4) - \frac{420 \cdot 4^6}{60 \cdot 58 \cdot 56} \Sigma_1(2) + \frac{105 \cdot 4^8}{60 \cdot 58 \cdot 56 \cdot 54} a(4, \mathscr{L}_{ne})^2 \right\}.$$

Proof. Proof of the equation (i). We start from Eqn. (3.4) with $\boldsymbol{\alpha} \in \Lambda_4$:

$$\sum\nolimits_{\mathbf{x} \in \Lambda_4} (\mathbf{x}, \boldsymbol{\alpha})^2 = \frac{4}{48} a(4, \mathscr{L}_{ne}) (\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \frac{1}{3} a(4, \mathscr{L}_{ne}).$$

From this and by Eqn. (5.1) we have

$$\sum_{\boldsymbol{\alpha} \in \Lambda_4} \sum_{\mathbf{x} \in \Lambda_4} (\mathbf{x}, \boldsymbol{\alpha})^2 = 2 \cdot 4^2 a(4, \mathscr{L}_{ne}) + 2^3 a(T_1, \mathscr{L}_{ne}) + 2a(T_2, \mathscr{L}_{ne})$$

$$= \sum_{\boldsymbol{\alpha} \in \Lambda_4} \left( \frac{1}{3} a(4, \mathscr{L}_{ne}) \right)$$

$$= \frac{1}{3} a(4, \mathscr{L}_{ne})^2.$$

The proof of the equation (ii) is similar to that of the equation (i) and we omit it.

The proof of the equation (iii). We begin with Eqn. (i) in Proposition 3.4:

$$\sum\nolimits_{\mathbf{y} \in \Lambda_6} \left[ (\mathbf{y}, \boldsymbol{\alpha})^4 - \frac{6 \cdot 6(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{52} (\mathbf{y}, \boldsymbol{\alpha})^2 + \frac{3 \cdot 6^2 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^2}{52 \cdot 50} \right]$$

$$= 192 \sum\nolimits_{\mathbf{x} \in \Lambda_4} \left[ (\mathbf{x}, \boldsymbol{\alpha})^4 - \frac{6 \cdot 4(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{52} (\mathbf{x}, \boldsymbol{\alpha})^2 + \frac{3 \cdot 4^2 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^2}{52 \cdot 50} \right].$$

We take a sum over $\boldsymbol{\alpha} \in \Lambda_4$ at the both sides in the above equation,

$$(*) \quad \sum\nolimits_{\boldsymbol{\alpha} \in \Lambda_4} \sum\nolimits_{\mathbf{y} \in \Lambda_6} \left[ (\mathbf{y}, \boldsymbol{\alpha})^4 - \frac{6 \cdot 6(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{52} (\mathbf{y}, \boldsymbol{\alpha})^2 + \frac{3 \cdot 6^2 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^2}{52 \cdot 50} \right]$$

$$= 192 \sum\nolimits_{\boldsymbol{\alpha} \in \Lambda_4} \sum\nolimits_{\mathbf{x} \in \Lambda_4} \left[ (\mathbf{x}, \boldsymbol{\alpha})^4 - \frac{6 \cdot 4(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{52} (\mathbf{x}, \boldsymbol{\alpha})^2 + \frac{3 \cdot 4^2 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^2}{52 \cdot 50} \right].$$

By changing the role of $\boldsymbol{\alpha}$ and $\mathbf{y}$ in Eqn. (5.2) the left-hand side of the equation $(*)$ can be written as

$$2 \cdot 3^4 a(T_2, \mathscr{L}_{ne}) + 2 \cdot 2^4 a(T_4, \mathscr{L}_{ne}) + 2a(T_5, \mathscr{L}_{ne})$$

$$- \frac{6^2 \cdot 4}{52} \{2 \cdot 3^2 a(T_2, \mathscr{L}_{ne}) + 2 \cdot 2^2 a(T_4, \mathscr{L}_{ne}) + 2a(T_5, \mathscr{L}_{ne})\}$$

$$+ \frac{3 \cdot 6^2 \cdot 4^2}{52 \cdot 50} a(4, \mathscr{L}_{ne}) a(6, \mathscr{L}_{ne})$$

$$= \Sigma_2(4) - \frac{6^2 \cdot 4}{52} \Sigma_2(2) + \frac{3 \cdot 6^2 \cdot 4^2}{52 \cdot 50} a(4, \mathscr{L}_{ne}) a(6, \mathscr{L}_{ne}).$$

By using Eqn. (5.1) many times the right-hand side of the equation $(*)$ is transformed into

$$192\Big\{(2\cdot 4^4 a(4,\mathscr{L}_{ne}) + 2\cdot 2^4 a(T_1,\mathscr{L}_{ne}) + 2a(T_2,\mathscr{L}_{ne}))$$

$$-\frac{6^2\cdot 4}{52}(2\cdot 4^2 a(4,\mathscr{L}_{ne}) + 2\cdot 2^2 a(T_1,\mathscr{L}_{ne}) + 2a(T_2,\mathscr{L}_{ne}))$$

$$+\frac{3\cdot 4^2\cdot 4^2}{52\cdot 50}a(4,\mathscr{L}_{ne})^2\Big\}$$

$$= 192\left\{\Sigma_1(4) - \frac{6\cdot 4\cdot 4}{52}\Sigma_1(2) + \frac{3\cdot 4^4}{52\cdot 50}a(4,\mathscr{L}_{ne})^2\right\}.$$

Thus the equation (iii) is proved. The proofs of the equations (iv) and (v) are similar to that of (iii), and we omit them. □

The above equations (i) to (v) are linear conditions on $r_1$, $r_2$, $r_4$, $r_5$ and we have basic relations.

Proposition 5.3.

$$r_2 = -16\cdot a(4,\mathscr{L}_{ne}) - 4\cdot a(T_1,\mathscr{L}_{ne}) + \frac{1}{6}a(4,\mathscr{L}_{ne})^2$$

$$r_4 = 484416\cdot a(4,\mathscr{L}_{ne}) + 216\cdot a(T_1,\mathscr{L}_{ne}) - \frac{4}{3}a(4,\mathscr{L}_{ne})^2$$

$$r_5 = 11166480\cdot a(4,\mathscr{L}_{ne}) - 828\cdot a(T_1,\mathscr{L}_{ne}) - \frac{49}{6}a(4,\mathscr{L}_{ne})^2.$$

Proposition 5.4. *We have*

(i) $\Sigma_3(2) = \frac{3}{4}a(6,\mathscr{L}_{ne})^2$,

(ii) $$\Sigma_3(4) - \frac{6^3}{52}\Sigma_3(2) + \frac{3\cdot 6^4}{52\cdot 50}a(6,\mathscr{L}_{ne})^2$$

$$= 192\left\{\Sigma_2(4) - \frac{6\cdot 4\cdot 6}{52}\Sigma_2(2) + \frac{3\cdot 4^2\cdot 6^2}{52\cdot 50}a(4,\mathscr{L}_{ne})a(6,\mathscr{L}_{ne})\right\},$$

(iii) $$\Sigma_3(6) - \frac{15\cdot 6^2}{56}\Sigma_3(4) + \frac{45\cdot 6^4}{56\cdot 54}\Sigma_3(2) - \frac{15\cdot 6^6}{56\cdot 54\cdot 52}a(6,\mathscr{L}_{ne})^2$$

$$= -552\Big\{\Sigma_2(6) - \frac{15\cdot 4\cdot 6}{56}\Sigma_2(4) + \frac{45\cdot 4^2\cdot 6^2}{56\cdot 54}\Sigma_2(2)$$

$$-\frac{15\cdot 4^3\cdot 6^3}{56\cdot 54\cdot 52}a(4,\mathscr{L}_{ne})a(6,\mathscr{L}_{ne})\Big\}.$$

PROOF. Proof of (i). Again we start from Eqn. (3.4) with $\boldsymbol{\alpha} \in \Lambda_6$:

$$\sum\nolimits_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{\alpha})^2 = \frac{6}{48} a(6, \mathscr{L}_{ne})(\boldsymbol{\alpha}, \boldsymbol{\alpha})$$
$$= \frac{3}{4} a(6, \mathscr{L}_{ne}).$$

By this equation and Eqn. (5.3) we have

$$\sum_{\boldsymbol{\alpha} \in \Lambda_6} \sum_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{\alpha})^2 = 2 \cdot 6^t a(6, \mathscr{L}_{ne}) + 2 \cdot 4^t a(T_4, \mathscr{L}_{ne}) + 2 \cdot 3^t a(T_7, \mathscr{L}_{ne})$$
$$+ 2 \cdot 2^t a(T_8, \mathscr{L}_{ne}) + 2a(T_9, \mathscr{L}_{ne})$$
$$= \sum_{\boldsymbol{\alpha} \in \Lambda_6} \left( \frac{3}{4} a(6, \mathscr{L}_{ne}) \right)$$
$$= \frac{3}{4} a(6, \mathscr{L}_{ne})^2.$$

Proof of (ii). The starting point is the same as that of (iii) in Proposition 5.2.

$$\sum\nolimits_{\mathbf{y} \in \Lambda_6} \left[ (\mathbf{y}, \boldsymbol{\alpha})^4 - \frac{6 \cdot 6(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{52} (\mathbf{y}, \boldsymbol{\alpha})^2 + \frac{3 \cdot 6^2 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^2}{52 \cdot 50} \right]$$
$$= 192 \sum\nolimits_{\mathbf{x} \in \Lambda_4} \left[ (\mathbf{x}, \boldsymbol{\alpha})^4 - \frac{6 \cdot 4(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{52} (\mathbf{x}, \boldsymbol{\alpha})^2 + \frac{3 \cdot 4^2 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^2}{52 \cdot 50} \right].$$

But this time we take a sum over $\boldsymbol{\alpha} \in \Lambda_6$ at the both sides in the above equation,

$$\sum\nolimits_{\boldsymbol{\alpha} \in \Lambda_6} \sum\nolimits_{\mathbf{y} \in \Lambda_6} \left[ (\mathbf{y}, \boldsymbol{\alpha})^4 - \frac{6 \cdot 6(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{52} (\mathbf{y}, \boldsymbol{\alpha})^2 + \frac{3 \cdot 6^2 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^2}{52 \cdot 50} \right]$$
$$= 192 \sum\nolimits_{\boldsymbol{\alpha} \in \Lambda_6} \sum\nolimits_{\mathbf{x} \in \Lambda_4} \left[ (\mathbf{x}, \boldsymbol{\alpha})^4 - \frac{6 \cdot 4(\boldsymbol{\alpha}, \boldsymbol{\alpha})}{52} (\mathbf{x}, \boldsymbol{\alpha})^2 + \frac{3 \cdot 4^2 \cdot (\boldsymbol{\alpha}, \boldsymbol{\alpha})^2}{52 \cdot 50} \right].$$

The disposition of the rest is similar to that of (iii) in Proposition 5.2 and we omit it.

We also omit the proof of (iii). □

As a consequence of Proposition 5.4 we obtain

PROPOSITION 5.5. *Let $a(T_j, \mathscr{L}_{ne})$, $7 \leq j \leq 9$ be the quantities considered above. Then we have the following expressions for them.*

$$r_7 = 1931424768000 - 4721280 \cdot a(4, \mathscr{L}_{ne}) + 11232 \cdot a(T_1, \mathscr{L}_{ne}) - 76 \cdot a(4, \mathscr{L}_{ne})^2$$

$$r_8 = 88000540992000 - 163026000 \cdot a(4, \mathscr{L}_{ne}) - 34848 \cdot a(T_1, \mathscr{L}_{ne}) + \frac{1760}{3} a(4, \mathscr{L}_{ne})^2$$

$$r_9 = 660902022144000 - 1200129408 \cdot a(4, \mathscr{L}_{ne}) + 34848 \cdot a(T_1, \mathscr{L}_{ne}) - \frac{2332}{3} a(4, \mathscr{L}_{ne})^2.$$

As to $a(T_3, \mathscr{L}_{ne})$, $a(T_6, \mathscr{L}_{ne})$, $a(T_{10}, \mathscr{L}_{ne})$ we have

PROPOSITION 5.6. *It holds that*

$$2a(T_1, \mathscr{L}_{ne}) + 2a(T_2, \mathscr{L}_{ne}) + a(T_3, \mathscr{L}_{ne}) + 2a(4, \mathscr{L}_{ne}) = a(4, \mathscr{L}_{ne})^2,$$

$$2a(T_2, \mathscr{L}_{ne}) + 2a(T_4, \mathscr{L}_{ne}) + 2a(T_5, \mathscr{L}_{ne}) + a(T_6, \mathscr{L}_{ne}) = a(4, \mathscr{L}_{ne})a(6, \mathscr{L}_{ne}),$$

$$2a(6, \mathscr{L}_{ne}) + 2a(T_4, \mathscr{L}_{ne}) + 2a(T_7, \mathscr{L}_{ne}) + 2a(T_8, \mathscr{L}_{ne}) + 2a(T_9, \mathscr{L}_{ne}) + a(T_{10}, \mathscr{L}_{ne}) = a(6, \mathscr{L}_{ne})^2.$$

PROOF. We use an equation

$$2v_4(\boldsymbol{\alpha}) + 2v_2(\boldsymbol{\alpha}) + 2v_1(\boldsymbol{\alpha}) + v_0(\boldsymbol{\alpha}) = |\Lambda_4(\mathscr{L}_{ne})|,$$

which counts the number of norm 4 vectors in $\mathscr{L}_{ne}$ in the both sides. $\boldsymbol{\alpha}$ is any element of $\Lambda_4(\mathscr{L}_{ne})$. From this we have

$$\sum_{\boldsymbol{\alpha} \in \Lambda_4(\mathscr{L}_{ne})} 2v_4(\boldsymbol{\alpha}) + 2v_2(\boldsymbol{\alpha}) + 2v_1(\boldsymbol{\alpha}) + v_0(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\alpha} \in \Lambda_4(\mathscr{L}_{ne})} a(4, \mathscr{L}_{ne}).$$

The lefthand side of the equation is

$$2a(4, \mathscr{L}_{ne}) + 2a(T_1, \mathscr{L}_{ne}) + 2a(T_2, \mathscr{L}_{ne}) + a(T_3, \mathscr{L}_{ne}),$$

and the righthand side is $a(4, \mathscr{L}_{ne})^2$. Thus we showed the first equation.

The last two equations are showed likewisely using the equalities

$$2\mu_3(\boldsymbol{a}) + 2\mu_2(\boldsymbol{a}) + 2\mu_1(\boldsymbol{a}) + \mu_0(\boldsymbol{a}) = |\Lambda_4(\mathscr{L}_{ne})|, \quad \boldsymbol{a} \in \Lambda_4(\mathscr{L}_{ne}),$$

$$2\lambda_6(\boldsymbol{a}) + 2\lambda_4(\boldsymbol{a}) + 2\lambda_3(\boldsymbol{a}) + 2\lambda_2(\boldsymbol{a}) + 2\lambda_1(\boldsymbol{a}) + \lambda_0(\boldsymbol{a})$$
$$= |\Lambda_6(\mathscr{L}_{ne})|, \quad \boldsymbol{a} \in \Lambda_6(\mathscr{L}_{ne}). \qquad \square$$

PROPOSITION 5.7.

$$a(T_3, \mathscr{L}_{ne}) = 6a(T_1, \mathscr{L}_{ne}) + 30a(4, \mathscr{L}_{ne}) + \frac{2}{3}a(4, \mathscr{L}_{ne})^2,$$

$$a(T_6, \mathscr{L}_{ne}) = 1232a(T_1, \mathscr{L}_{ne}) + 29114240a(4, \mathscr{L}_{ne}) - \frac{88}{3}a(4, \mathscr{L}_{ne})^2,$$

$$a(T_{10}, \mathscr{L}_{ne}) = -22896a(T_1, \mathscr{L}_{ne}) - 2297151456a(4, \mathscr{L}_{ne}) + 2840a(4, \mathscr{L}_{ne})^2$$
$$+ 1245769080192000.$$

**5.2. Some Results on the Siegel Theta Series.** According to Igusa [10] the graded ring of the Siegel modular forms of even weights is generated by the four algebraiclly independent modular forms: $\mathscr{E}_4(Z)$ Siegel Eisenstein series of degree 2 and weight 4, $\mathscr{E}_6(Z)$ Siegel Eisenstein series of degree 2 and weight 6, $\chi_{10}(Z)$ the Siegel cusp form of degree 2 and weight 10, and $\chi_{12}(Z)$ the cusp form of degree 2 and weight 12.

Let $M(2,k)$, $k \equiv 0 \pmod 2$ be the linear space of the Siegel modular forms of degree $g = 2$ and even weights $k$. Then by [10] we know that $\dim_{\mathbf{C}} M(2,24) = 8$ and $M(2,24)$ has a linear basis

$$M(2,24) = [\mathscr{E}_4(Z)^6, \mathscr{E}_4(Z)^3\mathscr{E}_6(Z)^2, \mathscr{E}_6(Z)^4, \chi_{10}(Z)\mathscr{E}_4(Z)^2\mathscr{E}_6(Z), \chi_{12}(Z)\mathscr{E}_4(Z)^3,$$
$$\chi_{12}\mathscr{E}_6(Z)^2, \chi_{10}(Z)^2\mathscr{E}_4(Z), \chi_{12}(Z)^2].$$

We reconstruct another basis of $M(2,24)$ which is easy to treat. We put

$$\psi_1(Z) = \Theta_2(Z, E_8^6), \qquad \psi_2(Z) = \Theta_2(Z, \mathscr{L}_{48}), \qquad \psi_3(Z) = \Theta_2(Z, \mathscr{L}_1),$$

$$\psi_4(Z) = \chi_{10}(Z)\mathscr{E}_4(Z)^2\mathscr{E}_6(Z), \quad \psi_5(Z) = \chi_{12}(Z)\mathscr{E}_4(Z)^3, \quad \psi_6(Z) = \chi_{12}(Z)\mathscr{E}_6(Z)^2,$$

$$\psi_7(Z) = \chi_{10}(Z)^2\mathscr{E}_4(Z), \qquad \psi_8(Z) = \chi_{12}(Z)^2.$$

Here the lattice $E_8$ is the unique 8-dimensional even unimodular lattice and $E_8^6$ is an orthogonal sum of six copies of $E_8$, $\mathscr{L}_{48}$ is any one of 48-dimensional even unimodular extremal lattices, some of which are recorded in Nebe's table [22] and the Fourier coefficients of the theta series of degree 2 associated with $\mathscr{L}_{48}$ are computed in [32]. The lattice $\mathscr{L}_1$ is described in Section 4. In [31] the Fourier coefficients for the Siegel theta series associated with the Leech lattice of degree two are given. Using this data we can compute the Fourier coefficients of $\Theta_2(Z, \mathscr{L}_1)$. In the process of computation we utilized the computer algebraic system [1].

We have

Lemma 5.8. *The series $\psi_1(Z), \psi_2(Z), \dots, \psi_8(Z)$ given above are linearly independent over* $\mathbf{C}$ *and therefore they are another basis of $M(2, 24)$.*

Proof. We consider the following linear combinations of $\psi_1(Z), \dots,$ $\psi_8(Z)$

$$\Psi_1(Z) = \psi_2(Z),$$

$$\Psi_2(Z) = \frac{\psi_1(Z)}{1440} + \frac{\psi_2(Z)}{1170} - \frac{29\psi_3(Z)}{18720} - 612288\psi_7(Z) + 3085152\psi_8(Z) + \frac{383\psi_4(Z)}{6} - \frac{73\psi_5(Z)}{12} - \frac{455\psi_6(Z)}{4},$$

$$\Psi_3(Z) = \frac{\psi_3(Z)}{393120} - \frac{\psi_2(Z)}{393120} - \frac{5612\psi_7(Z)}{3} - \frac{8188\psi_8(Z)}{3},$$

$$\Psi_4(Z) = \frac{5\psi_4(Z)}{6} + \frac{\psi_5(Z)}{24} + \frac{\psi_6(Z)}{8} - 61488\psi_7(Z) - 8832\psi_8(Z),$$

$$\Psi_5(Z) = \frac{7\psi_5(Z)}{216} - \frac{\psi_4(Z)}{12} + \frac{11\psi_6(Z)}{216} + 5616\psi_7(Z) - 4928\psi_8(Z),$$

$$\Psi_6(Z) = \frac{\psi_6(Z)}{1728} - \frac{\psi_5(Z)}{1728} - \frac{243\psi_7(Z)}{8} + \frac{227\psi_8(Z)}{8},$$

$$\Psi_7(Z) = \frac{5\psi_7(Z)}{6} + \frac{\psi_8(Z)}{6},$$

$$\Psi_8(Z) = \frac{\psi_8(Z)}{24} - \frac{\psi_7(Z)}{24}.$$

Some of the Fourier coefficients of the above obtained forms are given by

| $\Psi_i \backslash index$ | $S_0$ | $S_1$ | $S_2$ | $S_4$ | $S_5$ | $S_6$ | $T_1$ | $T_2$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | |
| 2 | | 1 | | | | | | |
| 3 | | | 1 | | | | | |
| 4 | | | | 1 | | | | |
| 5 | | | | | 1 | | | |
| 6 | | | | | | 1 | | |
| 7 | | | | | | | 1 | |
| 8 | | | | | | | | 1 |

For instance the Fourier coefficient of $\Psi_1(Z)$ at the index $S_0 = (0,0,0)$ is 1. The blanks are all zeros. By this data it is appearent that $\Psi_1(Z),\dots,\Psi_8(Z)$ are linearly independent. Since these forms $\Psi_1(Z),\dots,\Psi_8(Z)$ are all linear combinations of $\psi_1(Z),\psi_2(Z),\dots,\psi_8(Z)$. Therefore $\psi_1(Z),\psi_2(Z),\dots,\psi_8(Z)$ are also linearly independent. □

REMARK 5.9. By the arguement in Lemma 5.8 $\Psi_1(Z),\Psi_2(Z),\dots,\Psi_8(Z)$ is also a basis for $M(2,24)$. It is sufficient for the determination of an element $f(Z)$ in $M(2,24)$ to know the values of the Fourier coefficients of $f(Z)$ at the indices $S_0$, $S_1$, $S_2$, $S_4$, $S_5$, $S_6$, $T_1$, $T_2$.

THEOREM 5.10. *The Siegel theta series of degree* 2 *associated with an even unimodular* 48-*dimensional nearly extremal lattice* $\mathscr{L}_{ne}$ *is determined completely by the coefficients* $a(4,\mathscr{L}_{ne})$ *and* $a(T_1,\mathscr{L}_{ne})$.

PROOF. By Lemma 5.8 and Remark 5.9 we have only to show that the Fourier coefficients $a(T,\mathscr{L}_{ne})$ for $T$, which belong to seven indices $S_0$, $S_1$, $S_2$, $S_4$, $S_5$, $S_6$, $T_1$, $T_2$, are determined by the data $a(4,\mathscr{L}_{ne})$ and $a(T_1,\mathscr{L}_{ne})$.

Since $\mathscr{L}_{ne}$ is nearly extremal, it obviously holds that $a(S_0,\mathscr{L}_{ne}) = 1$ and $a(T,\mathscr{L}_{ne}) = 0$ for $T = S_1,S_2,S_4,S_5,S_6$. By Proposition 5.3 $r_2 = a(T_2,\mathscr{L}_{ne})$ is determined by $a(4,\mathscr{L}_{ne})$ and $a(T_1,\mathscr{L}_{ne})$. This shows that Theorem is true. □

PROPOSITION 5.11. *Let* $M_{ne}(2,24)$ *be the linear subspace of the Siegel modular forms of weight* 24 *and degree* 2 *spanned by the Siegel theta series of degree* 2 *associated with the even unimodular nearly extremal lattices of dimension* 48. *Then we have*

$$3 \le \dim_{\mathbf{C}} M_{ne}(2,24) \le 4.$$

PROOF. Any $\Theta_2(Z, \mathscr{L}_{ne})$ can be expressed as a linear combination of $\Psi_1(Z)$, $\Psi_3(Z)$, $\Psi_7(Z)$ and $\Psi_8(Z)$, and therefore we have $\dim_{\mathbf{C}} M_{ne}(2,24) \leq 4$. We take three theta series of degree 2 $\Theta_2(Z, \mathscr{L}_1)$, $\Theta_2(Z, \mathscr{L}_2)$, $\Theta_2(Z, \mathscr{L}_3)$. We show that these three theta series are linearly independent. To show this we begin with the expressions

$$\Theta_2(Z, \mathscr{L}_1) = \Psi_1(Z) + a(S_2, \mathscr{L}_1) \cdot \Psi_3(Z) + a(T_1, \mathscr{L}_1) \cdot \Psi_7(Z) + a(T_2, \mathscr{L}_1) \cdot \Psi_8(Z),$$

$$\Theta_2(Z, \mathscr{L}_2) = \Psi_1(Z) + a(S_2, \mathscr{L}_2) \cdot \Psi_3(Z) + a(T_1, \mathscr{L}_2) \cdot \Psi_7(Z) + a(T_2, \mathscr{L}_2) \cdot \Psi_8(Z),$$

$$\Theta_2(Z, \mathscr{L}_3) = \Psi_1(Z) + a(S_2, \mathscr{L}_3) \cdot \Psi_3(Z) + a(T_1, \mathscr{L}_3) \cdot \Psi_7(Z) + a(T_2, \mathscr{L}_3) \cdot \Psi_8(Z).$$

Using explicit values of the Fourier coefficients we observe that

$$\Theta_2(Z, \mathscr{L}_3) = \Psi_1(Z) + 9512\Psi_3(Z) + 830208\Psi_7(Z), \tag{5.4}$$

$$\Theta_2(Z, \mathscr{L}_2) - \Theta_2(Z, \mathscr{L}_3) = 19304\Psi_3(Z) + 451562496\Psi_7(Z) + 4775215104\Psi_8(Z), \tag{5.5}$$

$$\Theta_2(Z, \mathscr{L}_1) - 2\Theta_2(Z, \mathscr{L}_2) + \Theta_2(Z, \mathscr{L}_3) = 904396800\Psi_7(Z) + 8967094272\Psi_8(Z), \tag{5.6}$$

are linearly independent over $\mathbf{C}$, and consequently $\Theta_2(Z, \mathscr{L}_1)$, $\Theta_2(Z, \mathscr{L}_2)$, $\Theta_2(Z, \mathscr{L}_3)$ are linearly independent. Thus we have $3 \leq \dim_{\mathbf{C}} M_{ne}(2,24)$. $\square$

We find a (possible) relationship between $\Theta_2(Z, \mathscr{L}_{48})$ and $M_{ne}(2,24)$.

PROPOSITION 5.12. *We have*

$$\dim_{\mathbf{C}} M_{ne}(2,24) = 4 \Leftrightarrow \Theta_2(Z, \mathscr{L}_{48}) \in M_{ne}(2,24).$$

PROOF. Suppose that $\dim_{\mathbf{C}} M_{ne}(2,24) = 4$ holds. Then $M_{ne}(2,24)$ contains $\Psi_1(Z)$, $\Psi_3(Z)$, $\Psi_7(Z)$ and $\Psi_8(Z)$. By computation we verify that

$$\Theta_2(Z, \mathscr{L}_2) = \Theta_2(Z, \mathscr{L}_{48}) + 198816\Psi_3(Z) + 452392704\Psi_7(Z) + 4775215104\Psi_8(Z). \tag{5.7}$$

Thus $\Theta_2(Z, \mathscr{L}_{48}) \in M_{ne}(2,24)$.

Conversely suppose that $\Theta_2(Z, \mathscr{L}_{48}) \in M_{ne}(2,24)$ holds. Then by Eqn. (5.5) we must have

Table 1. Fourier coefficients of Siegel theta series of degree 2 for the three 48-dimensional lattices.

| $D$ | red. form | $E_8^6$ | $\mathscr{L}_{48}$ | $D_{48}^+$ |
|---|---|---|---|---|
| 0 | $S_0 = (0,0,0)$ | 1 | 1 | 1 |
| 0 | $S_1 = (1,0,0)$ | 1440 | 0 | 4512 |
| 0 | $S_2 = (2,0,0)$ | 876960 | 0 | 3113376 |
| 0 | $S_3 = (3,0,0)$ | 292072320 | 52416000 | 785791872 |
| 3 | $S_4 = (1,1,1)$ | 80640 | 0 | 830208 |
| 4 | $S_5 = (1,1,0)$ | 1909440 | 0 | 18688704 |
| 7 | $S_6 = (1,2,1)$ | 97597440 | 0 | 1095874560 |
| 8 | $S_7 = (1,2,0)$ | 1063808640 | 0 | 11818425984 |
| 11 | $S_8 = (1,3,1)$ | 48317057280 | 0 | 395905440000 |
| 12 | $^*S_9 = (1,3,0)$ | 321822247680 | 0 | 2730043534080 |
| 12 | $T_1 = (2,2,2)$ | 5025196800 | 0 | 70721268480 |
| 15 | $T_2 = (2,2,1)$ | 107183278080 | 0 | 1322720593920 |
| 16 | $^*T_3 = (2,2,0)$ | 544444943040 | 0 | 6904028416704 |
| 20 | $T_4 = (2,3,2)$ | 3827907590400 | 0 | 40673240668416 |
| 23 | $T_5 = (2,3,1)$ | 47740634634240 | 0 | 436829845985280 |
| 24 | $T_6 = (2,3,0)$ | 152782163124480 | 0 | 1488790303932672 |
| 27 | $^*T_7 = (3,3,3)$ | 154205041501440 | 1931424768000 | 1448517720814848 |
| 32 | $T_8 = (3,3,2)$ | 2847471563157120 | 88000540992000 | 21294667534098048 |
| 35 | $T_9 = (3,3,1)$ | 19149643452695040 | 660902022144000 | 132674797389046272 |
| 36 | $^*T_{10} = (3,3,0)$ | 40995846962035200 | 1245768975360000 | 306550760949545472 |

$$\Theta_2(Z,\mathscr{L}_2) - \Theta_2(Z,\mathscr{L}_3) = 19304\Psi_3(Z) + 451562496\Psi_7(Z) + 4775215104\Psi_8(Z) \in M_{ne}(2,24).$$

From Eqn. (5.6) we have that

$$904396800\Psi_7(Z) + 8967094272\Psi_8(Z) \in M_{ne}(2,24).$$

And by Eqn. (5.7) we have

$$198816\Psi_3(Z) + 452392704\Psi_7(Z) + 4775215104\Psi_8(Z) \in M_{ne}(2,24).$$

We consider that the square matrix of degree 3 formed by

$$\begin{pmatrix} 19304 & 451562496 & 4775215104 \\ 0 & 904396800 & 8967094272 \\ 198816 & 452392704 & 4775215104 \end{pmatrix}.$$

It is easy to verify that the determinant of this matrix is not zero, and this implies that the space $M_{ne}(2,24)$ contains $\Psi_1(Z)$, $\Psi_3(Z)$, $\Psi_7(Z)$. Finally from Eqn. (5.4) $\Psi_1(Z) \in M_{ne}(2,24)$, and we conclude that $\dim_{\mathbf{C}} M_{ne}(2,24) = 4$. $\square$

Table 2. Table of Siegel cusp forms of degree 2 and weight 24.

| $D$ | red. form | $\chi_{10}\mathscr{E}_4^2\mathscr{E}_6$ | $\chi_{12}\mathscr{E}_4^3$ | $\chi_{12}\mathscr{E}_6^2$ | $\chi_{10}^2\mathscr{E}_4$ | $\chi_{12}^2$ |
|---|---|---|---|---|---|---|
| 0 | $S_0 = (0,0,0)$ | 0 | 0 | 0 | 0 | 0 |
| 0 | $S_1 = (1,0,0)$ | 0 | 0 | 0 | 0 | 0 |
| 0 | $S_2 = (2,0,0)$ | 0 | 0 | 0 | 0 | 0 |
| 0 | $S_3 = (3,0,0)$ | 0 | 0 | 0 | 0 | 0 |
| 3 | $S_4 = (1,1,1)$ | 1 | 1 | 1 | 0 | 0 |
| 4 | $S_5 = (1,1,0)$ | $-2$ | 10 | 10 | 0 | 0 |
| 7 | $S_6 = (1,2,1)$ | $-40$ | 632 | $-1096$ | 0 | 0 |
| 8 | $S_7 = (1,2,0)$ | 84 | 7068 | $-10212$ | 0 | 0 |
| 11 | $S_8 = (1,3,1)$ | $-196149$ | 117195 | 310731 | 0 | 0 |
| 12 | $S_9 = (1,3,0)$ | 392128 | 1698496 | 2341312 | 0 | 0 |
| 12 | $^*T_1 = (2,2,2)$ | 71616 | 64704 | 61248 | 1 | 1 |
| 15 | $T_2 = (2,2,1)$ | $-283800$ | 705480 | 1896072 | $-4$ | 20 |
| 16 | $^*T_3 = (2,2,0)$ | 424448 | 4271360 | 8867840 | 6 | 102 |
| 20 | $T_4 = (2,3,2)$ | 16409640 | 28566840 | $-31761096$ | 216 | 24 |
| 23 | $T_5 = (2,3,1)$ | $-53687112$ | 222751704 | $-644284392$ | $-828$ | $-2004$ |
| 24 | $T_6 = (2,3,0)$ | 75122376 | 851066520 | $-1688188008$ | 1232 | $-2992$ |
| 27 | $^*T_7 = (3,3,3)$ | 1340083359 | 1330627743 | 350478495 | 11232 | 288 |
| 32 | $T_8 = (3,3,2)$ | 5241369600 | 14146776576 | 31195501056 | $-34848$ | 49632 |
| 35 | $T_9 = (3,3,1)$ | 5055327270 | 68612549670 | 162278554662 | 34848 | $-77088$ |
| 36 | $^*T_{10} = (3,3,0)$ | $-23305987440$ | 147671262000 | 333866773296 | $-22896$ | 386064 |

Table 3. Fourier coefficients of Siegel-theta series of degree 2 for the lattices $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$

| $d_T$ | red. form | $a(T,\mathscr{L}_1)$ | $a(T,\mathscr{L}_2)$ | $a(T,\mathscr{L}_3)$ |
|---|---|---|---|---|
| 0 | $S_0 = (0,0,0)$ | 1 | 1 | 1 |
| 0 | $S_2 = (2,0,0)$ | 393120 | 198816 | 4512 |
| $^*12$ | $T_1 = (2,2,2)$ | 1808352000 | 452392704 | 830208 |
| 15 | $T_2 = (2,2,1)$ | 18517524480 | 4775215104 | 0 |
| $^*16$ | $T_3 = (2,2,0)$ | 113890795200 | 29072188608 | 18688704 |
| 20 | $T_4 = (2,3,2)$ | 374979870720 | 141322739712 | 2337865728 |
| 23 | $T_5 = (2,3,1)$ | 1630347264000 | 1522683346944 | 49529487360 |
| 24 | $T_6 = (2,3,0)$ | 9140008550400 | 5186242363392 | 131789094912 |
| $^*27$ | $T_7 = (3,3,3)$ | 8641511424000 | 3069920673792 | 1917900029952 |
| 32 | $T_8 = (3,3,2)$ | 51559732224000 | 63013026582528 | 87247980036096 |
| 35 | $T_9 = (3,3,1)$ | 131992914493440 | 407335796736000 | 655500144279552 |
| $^*36$ | $T_{10} = (3,3,0)$ | 740211875020800 | 890959504490496 | 1235443037110272 |

Here we give the three tables. It may be useful to assist for understanding the present research. The asterisk $*$ denotes that the matrix is not primitive. If someone wants to study the Hecke operators in the theory of Siegel modular forms $f$ he or she will need much more the values of the Fourier coefficients $a(T,f)$ at non-primitive indices $T$.

## 6. Some Problems

The present author has tried to find some codes, but he could not find any one of them. So he leaves them as the problems.

Problem 1: Is there an indecomposable doubly even binary self-dual $[48, 24, 8]$ code?

Problem 2: Is there an indecomposable self-dual ternary $[48, 24, 9]$ or $[48, 24, 12]$ code?

Problem 3: Is there a nearly extremal even unimodular lattice which is constructed by a method other than coding theory.

## 7. Appendix

Let $\mathscr{L}_{24}$ be the class of 24-dimensional even unimodular lattices. Let $L$ be a member of $\mathscr{L}_{24}$ and we consider the subsets $\Lambda_2(L)$ and $\Lambda_4(L)$. We briefly show that Siegel theta series $\Theta_g(Z, L)$ of degree $g$ $(g = 1, 2, 3)$ is completely detemined by the Fourier coefficients $a(T, L)$, where $T$'s are associated with the vectors in $\Lambda_2(L)$ and $\Lambda_4(L)$. We take up the following Niemeier lattices: the overlattices of the root lattices of $E_8^3, D_{24}, A_{24}, E_7 \oplus A_{17}$. We name $L_1$, $L_2$, $L_3$, $L_4$. As $T$'s we take $\overline{S_i}$, where we append the third row and the third column for each $S_i$ $(i = 1, 2, 4, 5, 6, 7)$ in Table 1 to make three by three matices from two by two matrices, and $T(2)$, $T(3)$, $T(4)$, $T(5)$, $T_1(6)$, $T_2(6)$, $T(7)$, $T_2(8)$ that are used in [26]. We verify that $\Theta_3(Z, L_1)$, $\Theta_3(Z, L_2)$, $\Theta_3(Z, L_3)$, $\Theta_3(Z, L_4)$ are linearly independent over $\mathbf{C}$, which matches with fact $\dim_{\mathbf{C}}(3, 12) = 4$. We also verify that of these $\Theta_2(Z, L_1)$, $\Theta_2(Z, L_2)$, $\Theta_2(Z, L_3)$, $\Theta_2(Z, L_4)$ there are three linearly independent theta's. Note that $\Theta_2(Z, L_j) = \Phi(\Theta_3(Z, L_j)$ for $1 \le j \le 4$ and $\Phi$ is the Siegel operator. We find that

$$\frac{\Theta_2(Z, L_2)}{451584} - \frac{\Theta_2(Z, L_1)}{46080} + \frac{\Theta_2(Z, L_3)}{35280} - \frac{\Theta_2(Z, L_4)}{112896} = 0,$$

and

$$\begin{aligned}\frac{\Theta_3(Z, L_2)}{451584} - \frac{\Theta_3(Z, L_1)}{46080} + \frac{\Theta_3(Z, L_3)}{35280} - \frac{\Theta_3(Z, L_4)}{112896} \\ = e^{2\pi i\sigma(T(2)Z)} + 18e^{2\pi i\sigma(T(3)Z)} + 164e^{2\pi i\sigma(T(4)Z)} - 106e^{2\pi i\sigma(T(5)Z)} - 408e^{2\pi i\sigma(T_1(6)Z)} \\ - 54e^{2\pi i\sigma(T_2(6)Z)} - 980e^{2\pi i\sigma(T(7)Z)} - 1008e^{2\pi i\sigma(T_2(8)Z)} + \cdots.\end{aligned}$$

The last series is a Siegel susp form of degree 3 and weight 12. This cusp form is discussed in Miyawaki [20] along with a different context from the present one. He also obtained a large table of the Fourier coefficients of this cusp form [21].

In the class of 32-dimensional even unimodualr lattices we select seven lattices $M_1, \dots, M_7$ and we computed some Fourier coefficients of their Siegel theta series of degrees up to 3. As a consequence we obtain three linearly independent Siegel cusp forms $\varphi_1(Z)$, $\varphi_2(Z)$, $\varphi_3(Z)$ of weight 16 and degree 3. Here we only give the beginning terms of their Fourier expansions:

$$\begin{aligned}\varphi_1(Z) = {} & e^{2\pi i\sigma(T(2)Z)} + 18e^{2\pi i\sigma(T(3)Z)} + 164e^{2\pi i\sigma(T(4)Z)} + 134e^{2\pi i\sigma(T(5)Z)} \\ & - 168e^{2\pi i\sigma(T_1(6)Z)} + 4266e^{2\pi i\sigma(T_2(6)Z)} + 3340e^{2\pi i\sigma(T(7)Z)} \\ & - 1008e^{2\pi i\sigma(T_2(8)Z)} + \cdots, \\ \varphi_2(Z) = {} & e^{2\pi i\sigma(T(3)Z)} + 22e^{2\pi i\sigma(T(4)Z)} + 27e^{2\pi i\sigma(T(5)Z)} + 136e^{2\pi i\sigma(T_1(6)Z)} \\ & + 662e^{2\pi i\sigma(T_2(6)Z)} + 2328e^{2\pi i\sigma(T(7)Z)} - 2328e^{2\pi i\sigma(T_2(8)Z)} + \cdots, \\ \varphi_3(Z) = {} & e^{2\pi i\sigma(T(4)Z)} + 12e^{2\pi i\sigma(T_1(6)Z)} + 32e^{2\pi i\sigma(T(7)Z)} + 36e^{2\pi i\sigma(T_2(8)Z)} + \cdots.\end{aligned}$$

These examples may be consistent with the work of Tsuyumine [39], in which the dimensions of the spaces of Siegel modular forms of degree 3 and the various weights.

## Acknowledgment

The author expresses his sincere gratitude to the referee of this article for some critical comments and for pointing out some practical suggestions.

## References

[1] Bosma, W., Cannon, J. and Playoust, C., The Magma algebra system. I. The user language, J. Symbolic Comput. **24** (1997), 235–265.

[2] Conway, J. H. and Pless, V., On the enumeration of self-dual codes, J. Comb. Th. Ser. A **28** (1980), 26–53.

[3] Conway, J. H., Pless, V. and Sloane, N. J. A., The binary self-dual codes of length up to 32: A revised Enumeration, J. Comb. Th. Ser. A **60** (1992), 183–195.

[4] Conway, J. H. and Sloane, N. J. A., Sphere Packings, Lattices and Groups, Springer-Verlag 1988. Third Edition (1998).

[5] Elkies, N. and Kominers, S. D., Refined Configuration Results for Extremal Type II Lattices of Ranks 40 and 80, Proc. Amer. Math. Soc. **138** (2010), 105–108.

[6] Erokhin, V. A., Theta series of even unimodular 24-dimensional lattices, LOMI **86** (1979), 82–93, J. Soviet Meth. **17** (1981), 1999–2008.

[7] Erokhin, V. A., Theta series of even unimodular lattices, LOMI **116** (1982), 68–73, J. Soviet Meth. **26** (1984), 1012–1020.

[8] Freitag, E., Siegelsche Modulfunktionen, Springer-Verlag 1983.
[9] Hecke, E., Analytische Arithmetik der positiven quadratischen Formen, Kgl. Danske Vid. Selskab. Mat.-fys. Medd. **13** (1940).
[10] Igusa, J., On Siegel modular forms of genus two, Amer. J. Math. **84** (1962), 175–200.
[11] Igusa, J., Modular forms and projective invariants, Amer. J. Math. **89** (1967), 817–855.
[12] Kervaire, M., Unimodular lattices with a complete root system, L'enseign. Math. **40** (1994), 59–104.
[13] Kominers, D. D. and Abel, Z., Configurations of rank-40r extremal even unimodular lattices ($r = 1, 2, 3$), Journal de thorie des nombres de Bordeaux, **20** (2008), 365–371.
[14] Kominers, S. D., Configurations of extremal even unimodular lattices, Int. J. Number Theory, **5** (2009), 457–464.
[15] Leech, J., Notes on sphere packings, Can. J. Math. **19** (1967), 251–267.
[16] Leon, J. S., Pless, V. and Sloane, N. J. A., On ternary self-dual codes of length 24, IEEE Trans. Inform. Theory, IT-27 (1981), 176–180.
[17] MacWilliams, F. J. and Sloane, N. J. A., The Theory of Error-Correcting Codes, North-Holland (1977).
[18] Mallows, C. L., Odlyzko, A. M. and Sloane, N. J. A., An upper bound for modular forms, lattices and codes, J. Algebra **36** (1975), 68–76.
[19] Mallows, C. L., Pless, V. and Sloane, N. J. A., Self-Dual Codes over GF(3), SIAM J. Appl. Math. Vol. **31** (1976), 649–666.
[20] Miyawaki, I., Numerical examples of Siegel cusp forms of degree 3 and their zeta-functions, Mem. Fac. Sci. Kyushu Univ. Ser. A, Mathematics, Vol. **46**, No. 2, (1992), 307–399.
[21] Miyawaki, I., A table of the Fourier coefficients of the Siegel cusp form of weight 12 and degree 3, around 1992, unpublished.
[22] Nebe, G., http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES.
[23] Niemeier, H. V., Definite quadratische Formen der Dimension 24 und Diskriminante 1, J. Number Theory **5** (1973), 142–178.
[24] Oura, M. and Ozeki, M., A numerical study of Siegel theta series of various degrees for the 32-dimensional even unimodular extremal lattices, Kyushu J. Math. **70** (2016), no. 2, 281–314.
[25] Oura, M. and Ozeki, M., Distinguishing Siegel theta series of degree 4 for the 32-dimensional even unimodular extremal lattices, Abh. Math. Semin. Univ. Hambg. **86** (2016), no. 1, 19–53.
[26] Ozeki, M. and Washio, T., Table of the Fourier coefficients of Eisenstein series of degree 3, Proc. Japan Acad. Vol. **59**, Ser. A, No. 6 (1983), 252–255.
[27] Ozeki, M., On the structure of even unimodular extremal lattices of rank 40, Rocky Mountain J. Math. **19** (1989), 847–862.
[28] Ozeki, M., On the configurations of even unimodular lattices of rank 48, Arch. Math. **46** (1986), 247–287.
[29] Ozeki, M., Ternary code construction of even unimodular lattices, in Theorie des nombres, Quebec 1987, Gruter Berlin, 1989, 772–784.
[30] Ozeki, M., On a problem posed by R. Salvati Manni, Acta Arithm. **150** (2011), 1–22.
[31] Ozeki, M., Siegel Theta Series of Various Degrees for the Leech Lattice, Kyushu J. Math. **68** (2014), 53–91.
[32] Ozeki, M., A numerical study of Siegel theta series of various degrees for the 48-dimensional even unimodular extremal lattices, Tsukuba J. Math. **40** (2017), no. 2, 139–186.
[33] Salvati Manni, R., Slopes of cusp forms and theta series, J. Number Theory **83** (2000), 282–296.
[34] Schöneberg, B., Das Verhalten von mehrfachen Thetareihen bei Modulsubstitutionen, Math. Ann. **116** (1939), 511–523.
[35] Schöneberg, B., Elliptic Modular Functions, Springer (1974).
[36] Shimura, G., Introduction to the Arithmetic Theory of Automorphic Functions, Princeton Univ. Prss. (1971).

[37] Siegel, C. L., Einführung in die Theorie der Modulfunktionen $n$-ten Grades, Math. Ann. **116** (1939), 617–657.
[38] Sloane, N. J. A., Self-Dual codes and lattices, Proc. Symp. in Pure Math. **34** (1979), 273–308.
[39] Tsuyumine, S., On Siegel modular forms of degree 3, Amer. J. Math. **108** (1986), 755–862.
[40] Venkov, B. B., The classification of integral even unimodular 24-dimensional quadratic forms, Trudy Math. Inst. Steklov **148** (1978), 65–76, Proc. Steklov Inst. Math. **148** (1980), 63–74.
[41] Venkov, B. B., On even unimodular Euclidean lattices of dimension 32, LOMI **116** (1982), 44–45, 161–162, J. Soviet Meth. **26** (1984), 1860–1867.

Michio Ozeki
Emeritus Professor at the Faculty of Science
Yamagata University
4-8-27 Nakano Hirosaki 036-8155
Japan
E-mail: ozeki.mitio@ruby.plala.or.jp