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PARAHOLOMORPHIC COHOMOLOGY GROUPS OF HYPERBOLIC ADJOINT ORBITS

By

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Abstract. For a paracomplex manifold, we construct certain cohomology groups. The main purpose of this paper is to clarify a link between such cohomology groups of hyperbolic adjoint orbits and the de Rham cohomology groups of real flag manifolds. That establishes relation between a paraholomorphic invariant and a topological invariant.

Contents

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1. Introduction

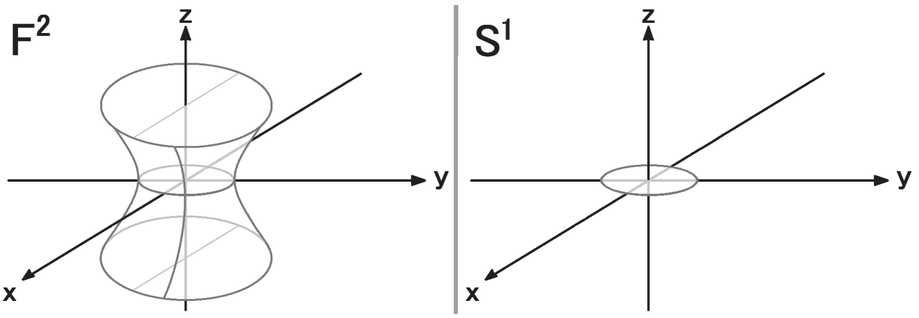
In 1952 Libermann [10] has introduced the notion of paracomplex manifold, and a paracomplex manifold M is a differentiable manifold endowed with a paracomplex structure I. Since the paracomplex structure I has properties like those of a complex structure, we can naturally formulate the real vector space $\mathscr{A}^{(r,s)}(M)$ of differential forms of type (r,s) on M, and get two kinds of linear operators $\partial : \mathscr{A}^{(r,s)}(M) \to \mathscr{A}^{(r+1,s)}(M)$ and $\bar{\partial} : \mathscr{A}^{(r,s)}(M) \to \mathscr{A}^{(r,s+1)}(M)$ such that $d = \partial + \bar{\partial}$, $\partial^2 = 0$, $(\bar{\partial})^2 = 0$ and $\partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$. Setting $\Omega^r(M) := \{\alpha \in \mathscr{A}^{(r,0)}(M) \mid \bar{\partial}\alpha = 0\}$, we see that $\Omega(M) := \bigoplus_{r=0}^{\infty} \Omega^r(M)$ forms a cochain complex with respect to ∂ (see Subsection 2.3 for detail), which enables us to construct cohomology groups

$$\mathscr{H}^0(M), \mathscr{H}^1(M), \mathscr{H}^2(M), \ldots$$

of the paracomplex manifold (M, I). Similarly, one obtains $\overline{\mathscr{H}}^0(M), \overline{\mathscr{H}}^1(M), \overline{\mathscr{H}}^2(M), \ldots$ from $\bar{\Omega}^s(M) := \{\beta \in \mathscr{A}^{(0,s)}(M) \mid \partial\beta = 0\}$. Remark here that these groups $\mathscr{H}^*(M)$, $\overline{\mathscr{H}}^*(M)$ are invariant under paraholomorphic diffeomorphisms. Now, let G be a connected real semisimple Lie group, and let S be a hyperbolic element of $\mathfrak{g}$ (see Definition 3.1 for the definition of hyperbolic element). The adjoint orbit $\operatorname{Ad} G(S) = G/L$ of G through S is called a *hyperbolic adjoint orbit* and G/L is a homogeneous paracomplex manifold of G. Furthermore, it is related with a real flag manifold G/Q^-. Taking this relation into account, we clarify a link between the cohomology groups $\mathscr{H}^*(G/L)$ of G/L and the de Rham cohomology groups $H^*(G/Q^-)$ of G/Q^-, which is the main result in this paper (see Theorem 4.1). For example, a hyperboloid F^2 of one sheet is a hyperbolic adjoint orbit, a circle S^1 is a real flag manifold and F^2 is related with S^1. Theorem 4.1 tells us that

$$\dim_{\mathbf{R}} \mathscr{H}^0(F^2) = \dim_{\mathbf{R}} \mathscr{H}^1(F^2) = 1, \quad \dim_{\mathbf{R}} \mathscr{H}^k(F^2) = 0 \quad \text{if } k \geq 2$$

(ref. Example 4.13), where we consider an $SL(2,\mathbf{R})$-invariant paracomplex structure $I_{SL(2,\mathbf{R})}$ of $F^2 = SL(2,\mathbf{R})/S(GL(1,\mathbf{R}) \times GL(1,\mathbf{R}))$ and such paracomplex structures are unique up to sign $\pm$.



This paper is organized as follows. In Section 2 we recall the definition of paracomplex manifold, and give some lemmas, propositions and so on. Then we construct cohomology groups $\mathscr{H}^*(M)$, $\overline{\mathscr{H}}^*(M)$ of a paracomplex manifold (M, I). In Section 3 we first recall the definition of hyperbolic element and observe that a hyperbolic adjoint orbit G/L is related with a real flag manifold G/Q^-. Next, we construct a G-invariant paracomplex structure I_G of G/L; furthermore we fix a paraholomorphic structure $\mathcal{S}_{G/L} = \{(O_g, \psi_g)\}_{g \in G}$ on $(G/L, I_G)$ and a differentiable structure $\mathcal{S}_{G/Q^-} = \{(O_g^+, \psi_g^+)\}_{g \in G}$ on G/Q^-. Finally in Section 4 we establish Theorem 4.1 and demonstrate it by investigating relation between $\mathcal{S}_{G/L}$ and $\mathcal{S}_{G/Q^-}$. As an appendix, we deduce that $F^2 = (F^2, I_{SL(2,\mathbf{R})})$ and a circular cylinder $S^1 \times \mathbf{R} = (S^1 \times \mathbf{R}, I_{S^1 \times \mathbf{R}})$ are diffeomorphic, but neither paraholomorphically nor anti-paraholomorphically diffeomorphic to each other, from the following data on cohomology groups:

$$\begin{cases} \dim_{\mathbf{R}} \mathscr{H}^1(F^2) = 1, & \dim_{\mathbf{R}} \overline{\mathscr{H}}^1(F^2) = 1, \\ \dim_{\mathbf{R}} \mathscr{H}^1(S^1 \times \mathbf{R}) = 1, & \dim_{\mathbf{R}} \overline{\mathscr{H}}^1(S^1 \times \mathbf{R}) = 0. \end{cases}$$

Here $I_{S^1 \times \mathbf{R}}$ is a paracomplex structure of $S^1 \times \mathbf{R}$ naturally defined by $(I_{S^1 \times \mathbf{R}})_{(p,x)}(u + v) := u - v$ for $(p, x) \in S^1 \times \mathbf{R}$ and $u \in T_p S^1$, $v \in T_x \mathbf{R}$.

Notation. Throughout this paper, for a Lie group G we denote its Lie algebra by the corresponding Fraktur small letter $\mathfrak{g}$; besides we always assume the differentiability of class C^∞ and utilize the following notation, where M is a differentiable manifold:

(n1) $T_p M$: the tangent vector space of M at a point $p \in M$,
(n2) $\mathfrak{X}(M)$: the real Lie algebra of vector fields on M,
(n3) $\mathcal{D}^k(M)$: the real vector space of differential forms of degree k on M,

(n4) $F^*\omega'$: the pullback by a differentiable mapping $F : M \to M'$ of a form $\omega' \in \mathcal{D}^k(M')$,
(n5) $\mathfrak{m} \oplus \mathfrak{n}$: the direct sum of vector spaces $\mathfrak{m}$ and $\mathfrak{n}$,
(n6) $f|_V$: the restriction of a mapping f to a set V,
(n7) $\mathbf{Z}_{\geq 0}$: the set of non-negative integers,
(n8) Ad, ad: the adjoint representation of G, $\mathfrak{g}$,
(n9) $C_G(S) := \{g \in G \mid \mathrm{Ad}\, g(S) = S\}$ for an element $S \in \mathfrak{g}$,
(n10) $N_G(\mathfrak{m}) := \{g \in G \mid \mathrm{Ad}\, g(\mathfrak{m}) \subset \mathfrak{m}\}$ for a vector subspace $\mathfrak{m} \subset \mathfrak{g}$,
(n11) G_0: the identity component of G,
(n12) $Z(G)$: the center of G,
(n13) τ_g: a diffeomorphic transformation of a homogeneous space G/H defined by $\tau_g(aH) := gaH$ for $aH \in G/H$, where $g \in G$.

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2. Cohomology Groups of Paracomplex Manifolds

2.1. Definitions of Paracomplex Manifold and Local Paraholomorphic Coordinate System. First of all, the definition of paracomplex manifold is as follows:

Definition 2.1 (cf. Libermann [10, p. 2518]). Let M be a differentiable manifold, and let I be a tensor field of type $(1,1)$ on M. With this setting,

(I) I is said to be an *almost paracomplex structure* of M, if
 (c1) $I^2 = \mathrm{id}$ (considering I as a linear transformation of vector fields),
 (c2) $\dim_{\mathbf{R}} T_p^+M = \dim_{\mathbf{R}} T_p^-M$ for all $p \in M$, where $T_p^{\pm}M := \{v \in T_pM \mid I_pv = \pm v\}$.
 In this case, (M, I) is called an *almost paracomplex manifold.*

(II) I is said to be a *paracomplex structure* of M, if the conditions (c1), (c2) above and
 (c3) $[IX, IY] - I[IX, Y] - I[X, IY] + [X, Y] = 0$ for all $X, Y \in \mathfrak{X}(M)$
 hold for I. In this case, (M, I) is called a *paracomplex manifold.*

In order to give the definition of local paraholomorphic coordinate system, we first show Lemma 2.2 which can be found, with some slight modifications, in

Kaneyuki-Kozai [5, p. 82, Proposition 1.1]. Here we note that a non-empty open subset $W \subset M$ is always an almost paracomplex manifold whenever M is an almost paracomplex manifold.

Lemma 2.2. *Let (M, I) be an almost paracomplex manifold of* $\dim_{\mathbf{R}} M = 2n$, *and let*

$$\mathfrak{X}^{\pm}(M) := \{A \in \mathfrak{X}(M) \mid IA = \pm A\}.$$

Then, the following three items hold:

(1) *For a given $X \in \mathfrak{X}(M)$, there exists a unique $(A, B) \in \mathfrak{X}^{+}(M) \times \mathfrak{X}^{-}(M)$ such that $X = A + B$.*

(2) *For any point $p \in M$, there exist an open neighborhood V of p and* $2n$ *vector fields A_i, B_i on V such that, for each $x \in V$, $\{(A_i)_x\}_{i=1}^{n}$ and $\{(B_i)_x\}_{i=1}^{n}$ are real bases of $T_x^{+}M$ and $T_x^{-}M$, respectively.*

(3) *(M, I) is a paracomplex manifold, namely the condition* (c3) *in Definition* 2.1 *holds if and only if $[\mathfrak{X}^{+}(M), \mathfrak{X}^{+}(M)] \subset \mathfrak{X}^{+}(M)$, $[\mathfrak{X}^{-}(M), \mathfrak{X}^{-}(M)] \subset \mathfrak{X}^{-}(M)$.*

Proof. (1) follows from $I^2 = \mathrm{id}$, $\mathfrak{X}^{+}(M) \cap \mathfrak{X}^{-}(M) = \{0\}$ and

$$X = \frac{1}{2}(X + IX) + \frac{1}{2}(X - IX).$$

(2) Since M is a manifold, there exists a coordinate neighborhood $(W, \phi = (x^1, \dots, x^{2n}))$ of M containing the point p. Putting

$$X_\ell := \frac{1}{2}\left(\frac{\partial}{\partial x^\ell} + I\frac{\partial}{\partial x^\ell}\right), \quad Y_\ell := \frac{1}{2}\left(\frac{\partial}{\partial x^\ell} - I\frac{\partial}{\partial x^\ell}\right)$$

for $1 \le \ell \le 2n$, we deduce that $X_\ell \in \mathfrak{X}^{+}(W)$, $Y_\ell \in \mathfrak{X}^{-}(W)$ and $\partial/\partial x^\ell = X_\ell + Y_\ell$, so that $T_p^{+}M$ and $T_p^{-}M$ are generated by $(X_\ell)_p$ and $(Y_\ell)_p$, $\ell = 1, \dots, 2n$, respectively (because $\{(\partial/\partial x^\ell)_p\}_{\ell=1}^{2n}$ is a basis of T_pM and $T_p^{\pm}M \subset T_pM$). Therefore there exist $\lambda_i^\ell, \mu_i^\ell \in \mathbf{R}$ such that $\{\sum_{\ell=1}^{2n} \lambda_i^\ell (X_\ell)_p\}_{i=1}^{n}$ and $\{\sum_{\ell=1}^{2n} \mu_i^\ell (Y_\ell)_p\}_{i=1}^{n}$ are bases of $T_p^{+}M$ and $T_p^{-}M$, respectively. Then, it turns out that $A_i := \sum_{\ell=1}^{2n} \lambda_i^\ell X_\ell \in \mathfrak{X}^{+}(W)$, $B_i := \sum_{\ell=1}^{2n} \mu_i^\ell Y_\ell \in \mathfrak{X}^{-}(W)$ for all $1 \le i \le n$, and $\{(A_i)_p\}_{i=1}^{n} \bigcup \{(B_i)_p\}_{i=1}^{n}$ is a basis of $T_pM = T_p^{+}M \oplus T_p^{-}M$. Accordingly we can get the conclusion by changing the W for a sufficiently small open neighborhood V of $p \in M$ (if necessary).

(3) A direct computation, together with (1) and (c3), enables us to conclude (3). □

Lemma 2.2 leads to Proposition 2.3 whose statement is similar to that of Proposition 1.2 in Kaneyuki-Kozai [5, p. 83]. See Proposition 2 in Cortés-Mayer-Mohaupt-Saueressig [2, p. 11] also.

PROPOSITION 2.3.

(A) *For an arbitrary* $2n$*-dimensional paracomplex manifold* (M, I), *there exists an atlas* $\{(O_\alpha, \psi_\alpha = (x_\alpha^1, \dots, x_\alpha^n, y_\alpha^1, \dots, y_\alpha^n))\}_{\alpha \in A}$ *of* M *such that on each* O_α,

$$\text{(a)} \qquad I\left(\frac{\partial}{\partial x_\alpha^i}\right) = \frac{\partial}{\partial x_\alpha^i}, \quad I\left(\frac{\partial}{\partial y_\alpha^i}\right) = -\frac{\partial}{\partial y_\alpha^i} \quad (1 \le i \le n).$$

(B) *From the above condition* (a) *it follows that on* $O_\alpha \cap O_\beta$,

$$\text{(b)} \qquad \frac{\partial x_\beta^j}{\partial y_\alpha^i} = 0 = \frac{\partial y_\beta^j}{\partial x_\alpha^i} \quad (1 \le i, j \le n)$$

whenever $O_\alpha \cap O_\beta \neq \emptyset$ $(\alpha, \beta \in A)$.

(C) *Conversely, let us consider the case where a differentiable manifold* M *admits an atlas* $\{(O_\alpha, \psi_\alpha = (x_\alpha^1, \dots, x_\alpha^n, y_\alpha^1, \dots, y_\alpha^n))\}_{\alpha \in A}$ *which the above condition* (b) *holds for. In this case; if one defines a tensor field* I_α *on each* O_α *by* $I_\alpha(\partial/\partial x_\alpha^i) := \partial/\partial x_\alpha^i$, $I_\alpha(\partial/\partial y_\alpha^i) := -\partial/\partial y_\alpha^i$ *for* $1 \le i \le n$, *then we can get a tensor field* I *on the whole* $M = \bigcup_{\alpha \in A} O_\alpha$ *by setting* $I|_{O_\alpha} := I_\alpha$ *for* $\alpha \in A$, *and* (M, I) *is a paracomplex manifold.*

PROOF. (A) Lemma 2.2-(2), (3) implies that both $M \ni p \mapsto T_p^{\pm} M$ are involutive distributions on M. By Frobenius's theorem, there exists an atlas $\{(O_\alpha, \psi_\alpha = (x_\alpha^1, \dots, x_\alpha^n, y_\alpha^1, \dots, y_\alpha^n))\}_{\alpha \in A}$ of M which the condition (a) holds for.

(B) Taking $I(\partial/\partial x_\alpha^i) = \partial/\partial x_\alpha^i$, $I(\partial/\partial x_\beta^j) = \partial/\partial x_\beta^j$, $I(\partial/\partial y_\beta^j) = -\partial/\partial y_\beta^j$ and

$$\frac{\partial}{\partial x_\alpha^i} = \sum_{j=1}^n \left(\frac{\partial x_\beta^j}{\partial x_\alpha^i} \frac{\partial}{\partial x_\beta^j} + \frac{\partial y_\beta^j}{\partial x_\alpha^i} \frac{\partial}{\partial y_\beta^j} \right)$$

into account, one has $\partial y_\beta^j / \partial x_\alpha^i = 0$. In a similar way, we have $\partial x_\beta^j / \partial y_\alpha^i = 0$.

(C) For each $\alpha \in A$, let us define a tensor field I_α of type $(1,1)$ on O_α as follows:

$$\text{(c)} \qquad I_\alpha\left(\frac{\partial}{\partial x_\alpha^i}\right) := \frac{\partial}{\partial x_\alpha^i}, \quad I_\alpha\left(\frac{\partial}{\partial y_\alpha^i}\right) := -\frac{\partial}{\partial y_\alpha^i}$$

$(1 \leq i \leq n)$. Then we see that on $O_\alpha \cap O_\beta \neq \emptyset$,

$$I_\beta\left(\frac{\partial}{\partial x_\alpha^i}\right) = I_\beta\left(\sum_{j=1}^n \frac{\partial x_\beta^j}{\partial x_\alpha^i}\frac{\partial}{\partial x_\beta^j} + \sum_{j=1}^n \frac{\partial y_\beta^j}{\partial x_\alpha^i}\frac{\partial}{\partial y_\beta^j}\right) = I_\beta\left(\sum_{j=1}^n \frac{\partial x_\beta^j}{\partial x_\alpha^i}\frac{\partial}{\partial x_\beta^j}\right) \quad (\because \text{(b)})$$

$$= \sum_{j=1}^n \frac{\partial x_\beta^j}{\partial x_\alpha^i}\frac{\partial}{\partial x_\beta^j} \quad (\because \text{(c)})$$

$$= \sum_{j=1}^n \frac{\partial x_\beta^j}{\partial x_\alpha^i}\frac{\partial}{\partial x_\beta^j} + \sum_{j=1}^n \frac{\partial y_\beta^j}{\partial x_\alpha^i}\frac{\partial}{\partial y_\beta^j} \quad (\because \text{(b)})$$

$$= \frac{\partial}{\partial x_\alpha^i}$$

and $I_\beta(\partial/\partial y_\alpha^i) = -\partial/\partial y_\alpha^i$ for all $1 \leq i \leq n$. This implies that one can get a tensor field I on $M = \bigcup_{\alpha \in A} O_\alpha$ by setting $I|_{O_\alpha} := I_\alpha$ for $\alpha \in A$. Needless to say, this I is an almost paracomplex structure of M. From (c) and Frobenius's theorem it follows that both $[\mathfrak{X}^\pm(M), \mathfrak{X}^\pm(M)] \subset \mathfrak{X}^\pm(M)$ hold for I. Therefore, the I is a paracomplex structure of M by Lemma 2.2-(3). □

Remark 2.4 (cf. Cortés-Mayer-Mohaupt-Saueressig [2, p. 11, Example 3]). Let N_1 and N_2 be differentiable manifolds with $\dim_{\mathbf{R}} N_1 = \dim_{\mathbf{R}} N_2$. Then, Proposition 2.3-(C) ensures that one can define a paracomplex structure $I_{N_1 \times N_2}$ of the product manifold $N_1 \times N_2$ by

$$(I_{N_1 \times N_2})_{(p_1,p_2)}(u_1 + u_2) := u_1 - u_2$$

for $p_i \in N_i$ and $u_i \in T_{p_i}N_i$, $i = 1, 2$. Hence, for example, a $2n$-dimensional Euclidean space $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$ and a circular cylinder $S^1 \times \mathbf{R}$ are paracomplex manifolds.

Let us give the definition of local paraholomorphic coordinate system.

Definition 2.5. Let (M, I) be a paracomplex manifold, and let

$$\{(O_\alpha, \psi_\alpha = (x_\alpha^1, \ldots, x_\alpha^n, y_\alpha^1, \ldots, y_\alpha^n))\}_{\alpha \in A}$$

be an atlas of M which the condition (a) in Proposition 2.3 holds for. Then we say that $(O_\alpha, \psi_\alpha = (x_\alpha^1, \ldots, x_\alpha^n, y_\alpha^1, \ldots, y_\alpha^n))$ is a *paraholomorphic coordinate neighborhood* of (M, I), and that $(x_\alpha^1, \ldots, x_\alpha^n, y_\alpha^1, \ldots, y_\alpha^n)$ is a *local paraholomorphic coordinate system* on O_α.

2.2. Paraholomorphic Mappings. We recall the definition of paraholomorphic mapping, and state some topics related to paraholomorphic mappings. Throughout Subsection 2.2, $M = (M, I)$ and $M' = (M', I')$ are paracomplex manifolds.

Definition 2.6. A differentiable mapping $F : M \to M'$ is said to be *paraholomorphic* (resp. *anti-paraholomorphic*), if

$$I'_{F(p)} \circ (dF)_p = (dF)_p \circ I_p \quad (\text{resp. } I'_{F(p)} \circ (dF)_p = -(dF)_p \circ I_p)$$

for all $p \in M$, where $(dF)_p$ stands for the differential of F at p.

One mentions the following:

Lemma 2.7. *For a given differentiable mapping $F : M \to M'$, it follows that*

(i) *F is paraholomorphic if and only if $(dF)_p(T_p^+ M) \subset T_{F(p)}^+ M'$, $(dF)_p(T_p^- M) \subset T_{F(p)}^- M'$ for all $p \in M$;*
(ii) *F is anti-paraholomorphic if and only if $(dF)_p(T_p^+ M) \subset T_{F(p)}^- M'$, $(dF)_p(T_p^- M) \subset T_{F(p)}^+ M'$ for all $p \in M$.*

Proof. Trivial. □

From Lemma 2.7 it is easy to see

Corollary 2.8. *For any diffeomorphism $\Psi : M \to M'$, it is paraholomorphic (resp. anti-paraholomorphic) if and only if its inverse Ψ^{-1} is paraholomorphic (resp. anti-paraholomorphic).*

We end this subsection with

Remark 2.9. Let $(O, \psi = (x^1, \dots, x^n, y^1, \dots, y^n))$ be a paraholomorphic coordinate neighborhood of (M, I), and let $I_{\mathbf{R}^n \times \mathbf{R}^n}$ be the paracomplex structure of $\mathbf{R}^{2n}$ given in Remark 2.4. Then, $\psi : (O, I) \to (\psi(O), I_{\mathbf{R}^n \times \mathbf{R}^n})$ is a paraholomorphic diffeomorphism.

2.3. Construction of Paraholomorphic Cohomology Groups. Our main goal in this subsection is to construct cohomology groups of a paracomplex manifold. Remark here that our approach is a little different from Krahe's [9]; and that Angella-Rossi [1] deals with D-Dolbeaut cohomology groups $H_{\partial}^{*,*}(X;\mathbf{R})$ and D-complex subgroups $H_I^{*\pm}(X;\mathbf{R})$ of the de Rham cohomology groups of a compact D-complex manifold X, but these groups $H_{\partial}^{*,*}(X;\mathbf{R})$, $H_I^{*\pm}(X;\mathbf{R})$ are different from our cohomology groups.

2.3.1. Cohomology Groups $\mathscr{H}^*(M)$. We introduce the notion of differential form of type (r,s).

Definition 2.10. Let (M,I) be a paracomplex manifold, and let $\omega \in \mathcal{D}^{r+s}(M)$. We say that ω is *of type* (r,s), if for each $p \in M$ there exists a paraholomorphic coordinate neighborhood $(O,(x^1,\dots,x^n,y^1,\dots,y^n))$ of (M,I) such that

(1) $p \in O$,

(2) ω is expressed as

$$\omega = \sum_{i_1<\cdots<i_r,\, j_1<\cdots<j_s} \omega_{i_1\cdots i_r j_1\cdots j_s}\, dx^{i_1}\wedge\cdots\wedge dx^{i_r}\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_s}$$

on O.

Remark 2.11. Here are comments on Definition 2.10.

(i) The property that ω is of type (r,s) does not depend on the choice of local paraholomorphic coordinate system $(x^1,\dots,x^n,y^1,\dots,y^n)$ on O, due to (b) in Proposition 2.3.

(ii) A differential form of type $(0,0)$ on M is a differentiable function on M.

(iii) If ω is a differential form of type (r,s), then $\omega_p(v_1,\dots,v_{r+s}) = 0$ for vectors $v_1,\dots,v_{r+s} \in T_pM$ of which more than r belong to T_p^+M or more than s belong to T_p^-M.

Let us denote by $\mathscr{A}^{(r,s)}(M)$ the real vector space of differential forms of type (r,s) on a paracomplex manifold (M,I), and show Lemma 2.12, Lemma 2.13 and Corollary 2.14 which enable us to construct cohomology groups.

Lemma 2.12. *The following three items hold*:

(1) $\mathscr{A}^{(r,s)}(M)\wedge\mathscr{A}^{(r',s')}(M) \subset \mathscr{A}^{(r+r',s+s')}(M)$ *for all* $r,r',s,s' \in \mathbf{Z}_{\geq 0}$.

(2) $d(\mathscr{A}^{(r,s)}(M)) \subset \mathscr{A}^{(r+1,s)}(M) \oplus \mathscr{A}^{(r,s+1)}(M)$ *for all* $r,s \in \mathbf{Z}_{\geq 0}$.

(3) $\mathscr{A}^{(r,s)}(M) = \{0\}$ *if* $r>n$ *or* $s>n$.

Here $\dim_{\mathbf{R}} M = 2n$.

PROOF. We prove (2) only, since (1) and (3) are immediate from Definition 2.10. Suppose that an $\omega \in \mathscr{A}^{(r,s)}(M)$ is expressed as

$$\omega = \sum_{i_1<\cdots<i_r,\, j_1<\cdots<j_s} \omega_{i_1\cdots i_r j_1\cdots j_s}\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_s}$$

in terms of a local paraholomorphic coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^n)$. Then, it follows that

$$\begin{aligned} d\omega = {} & \sum_{k=1}^{n} \sum_{i_1<\cdots<i_r,\, j_1<\cdots<j_s} \frac{\partial \omega_{i_1\cdots i_r j_1\cdots j_s}}{\partial x^k}\, dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_s} \\ & + (-1)^r \sum_{k=1}^{n} \sum_{i_1<\cdots<i_r,\, j_1<\cdots<j_s} \frac{\partial \omega_{i_1\cdots i_r j_1\cdots j_s}}{\partial y^k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dy^k \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_s}, \end{aligned}$$

and the 1st (resp. 2nd) term of right-hand side belongs to $\mathscr{A}^{(r+1,s)}(M)$ (resp. $\mathscr{A}^{(r,s+1)}(M)$). This assures (2). □

For every $\omega \in \mathscr{A}^{(r,s)}(M)$, its exterior derivative $d\omega$ decomposes into a sum of differential forms of types $(r+1, s)$ and $(r, s+1)$, which we denote by $\partial\omega$ and $\bar{\partial}\omega$, respectively (cf. Lemma 2.12-(2)). With this notation we assert

LEMMA 2.13. *The following four items hold*:

(i) $d = \partial + \bar{\partial}$.

(ii) $\partial : \mathscr{A}^{(r,s)}(M) \to \mathscr{A}^{(r+1,s)}(M)$ *and* $\bar{\partial} : \mathscr{A}^{(r,s)}(M) \to \mathscr{A}^{(r,s+1)}(M)$ *are linear mappings for all* $r, s \in \mathbf{Z}_{\geq 0}$.

(iii) $\partial^2 = 0$, $(\bar{\partial})^2 = 0$ *and* $\partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$.

(iv) *For all* $r, r', s, s' \in \mathbf{Z}_{\geq 0}$, $\omega \in \mathscr{A}^{(r,s)}(M)$ *and* $\eta \in \mathscr{A}^{(r',s')}(M)$, *it follows that*

$$\partial(\omega \wedge \eta) = (\partial\omega) \wedge \eta + (-1)^{r+s} \omega \wedge (\partial\eta),$$

$$\bar{\partial}(\omega \wedge \eta) = (\bar{\partial}\omega) \wedge \eta + (-1)^{r+s} \omega \wedge (\bar{\partial}\eta).$$

PROOF. We only confirm (iii). For any $\omega \in \mathscr{A}^{(r,s)}(M)$, one obtains

$$0 = d(d\omega) = \partial(\partial\omega) + (\partial \circ \bar{\partial} + \bar{\partial} \circ \partial)\omega + \bar{\partial}(\bar{\partial}\omega)$$

from $0 = d^2$ and (i). Hence, we can conclude $\partial(\partial\omega) = 0$, $(\partial \circ \bar{\partial} + \bar{\partial} \circ \partial)\omega = 0$ and $\bar{\partial}(\bar{\partial}\omega) = 0$ by virtue of (ii), $\partial(\partial\omega) \in \mathscr{A}^{(r+2,s)}(M)$, $(\partial \circ \bar{\partial} + \bar{\partial} \circ \partial)\omega \in \mathscr{A}^{(r+1,s+1)}(M)$ and $\bar{\partial}(\bar{\partial}\omega) \in \mathscr{A}^{(r,s+2)}(M)$. □

Two Lemmas 2.12 and 2.13 lead to

Corollary 2.14. *For an* $r \in \mathbf{Z}_{\geq 0}$, *we set*

$$\Omega^r(M) := \{\alpha \in \mathscr{A}^{(r,0)}(M) \mid \bar{\partial}\alpha = 0\} \quad (= \ker(\bar{\partial} : \mathscr{A}^{(r,0)}(M) \to \mathscr{A}^{(r,1)}(M))).$$

Then, the following five items hold:

(1) $\Omega^r(M)$ *is a real vector space for each* $r \in \mathbf{Z}_{\geq 0}$.

(2) *Let* $(O, (x^1, \dots, x^n, y^1, \dots, y^n))$ *be a paraholomorphic coordinate neighborhood of* (M, I), *and let* $\alpha \in \Omega^r(M)$. *Then,* α *is expressed as*

$$\alpha = \sum_{i_1 < \cdots < i_r} \alpha_{i_1 \cdots i_r} \, dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \quad \alpha_{i_1 \cdots i_r} = \alpha_{i_1 \cdots i_r}(x^1, \dots, x^n)$$

on O. *Here,* $\alpha_{i_1 \cdots i_r} = \alpha_{i_1 \cdots i_r}(x^1, \dots, x^n)$ *means that the function* $\alpha_{i_1 \cdots i_r}$ *is independent of the variables* $y^1, \dots, y^n$.

(3) $\partial(\Omega^r(M)) \subset \Omega^{r+1}(M)$ *for all* $r \in \mathbf{Z}_{\geq 0}$.

(4) $\Omega^r(M) \wedge \Omega^{r'}(M) \subset \Omega^{r+r'}(M)$ *for all* $r, r' \in \mathbf{Z}_{\geq 0}$.

(5) $\Omega^r(M) = \{0\}$ *if* $r > n$.

Here $\dim_{\mathbf{R}} M = 2n$.

Let $\Omega(M)$ be the direct sum of real vector spaces $\Omega^r(M)$, $r \in \mathbf{Z}_{\geq 0}$. Lemma 2.13-(iii) and Corollary 2.14-(3) allow one to regard $\Omega(M) = \bigoplus_{r=0}^{\infty} \Omega^r(M)$ as a cochain complex with coboundary operator ∂. We denote by $\mathscr{H}^r(M)$ the r-dimensional cohomology group of this cochain complex—that is, for $r \in \mathbf{Z}_{\geq 0}$ we set

$$(2.15) \qquad \begin{cases} \mathscr{Z}^r(M) := \{\alpha \in \Omega^r(M) \mid \partial\alpha = 0\}, \\ \mathscr{B}^r(M) := \{\partial\beta \mid \beta \in \Omega^{r-1}(M)\} \ (r > 0), \quad \mathscr{B}^0(M) := \{0\}, \end{cases}$$

and $\mathscr{H}^r(M) := \mathscr{Z}^r(M)/\mathscr{B}^r(M)$ (the quotient linear space). With this setting, one can demonstrate the proposition below, because Corollary 2.14-(4) and Lemma 2.13-(iv) yield

$$\mathscr{Z}^r(M) \wedge \mathscr{Z}^{r'}(M) \subset \mathscr{Z}^{r+r'}(M), \quad \mathscr{Z}^r(M) \wedge \mathscr{B}^{r'}(M) \subset \mathscr{B}^{r+r'}(M) \quad (r, r' \in \mathbf{Z}_{\geq 0}).$$

Proposition 2.16. *Let* (M, I) *be a* $2n$*-dimensional paracomplex manifold. Then,*

(1) $\dim_{\mathbf{R}} \mathscr{H}^0(M)$ *is equal to the number of connected components of* M.

(2) $\dim_{\mathbf{R}} \mathscr{H}^r(M) = 0$ *if* $r > n$.

(3) $\mathscr{H}(M) := \bigoplus_{r=0}^{\infty} \mathscr{H}^r(M)$ *forms a real algebra with respect the following product*:

$$[\alpha] \cdot [\gamma] := [\alpha \wedge \gamma] \quad \text{for } [\alpha] \in \mathscr{H}^r(M),\ [\gamma] \in \mathscr{H}^{r'}(M).$$

2.3.2. Cohomology Groups $\overline{\mathscr{H}}^*(M)$. In the preceding paragraph we have constructed the cohomology group $\mathscr{H}^r(M)$ of a paracomplex manifold (M, I). Now, let

$$\overline{\Omega}^s(M) := \{\beta \in \mathscr{A}^{(0,s)}(M) \mid \partial\beta = 0\} \quad (s \in \mathbf{Z}_{\geq 0}).$$

In a similar way, one can see that $\overline{\Omega}(M) := \bigoplus_{s=0}^{\infty} \overline{\Omega}^s(M)$ is a cochain complex with coboundary operator $\bar{\partial}$, and define the s-dimensional cohomology group $\overline{\mathscr{H}}^s(M)$ of this cochain complex. Here $\overline{\mathscr{Z}}^s(M) := \{\beta \in \overline{\Omega}^s(M) \mid \bar{\partial}\beta = 0\}$, $\overline{\mathscr{B}}^s(M) := \{\bar{\partial}\gamma \mid \gamma \in \overline{\Omega}^{s-1}(M)\}$ $(s > 0)$, $\overline{\mathscr{B}}^0(M) := \{0\}$ and $\overline{\mathscr{H}}^s(M) := \overline{\mathscr{Z}}^s(M)/\overline{\mathscr{B}}^s(M)$. Note that $\overline{\mathscr{H}}(M) := \bigoplus_{s=0}^{\infty} \overline{\mathscr{H}}^s(M)$ similarly forms a real algebra, and that $\mathscr{H}^r(M)$ is irrelevant to $\overline{\mathscr{H}}^s(M)$ in general, because we can prove

Theorem 2.17. *Let N_1 and N_2 be connected differentiable manifolds with $\dim_{\mathbf{R}} N_1 = \dim_{\mathbf{R}} N_2$. For the paracomplex structure $I_{N_1 \times N_2}$ of $N_1 \times N_2$ given in Remark* 2.4, *it follows that*

$$\dim_{\mathbf{R}} \mathscr{H}^k(N_1 \times N_2) = \dim_{\mathbf{R}} H^k(N_1), \quad \dim_{\mathbf{R}} \overline{\mathscr{H}}^k(N_1 \times N_2) = \dim_{\mathbf{R}} H^k(N_2)$$

for all $k \in \mathbf{Z}_{\geq 0}$. Here $H^k(N_j)$ stands for the k-dimensional de Rham cohomology group of N_j.

Proof. We fix coordinate neighborhoods $(U, (x^1, \ldots, x^n))$ of N_1 and $(W, (y^1, \ldots, y^n))$ of N_2. Then, $(U \times W, (x^1, \ldots, x^n, y^1, \ldots, y^n))$ is a paraholomorphic coordinate neighborhood of $(N_1 \times N_2, I_{N_1 \times N_2})$. Any $\alpha \in \Omega^r(N_1 \times N_2)$ and $\beta \in \overline{\Omega}^s(N_1 \times N_2)$ are expressed as

$$\alpha = \sum_{i_1 < \cdots < i_r} \alpha_{i_1 \cdots i_r}\, dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \quad \alpha_{i_1 \cdots i_r} = \alpha_{i_1 \cdots i_r}(x^1, \ldots, x^n),$$

$$\beta = \sum_{j_1 < \cdots < j_s} \beta_{j_1 \cdots j_s}\, dy^{j_1} \wedge \cdots \wedge dy^{j_s}, \quad \beta_{j_1 \cdots j_s} = \beta_{j_1 \cdots j_s}(y^1, \ldots, y^n)$$

on $U \times W$, respectively (cf. Corollary 2.14-(2)). Therefore one may assume that α and β belong to $\mathcal{D}^r(N_1)$ and $\mathcal{D}^s(N_2)$, respectively; and conversely, any $\omega \in \mathcal{D}^r(N_1)$ and $\eta \in \mathcal{D}^s(N_2)$ belong to $\Omega^r(N_1 \times N_2)$ and $\overline{\Omega}^s(N_1 \times N_2)$, respectively. Furthermore, on this assumption we have

$$\Omega^{r+1}(N_1 \times N_2) \ni \partial\alpha = \sum_{k=1}^{n} \sum_{i_1<\cdots<i_r} \frac{\partial \alpha_{i_1\cdots i_r}}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} = d\alpha \in \mathcal{D}^{r+1}(N_1),$$

$$\overline{\Omega}^{s+1}(N_1 \times N_2) \ni \overline{\partial}\beta = \sum_{k=1}^{n} \sum_{j_1<\cdots<j_s} \frac{\partial \beta_{j_1\cdots j_s}}{\partial y^k} dy^k \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_s} = d\beta \in \mathcal{D}^{s+1}(N_2).$$

Consequently we deduce that $\dim_{\mathbf{R}} \mathscr{H}^r(N_1 \times N_2) = \dim_{\mathbf{R}} H^r(N_1)$ for all $r \in \mathbf{Z}_{\geq 0}$, and that $\dim_{\mathbf{R}} \overline{\mathscr{H}}^s(N_1 \times N_2) = \dim_{\mathbf{R}} H^s(N_2)$ for all $s \in \mathbf{Z}_{\geq 0}$. $\square$

Theorem 2.17 enables us to compute cohomology groups $\mathscr{H}^*(M)$, $\overline{\mathscr{H}}^*(M)$ in some cases.

Example 2.18. For a circular cylinder $S^1 \times \mathbf{R}$,

$$\begin{cases} \dim_{\mathbf{R}} \mathscr{H}^0(S^1 \times \mathbf{R}) = \dim_{\mathbf{R}} \mathscr{H}^1(S^1 \times \mathbf{R}) = 1, & \dim_{\mathbf{R}} \mathscr{H}^k(S^1 \times \mathbf{R}) = 0 \text{ if } k \geq 2, \\ \dim_{\mathbf{R}} \overline{\mathscr{H}}^0(S^1 \times \mathbf{R}) = 1, & \dim_{\mathbf{R}} \overline{\mathscr{H}}^j(S^1 \times \mathbf{R}) = 0 \text{ if } j \geq 1. \end{cases}$$

cf. Remark 2.4.

2.4. Paraholomorphic Diffeomorphisms and Cohomology Groups. The following proposition is easy to prove, but we confirm it for the sake of completeness (see (2.15) for $\mathscr{Z}^k(M)$, $\mathscr{B}^k(M)$):

Proposition 2.19. *Let (M, I) and (M', I') be paracomplex manifolds, and let $\Psi : M \to M'$ be a paraholomorphic diffeomorphism. Then,*

(i) $\Psi^*(\mathscr{A}^{(r,s)}(M')) = \mathscr{A}^{(r,s)}(M)$ *for all* $r, s \in \mathbf{Z}_{\geq 0}$.

(ii) $\partial \circ \Psi^* = \Psi^* \circ \partial$ *and* $\overline{\partial} \circ \Psi^* = \Psi^* \circ \overline{\partial}$.

(iii) $\Psi^*(\Omega^k(M')) = \Omega^k(M)$ *and* $\Psi^*(\overline{\Omega}^k(M')) = \overline{\Omega}^k(M)$ *for all* $k \in \mathbf{Z}_{\geq 0}$.

(iv) *For every* $k \in \mathbf{Z}_{\geq 0}$, $\Psi^*(\mathscr{Z}^k(M')) = \mathscr{Z}^k(M)$, $\Psi^*(\mathscr{B}^k(M')) = \mathscr{B}^k(M)$, $\Psi^*(\overline{\mathscr{Z}}^k(M')) = \overline{\mathscr{Z}}^k(M)$ *and* $\Psi^*(\overline{\mathscr{B}}^k(M')) = \overline{\mathscr{B}}^k(M)$.

Therefore the mapping $\mathscr{H}^r(M') \ni [\alpha'] \mapsto [\Psi^*\alpha'] \in \mathscr{H}^r(M)$ is a linear isomorphism for every $r \in \mathbf{Z}_{\geq 0}$, and moreover it induces an algebra isomorphism of $\mathscr{H}(M')$ onto $\mathscr{H}(M)$. Similarly, $\overline{\mathscr{H}}^s(M') \cong \overline{\mathscr{H}}^s(M)$ and $\overline{\mathscr{H}}(M') \cong \overline{\mathscr{H}}(M)$ via Ψ^*.

Proof. (i) Let us take any $\omega' \in \mathscr{A}^{(r,s)}(M')$ and verify that $\Psi^*\omega' \in \mathscr{A}^{(r,s)}(M)$. From $\omega' \in \mathcal{D}^{r+s}(M')$ it is natural that $\Psi^*\omega' \in \mathcal{D}^{r+s}(M)$. For each $p \in M$, there exists a paraholomorphic coordinate neighborhood $(O', (x^1, \ldots, x^n,$

$y^1, \dots, y^n)$) of (M', I') such that (1) $\Psi(p) \in O'$ and (2) ω' is expressed as

$$\omega' = \sum_{i_1 < \cdots < i_r,\, j_1 < \cdots < j_s} \omega'_{i_1 \cdots i_r j_1 \cdots j_s}\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_s}$$

on O'. Since $\Psi : M \to M'$ is a paraholomorphic diffeomorphism we set

$$V := \Psi^{-1}(O'), \quad u^i := x^i \circ \Psi, \quad v^i := y^i \circ \Psi \quad (1 \le i \le n),$$

and assert that $(V, (u^1, \dots, u^n, v^1, \dots, v^n))$ is a paraholomorphic coordinate neighborhood of (M, I) containing the point p, and that

$$\Psi^*\omega' = \sum_{i_1 < \cdots < i_r,\, j_1 < \cdots < j_s} (\omega'_{i_1 \cdots i_r j_1 \cdots j_s} \circ \Psi)\, du^{i_1} \wedge \cdots \wedge du^{i_r} \wedge dv^{j_1} \wedge \cdots \wedge dv^{j_s}$$

on V. This implies $\Psi^*\omega' \in \mathscr{A}^{(r,s)}(M)$.

(ii) For any $\omega' \in \mathscr{A}^{(r,s)}(M')$, it follows from $d = \partial + \bar{\partial}$ and $d \circ \Psi^* = \Psi^* \circ d$ that

$$\partial(\Psi^*\omega') + \bar{\partial}(\Psi^*\omega') = d(\Psi^*\omega') = \Psi^*(d\omega') = \Psi^*(\partial\omega') + \Psi^*(\bar{\partial}\omega').$$

This, combined with (i), $\partial(\Psi^*\omega'), \Psi^*(\partial\omega') \in \mathscr{A}^{(r+1,s)}(M)$ and $\bar{\partial}(\Psi^*\omega'), \Psi^*(\bar{\partial}\omega') \in \mathscr{A}^{(r,s+1)}(M)$, assures that $\partial(\Psi^*\omega') = \Psi^*(\partial\omega')$ and $\bar{\partial}(\Psi^*\omega') = \Psi^*(\bar{\partial}\omega')$.

(iii) comes from (i) and (ii).

(iv) comes from (ii) and (iii). □

By arguments similar to those in the proof of Proposition 2.19 we deduce

PROPOSITION 2.20. *Let (M, I) and (M', I') be paracomplex manifolds. Suppose that there exists an anti-paraholomorphic diffeomorphism $\Xi : M \to M'$. Then,*

(i) $\Xi^*(\mathscr{A}^{(r,s)}(M')) = \mathscr{A}^{(s,r)}(M)$ *for all* $r, s \in \mathbf{Z}_{\ge 0}$.

(ii) $\partial \circ \Xi^* = \Xi^* \circ \bar{\partial}$ *and* $\bar{\partial} \circ \Xi^* = \Xi^* \circ \partial$.

(iii) $\Xi^*(\Omega^r(M')) = \bar{\Omega}^r(M)$ *for all* $r \in \mathbf{Z}_{\ge 0}$.

(iv) *The mapping* $\mathscr{H}^r(M') \ni [\alpha'] \mapsto [\Xi^*\alpha'] \in \overline{\mathscr{H}}^r(M)$ *is a linear isomorphism for every* $r \in \mathbf{Z}_{\ge 0}$.

REMARK 2.21. Proposition 2.20-(iv) and Example 2.18 imply that for the paracomplex structure $I_{S^1 \times \mathbf{R}}$ of $S^1 \times \mathbf{R}$ given in Remark 2.4, there are no anti-paraholomorphic diffeomorphisms of $S^1 \times \mathbf{R}$ onto itself, because of $\dim_{\mathbf{R}} \mathscr{H}^1(S^1 \times \mathbf{R}) \neq \dim_{\mathbf{R}} \overline{\mathscr{H}}^1(S^1 \times \mathbf{R})$.

3. Structures on Hyperbolic Adjoint Orbits and Real Flag Manifolds

3.1. The Definition of Hyperbolic Element. In this subsection we establish Proposition 3.7 which will play a role later and is applicable to hyperbolic adjoint orbits. Here, the definition of hyperbolic adjoint orbit is as follows:

Definition 3.1 (cf. Kobayashi [7, p. 5]). Let $\mathfrak{g}$ be a real semisimple Lie algebra, and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then, an element $S \in \mathfrak{g}$ is said to be *hyperbolic*, if $\operatorname{ad} S$ is a semisimple linear transformation of $\mathfrak{g}$ and all the eigenvalues of $\operatorname{ad} S$ in $\mathfrak{g}$ are real. The adjoint orbit $\operatorname{Ad} G(S) = G/C_G(S)$ of G through a hyperbolic element $S \in \mathfrak{g}$ is called a *hyperbolic adjoint orbit*.

Remark 3.2 (e.g. Helgason [3, p. 431, Theorem 7.2-(ii)]). An element $X \in \mathfrak{g}$ is hyperbolic if and only if there exists a Cartan involution θ_* of $\mathfrak{g}$ such that $\theta_*(X) = -X$.

From Lemma 3.3 we will generalize known facts and obtain Proposition 3.7.

Lemma 3.3. *Let P be a Lie group, let Q be a closed subgroup of P, and let R be a subgroup of P such that $Q_0 \subset R \subset Q$, where R is not necessary closed in P. Then,*

(1) *R is an open and closed subgroup of Q.*
(2) *R is a closed subgroup of P.*

Proof. (1) It suffices to confirm that R is an open subgroup of Q (cf. the proof of Proposition 1.93-(a) in Knapp [6, p. 84]). It is obvious that R is a subgroup of Q. Let us prove that R is open in Q. Since R is a group and $Q_0 \subset R$, we see that $xQ_0 \subset R$ for all $x \in R$. So, it follows from $e \in Q_0$ that

$$R = \bigcup_{x \in R} xQ_0,$$

where e is the identity element of P. By virtue of $R \subset Q$, the left translation of Q by any $x \in R$ is a homeomorphism of Q onto itself. Therefore xQ_0 is open in Q because so is Q_0. Consequently, $R = \bigcup_{x \in R} xQ_0$ is an open subset of Q.

(2) Q is closed in P, and R is closed in Q by (1). Thus R is closed in P. □

We want to state Proposition 3.7. First, let us fix the notation and setting of the proposition. Let G be a connected real semisimple Lie group, let S be a hyperbolic element of $\mathfrak{g}$, and let L be a subgroup of G such that $C_G(S)_0 \subset L \subset C_G(S)$. Then we set

$$(3.4) \qquad \begin{cases} \mathfrak{g}^{\lambda} := \{X \in \mathfrak{g} \mid \mathrm{ad}\, S(X) = \lambda X\} & \text{for } \lambda \in \mathbf{R}, \\ \mathfrak{u}^{\pm} := \bigoplus_{\lambda > 0} \mathfrak{g}^{\pm\lambda}, \quad U^{\pm} := \exp \mathfrak{u}^{\pm}, \quad Q^{\pm} := LU^{\pm}, \end{cases}$$

where $\mathfrak{g}^{\lambda} = \{0\}$ in the case where λ is different from the eigenvalues of $\mathrm{ad}\, S$ and $\exp : \mathfrak{g} \to G$ is the exponential mapping. In addition, let θ_* be a Cartan involution of $\mathfrak{g}$ satisfying $\theta_*(S) = -S$. Since the Lie group G is semisimple, θ_* is liftable to G. We denote its lift by θ, and define a closed subgroup K of G by

$$(3.5) \qquad K := G^{\theta} = \{k \in G \mid \theta(k) = k\}.$$

Next, let us prepare for the proof of Proposition 3.7.

Lemma 3.6. *With the setting* (3.4) *and* (3.5); *the following seven items hold*:

(i) *L is a closed subgroup of G with $\mathfrak{l} = \mathfrak{c}_{\mathfrak{g}}(S) = \mathfrak{g}^0$.*
(ii) $\mathfrak{g} = \bigoplus_{\lambda \in \mathbf{R}} \mathfrak{g}^{\lambda} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}^+$.
(iii) $\mathfrak{l} \oplus \mathfrak{u}^+ = \bigoplus_{\mu \geq 0} \mathfrak{g}^{\mu}$ *and* $\mathfrak{l} \oplus \mathfrak{u}^- = \bigoplus_{\mu \geq 0} \mathfrak{g}^{-\mu}$.
(iv) $\mathrm{Ad}\, x(\mathfrak{g}^{\lambda}) \subset \mathfrak{g}^{\lambda}$ *for all* $(x, \lambda) \in C_G(S) \times \mathbf{R}$.
(v) $[\mathfrak{g}^{\lambda}, \mathfrak{g}^{\mu}] \subset \mathfrak{g}^{\lambda+\mu}$ *for all* $\lambda, \mu \in \mathbf{R}$.
(vi) $\theta_*(\mathfrak{g}^{\lambda}) = \mathfrak{g}^{-\lambda}$ *for all* $\lambda \in \mathbf{R}$.
(vii) *Both* $\mathfrak{u}^+$ *and* $\mathfrak{u}^-$ *are subalgebras of* $\mathfrak{g}$ *such that* $\mathrm{Ad}\, x(\mathfrak{u}^+) \subset \mathfrak{u}^+$, $\mathrm{Ad}\, x(\mathfrak{u}^-) \subset \mathfrak{u}^-$ *for all* $x \in C_G(S)$, *and* $\theta_*(\mathfrak{u}^+) = \mathfrak{u}^-$, $\theta_*(\mathfrak{u}^-) = \mathfrak{u}^+$.

Proof. (i) follows by Lemma 3.3-(2), $C_G(S)_0 \subset L \subset C_G(S)$ and (3.4).

(ii) Since the element $S \in \mathfrak{g}$ is hyperbolic, one obtains $\mathfrak{g} = \bigoplus_{\lambda \in \mathbf{R}} \mathfrak{g}^{\lambda}$ from (3.4). Hence, we can assert (ii) because $\bigoplus_{\lambda \in \mathbf{R}} \mathfrak{g}^{\lambda} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}^+$ is an easy consequence of (3.4) and $\mathfrak{l} = \mathfrak{g}^0$.

(iii) is immediate from (3.4) and $\mathfrak{l} = \mathfrak{g}^0$.

(iv) One has (iv) by a direct computation and (3.4).

(v) comes from (3.4) and the Jacobi identity in $\mathfrak{g}$.

(vi) From $\theta_*(S) = -S$ and (3.4) we deduce $\theta_*(\mathfrak{g}^{\lambda}) = \mathfrak{g}^{-\lambda}$.

(vii) Since $\mathfrak{u}^{\pm} = \bigoplus_{\lambda > 0} \mathfrak{g}^{\pm\lambda}$, (v) implies that $[\mathfrak{u}^{\pm}, \mathfrak{u}^{\pm}] \subset \mathfrak{u}^{\pm}$, and thus both $\mathfrak{u}^{\pm}$ are subalgebras of $\mathfrak{g}$; moreover, (iv) and (vi) imply that $\mathrm{Ad}\, x(\mathfrak{u}^{\pm}) \subset \mathfrak{u}^{\pm}$ and $\theta_*(\mathfrak{u}^{\pm}) = \mathfrak{u}^{\mp}$, respectively. □

Now, we are in a position to state

Proposition 3.7. *With the setting* (3.4) *and* (3.5); *the following eight items hold*:

(1) U^s *is a simply connected, closed nilpotent subgroup of* G *whose Lie algebra is* $\mathfrak{u}^s$, *and* $\exp : \mathfrak{u}^s \to U^s$ *is a diffeomorphism, for each* $s = \pm$.
(2) $\theta(U^-) = U^+$ *and* $\theta(U^+) = U^-$.
(3) $Q^s = LU^s$ *is a closed subgroup of* G *such that* $Q^s = L \ltimes U^s = U^s \rtimes L$ (*semidirect*) *and* $\mathfrak{q}^s = \mathfrak{l} \oplus \mathfrak{u}^s$, *for each* $s = \pm$.
(4) *Both the product mappings* $U^+ \times U^- \times L \ni (u^+, u^-, x) \mapsto u^+u^-x \in G$ *and* $U^- \times U^+ \times L \ni (u^-, u^+, x) \mapsto u^-u^+x \in G$ *are embeddings whose images are open subsets of* G.[1]
(5) K *is a connected closed subgroup of* G, *and* $G = KQ^s$ *for each* $s = \pm$.
(6) *The mapping* $K/(K \cap Q^s) \ni k(K \cap Q^s) \mapsto kQ^s \in G/Q^s$ *is a diffeomorphism for each* $s = \pm$.
(7) *If* $\theta(L) \subset L$, *then* $K \cap Q^s = K \cap L$ *holds for each* $s = \pm$.
(8) *The center* $Z(G)$ *is finite if and only if all* K, G/Q^+ *and* G/Q^- *are compact.*

Proof. We investigate the case of $s = +$ only.

Let us prepare for the proof. From $\theta_*(S) = -S$, one can obtain a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $S \in \mathfrak{p}$. We fix a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ containing S, and denote by $\triangle = \triangle(\mathfrak{g}, \mathfrak{a})$ the (non-zero restricted) root system of $\mathfrak{g}$ relative to $\mathfrak{a}$. Moreover, we define a lexicographic linear ordering of the dual space $\mathfrak{a}^*$ such that $\alpha(S) \geq 0$ for all positive roots α. Let $\triangle_+$ be the subset of $\triangle$ consisting of all positive roots relative to this ordering. Setting $\mathfrak{g}_\beta := \{X \in \mathfrak{g} \mid \operatorname{ad} H(X) = \beta(H)X \text{ for all } H \in \mathfrak{a}\}$ for $\beta \in \triangle$ and $\mathfrak{n}_+ := \bigoplus_{\alpha \in \triangle_+} \mathfrak{g}_\alpha$, one has an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_+$; moreover it follows from $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{c}_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \triangle_+} \mathfrak{g}_{\pm\alpha}$, $S \in \mathfrak{a}$ and (3.4) that

$$\mathfrak{u}^+ = \bigoplus_{\lambda > 0} \mathfrak{g}^\lambda \subset \bigoplus_{\alpha \in \triangle_+} \mathfrak{g}_\alpha = \mathfrak{n}_+ \subset \bigoplus_{\mu \geq 0} \mathfrak{g}^\mu = \mathfrak{l} \oplus \mathfrak{u}^+. \tag{3.8}$$

Denote by $G = KAN_+$ the Iwasawa decomposition of G corresponding to the $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}_+$. On the one hand; we obtain $A \subset C_G(S)_0$ from $\mathfrak{a} \subset \mathfrak{c}_{\mathfrak{g}}(S)$ and $A = \exp \mathfrak{a}$. On the other hand; since both $\mathfrak{n}_+ \cap \mathfrak{c}_{\mathfrak{g}}(S)$ and $\mathfrak{u}^+$ are subalgebras of $\mathfrak{n}_+$ and $\mathfrak{n}_+ = (\mathfrak{n}_+ \cap \mathfrak{c}_{\mathfrak{g}}(S)) \oplus \mathfrak{u}^+$, one can see that $N_+ = \exp \mathfrak{n}_+ =$

[1] In case of $L = C_G(S)$ this is called the Gel'fand-Naimark decomposition.

$\exp(\mathfrak{n}_+ \cap \mathfrak{c}_\mathfrak{g}(S)) \exp \mathfrak{u}^+ \subset C_G(S)_0 U^+$ by Lemma 6.2 in Kostant [8, p. 124]. Then, it turns out that

$$A \subset C_G(S)_0, \quad N_+ \subset C_G(S)_0 U^+. \tag{3.9}$$

Now, we are ready to prove the eight items.

(1) N_+ is a simply connected closed nilpotent subgroup of G and $\exp : \mathfrak{n}_+ \to N_+$ is a diffeomorphism. Hence (1) holds because of $\mathfrak{u}^+ \subset \mathfrak{n}_+$ and $U^+ = \exp \mathfrak{u}^+$.

(2) The above (1) and $\theta_*(\mathfrak{u}^\mp) = \mathfrak{u}^\pm$ provide us with $\theta(U^\mp) = U^\pm$.

(3) It suffices to get the conclusion in case of $L = C_G(S)$, because one can generalize the conclusion from Lemma 3.3-(1). First of all, let us prove that $C_G(S)U^+ = C_G(S) \ltimes U^+$, namely

$$C_G(S)U^+ \text{ is the semidirect product of groups } C_G(S) \text{ and } U^+, \text{ with } U^+ \text{ normal.}$$

On the one hand; (1) and Lemma 3.6-(vii) imply that

$$xU^+x^{-1} \subset U^+ \quad \text{for all } x \in C_G(S). \tag{3.10}$$

On the other hand; we can assert that

$$C_G(S) \cap U^+ = \{e\}. \tag{3.11}$$

Indeed, let us take an arbitrary $y \in C_G(S) \cap U^+$. By virtue of $y \in U^+$ and (1), there exists a unique $Y \in \mathfrak{u}^+$ satisfying $y = \exp Y$. It follows from $y \in C_G(S)$ that $\mathrm{Ad}\, y(S) = S$. So, for any $t \in \mathbf{R}$ we obtain $y(\exp tS)y^{-1} = \exp tS$, and then $y = (\exp tS)y(\exp tS)^{-1}$. Therefore $\exp Y = \exp \mathrm{Ad}(\exp tS)Y$. This, together with $Y, \mathrm{Ad}(\exp tS)Y \in \mathfrak{u}^+$ and (1), assures that $Y = \mathrm{Ad}(\exp tS)Y$. Differentiating this $Y = \mathrm{Ad}(\exp tS)Y$ at $t = 0$ we have $0 = [S, Y]$. Thus $Y \in \mathfrak{c}_\mathfrak{g}(S) \cap \mathfrak{u}^+ = \{0\}$, and $y = \exp Y = e$. For this reason (3.11) holds. By (3.10) and (3.11) we see that $C_G(S)U^+ = C_G(S) \ltimes U^+ = U^+ \rtimes C_G(S)$; besides $C_G(S)U^+$ is a subgroup of G.

At this stage, the rest of proof of (3) is to demonstrate the following items (A) and (B):

(A) $C_G(S)U^+$ is a closed subset of G,

(B) $\mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+$ is the Lie algebra of $C_G(S)U^+$.

(A) Since $N_G(\bigoplus_{\mu \geq 0} \mathfrak{g}^\mu)$ is closed in G, we are going to conclude (A) by showing

$$C_G(S)U^+ = N_G\left(\bigoplus_{\mu \geq 0} \mathfrak{g}^\mu\right). \tag{A$'$}$$

Our first aim is to verify

$$C_G(S)U^+ \subset N_G\left(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu\right). \tag{3.12}$$

On the one hand; it is immediate from $\mathrm{Ad}\,x(\mathfrak{g}^\lambda)\subset\mathfrak{g}^\lambda$ that $\mathrm{Ad}\,x\left(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu\right)\subset \bigoplus_{\mu\geq 0}\mathfrak{g}^\mu$ for all $x\in C_G(S)$, so that

$$C_G(S)\subset N_G\left(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu\right).$$

On the other hand; since $[\mathfrak{g}^\lambda,\mathfrak{g}^\mu]\subset\mathfrak{g}^{\lambda+\mu}$ for all $\lambda,\mu\in\mathbf{R}$ and $\mathfrak{u}^+=\bigoplus_{\lambda>0}\mathfrak{g}^\lambda$ we confirm that $[\mathfrak{u}^+,\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu]=[\bigoplus_{\lambda>0}\mathfrak{g}^\lambda,\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu]\subset\bigoplus_{\nu>0}\mathfrak{g}^\nu$, and therefore (1) yields $U^+\subset N_G(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu)$. Consequently one obtains $C_G(S)U^+\subset N_G(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu)$ because $N_G(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu)$ is a group. Hence, we have shown (3.12). Our second aim is to confirm that the converse inclusion also holds, namely

$$N_G\left(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu\right)\subset C_G(S)U^+. \tag{3.13}$$

Take any $g\in N_G(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu)$. By $g\in G=KAN_+$ there exists a unique $(k,a,n)\in K\times A\times N_+$ satisfying

$$g=kan.$$

Here (3.9) and (3.12) imply $k=g(an)^{-1}\in K\cap N_G(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu)$. Accordingly $\mathrm{Ad}\,k(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu)\subset\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu$. So, $\theta(k)=k$ and $\theta_*(\mathfrak{g}^\lambda)\subset\mathfrak{g}^{-\lambda}$ give $\mathrm{Ad}\,k(\bigoplus_{\nu\geq 0}\mathfrak{g}^{-\nu})\subset\bigoplus_{\nu\geq 0}\mathfrak{g}^{-\nu}$. Therefore it follows from $\mathfrak{c}_\mathfrak{g}(S)=\mathfrak{g}^0$ and $\mathfrak{u}^+=\bigoplus_{\lambda>0}\mathfrak{g}^\lambda$ that

$$\begin{cases}\mathrm{Ad}\,k(\mathfrak{c}_\mathfrak{g}(S))=\mathrm{Ad}\,k(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu\cap\bigoplus_{\nu\geq 0}\mathfrak{g}^{-\nu})\subset(\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu\cap\bigoplus_{\nu\geq 0}\mathfrak{g}^{-\nu})=\mathfrak{c}_\mathfrak{g}(S),\\ \mathrm{Ad}\,k(\mathfrak{u}^+)=\mathrm{Ad}\,k([S,\mathfrak{u}^+])\subset[\mathrm{Ad}\,k(S),\bigoplus_{\mu\geq 0}\mathfrak{g}^\mu]\subset[\mathrm{Ad}\,k(S),\mathfrak{c}_\mathfrak{g}(S)\oplus\mathfrak{u}^+]\subset\mathfrak{u}^+,\end{cases}$$

where we remark that $\mathrm{ad}\,S:\mathfrak{u}^+\to\mathfrak{u}^+$ is linear isomorphic and $\mathrm{Ad}\,k(S)$ belongs to the center of $\mathfrak{c}_\mathfrak{g}(S)$. Note that $\mathfrak{c}_\mathfrak{g}(S)=(\mathfrak{k}\cap\mathfrak{c}_\mathfrak{g}(S))\oplus(\mathfrak{p}\cap\mathfrak{c}_\mathfrak{g}(S))$ and $\mathfrak{a}$ is a maximal Abelian subspace in $\mathfrak{p}\cap\mathfrak{c}_\mathfrak{g}(S)$. By virtue of $\mathrm{Ad}\,k(\mathfrak{c}_\mathfrak{g}(S))\subset\mathfrak{c}_\mathfrak{g}(S)$ and $\mathfrak{c}_\mathfrak{g}(S)=\mathfrak{a}\oplus\mathfrak{c}_\mathfrak{k}(\mathfrak{a})\oplus\bigoplus_{\gamma\in\blacktriangle}\mathfrak{g}_\gamma$, one has $\mathrm{Ad}\,k(\mathfrak{a})\subset\mathfrak{p}\cap\mathfrak{c}_\mathfrak{g}(S)$ and there exists an $x_0\in K\cap C_G(S)_0$ satisfying

$$\mathrm{Ad}(x_0k)(\mathfrak{a})=\mathfrak{a},\quad {}^t\mathrm{Ad}(x_0k)^{-1}(\blacktriangle_+)\subset\blacktriangle_+,$$

where $\blacktriangle:=\{\gamma\in\triangle\,|\,\gamma(S)=0\}$ and $\blacktriangle_+:=\blacktriangle\cap\triangle_+$. From $\mathrm{Ad}\,k(\mathfrak{u}^+)\subset\mathfrak{u}^+$ and $x_0\in C_G(S)$ we obtain $\mathrm{Ad}(x_0k)(\mathfrak{u}^+)\subset\mathfrak{u}^+$. This, combined with $\mathrm{Ad}(x_0k)(\mathfrak{a})=\mathfrak{a}$ and $\mathfrak{u}^+=\bigoplus_{\alpha\in\triangle_+-\blacktriangle_+}\mathfrak{g}_\alpha$, assures that ${}^t\mathrm{Ad}(x_0k)^{-1}(\triangle_+-\blacktriangle_+)\subset\triangle_+-\blacktriangle_+$. Consequently it follows that ${}^t\mathrm{Ad}(x_0k)^{-1}(\triangle_+) \subset \triangle_+$, and so $\mathrm{Ad}(x_0k) = \mathrm{id}$ on $\mathfrak{a}$ (e.g. Theorem 4.3.18 in Varadarajan [12, pp. 282–283]). Hence we see that

$$k \in C_G(S)$$

in view of $S \in \mathfrak{a}$ and $x_0 \in C_G(S)$. Therefore $g = kan \in C_G(S)C_G(S)_0 C_G(S)_0 U^+ \subset C_G(S)U^+$ by (3.9), and we have verified (3.13). This and (3.12) provide us with (A′). So, (A) follows.

(B) Let us show that $\mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+$ is the Lie algebra of $C_G(S)U^+$. From (A′) it follows that the Lie algebra of $C_G(S)U^+$ coincides with $\mathfrak{n}_\mathfrak{g}(\bigoplus_{\mu \geq 0} \mathfrak{g}^\mu)$. Accordingly it suffices to show that

$$\mathfrak{n}_\mathfrak{g}(\mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+) \subset \mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+, \tag{3.14}$$

since $\bigoplus_{\mu \geq 0} \mathfrak{g}^\mu = \mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+$. For an $X \in \mathfrak{g}$ suppose that $[X, \mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+] \subset \mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+$. By $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+$ there exists a unique $(X^-, X^0, X^+) \in \mathfrak{u}^- \times \mathfrak{c}_\mathfrak{g}(S) \times \mathfrak{u}^+$ such that $X = X^- + X^0 + X^+$. The supposition and $S \in \mathfrak{c}_\mathfrak{g}(S)$ imply $\mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+ \ni [X, S] = [X^-, S] + [X^+, S]$. Thus it follows from $[X^\pm, S] \in \mathfrak{u}^\pm$ that $[X^-, S] = 0$, so that $X^- \in \mathfrak{c}_\mathfrak{g}(S) \cap \mathfrak{u}^- = \{0\}$. Therefore $X = X^0 + X^+ \in \mathfrak{c}_\mathfrak{g}(S) \oplus \mathfrak{u}^+$, and (3.14) holds. That enables us to complete the proof of (3).

(4) We only prove that the mapping $U^- \times U^+ \times L \ni (b, a, x) \mapsto bax \in G$ is injective, since $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{u}^+ \oplus \mathfrak{l}$. First, let us verify

$$U^- \cap Q^+ = \{e\}. \tag{3.15}$$

Take any $y \in U^- \cap Q^+$. By $y \in U^-$ and (1) we get a unique $Y \in \mathfrak{u}^-$ such that $y = \exp Y$. (A′), $y \in Q^+ = LU^+$ and $L \subset C_G(S)$ allow us to show $y \in N_G(\bigoplus_{\mu \geq 0} \mathfrak{g}^\mu)$. Then, it follows from $S \in \mathfrak{g}^0$ that $\mathrm{Ad}\, y(S) \in \bigoplus_{\mu \geq 0} \mathfrak{g}^\mu$, and moreover

$$\bigoplus_{\mu \geq 0} \mathfrak{g}^\mu \ni \mathrm{Ad}\, y(S) - S = \sum_{n \geq 1} \frac{1}{n!} (\mathrm{ad}\, Y)^n S \in \bigoplus_{\lambda > 0} \mathfrak{g}^{-\lambda}$$

because $Y \in \mathfrak{u}^- = \bigoplus_{\lambda > 0} \mathfrak{g}^{-\lambda}$. This implies $\mathrm{Ad}\, y(S) - S = 0$, and $y \in C_G(S) \cap U^- = \{e\}$ by (3). Therefore (3.15) holds. Now, let us prove that $U^- \times U^+ \times L \ni (b, a, x) \mapsto bax \in G$ is injective. Suppose that

$$bax = b'a'x'$$

for $(b, a, x), (b', a', x') \in U^- \times U^+ \times L$. From (3) we deduce that $Q^+ \ni a'x'(ax)^{-1} = (b')^{-1} b \in U^-$, so that $a'x'(ax)^{-1} = e$ and $(b')^{-1} b = e$ due to (3.15). Moreover, it follows from $a'x'(ax)^{-1} = e$ and (3) that $a = a'$ and $x = x'$. Hence $b = b'$, $a = a'$ and $x = x'$.

(5) Since G is connected, K is also connected. In order to show $G = KQ^+$, it is enough to confirm that $G \subset KQ^+$, which comes from $G = KAN_+$, (3.9) and $AN_+ \subset C_G(S)_0 U^+ \subset LU^+ = Q^+$.

(6) is a consequence of (5).

(7) Suppose that $\theta(L) \subset L$. Let us prove $K \cap Q^+ \subset K \cap L$. For any $k \in K \cap Q^+$ there exists a unique $(x, u) \in L \times U^+$ such that

$$k = xu$$

because of $k \in Q^+$ and (3). Since $\theta(k) = k$ we see that $\theta(x)\theta(u) = xu$, and $\theta(u)u^{-1}(x^{-1}\theta(x)) = e$. Therefore the supposition, $(\theta(u), u^{-1}, x^{-1}\theta(x)) \in U^- \times U^+ \times L$ and (4) yield $\theta(u) = u^{-1} = x^{-1}\theta(x) = e$, especially $u = e$. Hence $k = x \in K \cap L$, and $K \cap Q^+ \subset K \cap L$. The converse inclusion $K \cap L \subset K \cap Q^+$ is obvious.

(8) It is known that the center $Z(G)$ is finite if and only if K is compact (e.g. Theorem 1.1-(i) in Helgason [3, pp. 252–253]). Thus (6) allows us to get the conclusion. □

Remark 3.16. Let us comment on Proposition 3.7, where $s = \pm$.

(i) $\mathfrak{q}^s = \mathfrak{l} \oplus \mathfrak{u}^s$ contains a Borel subalgebra $\mathfrak{a} \oplus \mathfrak{c}_{\mathfrak{k}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \triangle_+} \mathfrak{g}_{s\alpha}$ of $\mathfrak{g}$ and is a parabolic subalgebra of $\mathfrak{g}$ whose Levi factor and unipotent radical are $\mathfrak{l}$ and $\mathfrak{u}^s$, respectively.

(ii) In general $C_G(S)$ is not connected (cf. Example 4.13). For this reason, the condition $C_G(S)_0 \subset L \subset C_G(S)$ makes a sense.

(iii) The condition $\theta(L) \subset L$ in (7) always holds whenever $L = C_G(S)_0$ or $L = C_G(S)$.

(iv) In case of $L = C_G(S)$, it turns out that $Q^s = N_G(\mathfrak{l} \oplus \mathfrak{u}^s)$ and G/Q^s is a *real flag manifold*; besides G/Q^s is compact by virtue of $Z(G) \subset Q^s$.

(v) G/Q^s is called an *R-space*, if $L = C_G(S)$ and there exists a connected complex semisimple Lie group $G_{\mathbf{C}}$ such that (a) G is a closed subgroup of $G_{\mathbf{C}}$ and (b) $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbf{C}}$. Here Takeuchi [11, p. 100] has introduced the notion of R-space. If G/Q^s is an R-space, then it is compact because the inclusion $Z(G) \subset Z(G_{\mathbf{C}})$ forces $Z(G)$ to be finite.

3.2. Paraholomorphic Structures on Hyperbolic Adjoint Orbits. In this subsection, we first construct a G-invariant paracomplex structure I_G of the homogeneous space G/L and afterwards fix a paraholomorphic structure $\mathcal{S}_{G/L} = \{(O_g, \psi_g)\}_{g \in G}$ on $(G/L, I_G)$, where the setting in (3.4) and (3.5) remains valid here.

Now, let us construct a G-invariant paracomplex structure I_G of G/L. Define an involutive linear transformation $\imath$ of the real vector space $\mathfrak{u}^+ \oplus \mathfrak{u}^-$ by

$$\imath(A) := A \quad \text{for } A \in \mathfrak{u}^+, \qquad \imath(B) := -B \quad \text{for } B \in \mathfrak{u}^-. \tag{3.17}$$

Since $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}^+ \oplus \mathfrak{u}^-$ and $\operatorname{Ad} L(\mathfrak{u}^\pm) \subset \mathfrak{u}^\pm$, the homogeneous space G/L is reductive. Then, one can identify $T_o(G/L)$ with $\mathfrak{u}^+ \oplus \mathfrak{u}^-$, where o is the origin of G/L, and see that $\operatorname{Ad} x \circ \imath = \imath \circ \operatorname{Ad} x$ on $\mathfrak{u}^+ \oplus \mathfrak{u}^-$ for all $x \in L$ ($\because$ (3.17)); besides Proposition 3.7-(1), (2) assures $\dim_{\mathbf{R}} \mathfrak{u}^+ = \dim_{\mathbf{R}} \mathfrak{u}^-$. Accordingly we can obtain a G-invariant almost paracomplex structure I_G of G/L by setting

$$(I_G)_p v := (d\tau_g)_o(\imath((d\tau_{g^{-1}})_p v)) \quad \text{for } p = gL \in G/L \text{ and } v \in T_p(G/L). \tag{3.18}$$

About this I_G one has $T_p^\pm(G/L) = (d\tau_g)_o \mathfrak{u}^\pm$ for all $p = gL \in G/L$. Hence, it follows from $[\mathfrak{u}^\pm, \mathfrak{u}^\pm] \subset \mathfrak{u}^\pm$ that $[\mathfrak{X}^\pm(G/L), \mathfrak{X}^\pm(G/L)] \subset \mathfrak{X}^\pm(G/L)$, so that I_G is a G-invariant paracomplex structure of G/L due to Lemma 2.2-(3).

REMARK 3.19. We have constructed a G-invariant paracomplex structure of G/L. In fact, one can construct a G-invariant paraKähler metric on G/L further (cf. Theorem 3.8 in Hou-Deng-Kaneyuki-Nishiyama [4, p. 225]). In addition, it is known that in some cases G-invariant paracomplex structures of G/L are unique up to sign $\pm$ (e.g. Proposition 4.4 in Kaneyuki-Kozai [5, p. 96]).

Next, let us fix a paraholomorphic structure $\mathcal{S}_{G/L}$ on $(G/L, I_G)$, where I_G is the paracomplex structure constructed above. Take any real bases $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ of $\mathfrak{u}^+$ and $\mathfrak{u}^-$, respectively. In the first place, we are going to set a coordinate neighborhood of G/L containing the origin o. Proposition 3.7-(4) tells us that $O := U^+U^-L/L$ is an open neighborhood of $o \in G/L$, and moreover, for each $p \in O$ there exists a unique $(u^+, u^-) \in U^+ \times U^-$ satisfying $p = u^+u^-L$. Proposition 3.7-(1) then enables us to obtain unique $x^i, y^i \in \mathbf{R}$ such that $u^+ = \exp(\sum_{i=1}^n x^i A_i)$ and $u^- = \exp(\sum_{i=1}^n y^i B_i)$. Therefore one can define a mapping $\psi : O \to \mathbf{R}^{2n}$ by

$$\psi(p) := (x^1, \dots, x^n, y^1, \dots, y^n) \quad \text{for } p = \exp\Big(\sum\nolimits_{i=1}^n x^i A_i\Big) \exp\Big(\sum\nolimits_{i=1}^n y^i B_i\Big) L \in O;$$

and $(O, \psi = (x^1, \dots, x^n, y^1, \dots, y^n))$ is a coordinate neighborhood of G/L containing o. Furthermore, (3.17) and (3.18) imply that

$$I_G\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial}{\partial x^i}, \quad I_G\Big(\frac{\partial}{\partial y^i}\Big) = -\frac{\partial}{\partial y^i} \quad (1 \le i \le n) \tag{3.20}$$

on O. In the second place, for $g \in G$ we put $O_g := \tau_g(O)$, $\psi_g := \psi \circ \tau_{g^{-1}}$ and denote the local coordinate system in (O_g, ψ_g) by $(x_g^1, \dots, x_g^n, y_g^1, \dots, y_g^n)$. Since (3.20) and I_G is G-invariant, we conclude that

$$\mathcal{S}_{G/L} := \{(O_g, \psi_g = (x_g^1, \dots, x_g^n, y_g^1, \dots, y_g^n))\}_{g \in G} \tag{3.21}$$

is an atlas of G/L such that $I_G(\partial/\partial x_g^i) = \partial/\partial x_g^i$, $I_G(\partial/\partial y_g^i) = -\partial/\partial y_g^i$ $(1 \leq i \leq n)$ on each O_g; and this $(O_g, \psi_g = (x_g^1, \dots, x_g^n, y_g^1, \dots, y_g^n))$ is a paraholomorphic coordinate neighborhood of $(G/L, I_G)$.

We end this subsection with showing

Proposition 3.22. *With the setting* (3.4) *and* (3.5); *suppose that* $\theta(L) \subset L$. *Then for the invariant paracomplex structure* I_G *of* G/L *given in* (3.18), *there exists an anti-paraholomorphic diffeomorphism* Θ *of* G/L *onto itself.*

Proof. The supposition allows us to define a diffeomorphic transformation Θ of G/L by $\Theta(gL) := \theta(g)L$ for $gL \in G/L$. This Θ is anti-paraholomorphic by means of $\theta_*(S) = -S$ and (3.17). □

3.3. Differentiable Structures on Real Flag Manifolds. We obey the same setting as in Subsection 3.2. Let us fix a differentiable structure $\mathcal{S}_{G/Q^-}$ on the homogeneous space G/Q^-. Recall that $\{A_i\}_{i=1}^n$ is a real basis of $\mathfrak{u}^+$. Proposition 3.7-(1), (3), (4) enables us to define a coordinate neighborhood (O^+, ψ^+) of G/Q^- containing the origin eQ^- as follows:

$$\begin{cases} O^+ := U^+Q^-/Q^-, \\ \psi^+(q) := (z^1, \dots, z^n) \quad \text{for } q = \exp(\sum_{i=1}^n z^i A_i)Q^- \in O^+. \end{cases}$$

For $g \in G$ we set $O_g^+ := \tau_g(O^+)$, $\psi_g^+ := \psi^+ \circ \tau_{g^{-1}}$ and denote the local coordinate system in (O_g^+, ψ_g^+) by $(z_g^1, \dots, z_g^n)$. Then, it turns out that

$$\mathcal{S}_{G/Q^-} := \{(O_g^+, \psi_g^+ = (z_g^1, \dots, z_g^n))\}_{g \in G} \tag{3.23}$$

is an atlas of G/Q^-.

4. The Main Result and Its Related Topics

The main result in this paper is as follows (see Paragraph 2.3.1, (3.4) for $\mathscr{H}^r(G/L)$, Q^-):

Theorem 4.1. *Let G be a connected real semisimple Lie group, let S be a hyperbolic element of* $\mathfrak{g}$, *and let L be a subgroup of G such that*

$$C_G(S)_0 \subset L \subset C_G(S).$$

Then, for the invariant paracomplex structure I_G of G/L given in (3.18), *the cohomology group $\mathscr{H}^r(G/L)$ is linear isomorphic to the de Rham cohomology group $H^r(G/Q^-)$ for every $r \in \mathbf{Z}_{\geq 0}$.*

This theorem follows by Proposition 4.6 in Subsection 4.1.

4.1. A Link between Paraholomorphic Cohomology Groups of Hyperbolic Adjoint Orbits and the de Rham Cohomology Groups of Real Flag Manifolds. We obey the same setting as in Subsections 3.2 and 3.3.

Our first aim is to show Lemma 4.5, and we will deduce Proposition 4.6 from the lemma. Since $L \subset Q^-$ we can consider a surjection $\mathrm{Pr} : G/L \to G/Q^-$ defined by

$$\mathrm{Pr}(gL) := gQ^- \quad \text{for } gL \in G/L.$$

In Subsections 3.2 and 3.3 we have defined atlases $\mathcal{S}_{G/L} = \{(O_g, \psi_g = (x_g^1, \dots, x_g^n, y_g^1, \dots, y_g^n))\}_{g \in G}$ and $\mathcal{S}_{G/Q^-} = \{(O_g^+, \psi_g^+ = (z_g^1, \dots, z_g^n))\}_{g \in G}$, respectively. By means of these definitions, we can assert that for each $g \in G$ one has

$$\text{(4.2)} \qquad \mathrm{Pr}^{-1}(O_g^+) = O_g; \quad hu^- L \in O_g \quad \text{for all } (hL, u^-) \in O_g \times U^-$$

(cf. Proposition 3.7-(3)), and that Pr is expressed as

$$\text{(4.3)} \qquad \mathrm{Pr} : (x_g^1, \dots, x_g^n, y_g^1, \dots, y_g^n) \to (z_g^1, \dots, z_g^n), \quad x_g^i = z_g^i \circ \mathrm{Pr} \ (1 \leq i \leq n)$$

on O_g, so that

$$\text{(4.4)} \qquad (d\,\mathrm{Pr})_p : T_p^+(G/L) \to T_{\mathrm{Pr}(p)}(G/Q^-) \text{ is a linear isomorphism for any } p \in G/L.$$

Lemma 4.5. $\mathrm{Pr}^*(\mathcal{D}^r(G/Q^-)) \subset \Omega^r(G/L)$ *for all* $r \in \mathbf{Z}_{\geq 0}$.

Proof. Take an arbitrary $\xi \in \mathcal{D}^r(G/Q^-)$. We want to show that $\mathrm{Pr}^* \xi \in \Omega^r(G/L)$. It is clear that $\mathrm{Pr}^* \xi \in \mathcal{D}^r(G/L)$. Hence it suffices to confirm that $\mathrm{Pr}^* \xi$

is of type $(r,0)$ and $\bar{\partial}(\mathrm{Pr}^* \xi) = 0$. Suppose that ξ is expressed as

$$\xi = \sum\nolimits_{i_1 < \cdots < i_r} \xi_{i_1 \cdots i_r} \, dz_g^{i_1} \wedge \cdots \wedge dz_g^{i_r}, \quad \xi_{i_1 \cdots i_r} = \xi_{i_1 \cdots i_r}(z_g^1, \ldots, z_g^n)$$

on $(O_g^+, (z_g^1, \ldots, z_g^n))$. Then, in view of (4.2) and (4.3), $\mathrm{Pr}^* \xi$ is expressed as

$$\mathrm{Pr}^* \xi = \sum\nolimits_{i_1 < \cdots < i_r} f_{i_1 \cdots i_r} \, dx_g^{i_1} \wedge \cdots \wedge dx_g^{i_r}, \quad f_{i_1 \cdots i_r} = f_{i_1 \cdots i_r}(x_g^1, \ldots, x_g^n, y_g^1, \ldots, y_g^n)$$

on $(O_g, (x_g^1, \ldots, x_g^n, y_g^1, \ldots, y_g^n))$. Accordingly the differential form $\mathrm{Pr}^* \xi$ is of type $(r,0)$. Furthermore, since $\mathrm{Pr}(gu^+u^-L) = \mathrm{Pr}(gu^+L)$ for all $u^\pm \in U^\pm$ we see that the function $f_{i_1 \cdots i_r} = \xi_{i_1 \cdots i_r} \circ \mathrm{Pr}$ is independent of the variables $y_g^1, \ldots, y_g^n$, and $\partial f_{i_1 \cdots i_r} / \partial y_g^k = 0$. So, it follows that

$$\bar{\partial}(\mathrm{Pr}^* \xi) = \sum_{k=1}^{n} \sum_{i_1 < \cdots < i_r} \frac{\partial f_{i_1 \cdots i_r j_1 \cdots j_s}}{\partial y_g^k} \, dy_g^k \wedge dx_g^{i_1} \wedge \cdots \wedge dx_g^{i_r} = 0. \qquad \square$$

Now, let us demonstrate

Proposition 4.6. *For each* $r \in \mathbf{Z}_{\geq 0}$ *there exists a linear isomorphism* $\zeta_r : \Omega^r(G/L) \to \mathcal{D}^r(G/Q^-)$ *such that*

(1) $\mathrm{Pr}^* \circ \zeta_r = \mathrm{id}$ *on* $\Omega^r(G/L)$,

(2) $\zeta_r \circ \mathrm{Pr}^* = \mathrm{id}$ *on* $\mathcal{D}^r(G/Q^-)$,

(3) $d \circ \zeta_r = \zeta_{r+1} \circ \partial$.

Proof. Take any $\alpha \in \Omega^r(G/L)$. Fix an arbitrary $q \in G/Q^-$ and $w_1, \ldots, w_r \in T_q(G/Q^-)$. For each point $p \in \mathrm{Pr}^{-1}(q)$ there exist unique $v_1, \ldots, v_r \in T_p^+(G/L)$ satisfying $(d\,\mathrm{Pr})_p v_a = w_a$ $(1 \leq a \leq r)$ by (4.4), and then we put

$$(\zeta_r(\alpha))_q(w_1, \ldots, w_r) := \alpha_p(v_1, \ldots, v_r). \tag{4.7}$$

Our first aim is to confirm that (4.7) is well-defined. Suppose that $p' \in \mathrm{Pr}^{-1}(q)$ and $v_a' \in T_{p'}^+(G/L)$ satisfies $(d\,\mathrm{Pr})_{p'} v_a' = w_a$ $(1 \leq a \leq r)$. For the aim, it suffices to verify

$$\alpha_p(v_1, \ldots, v_r) = \alpha_{p'}(v_1', \ldots, v_r'). \tag{4.8}$$

On the one hand; if the point p is expressed as $p = hL$, then it follows from $\mathrm{Pr}(p) = q = \mathrm{Pr}(p')$ and $Q^- = U^-L$ that there exists an $u^- \in U^-$ satisfying

$p' = hu^- L$. On the other hand; by virtue of $q \in G/Q^- = \bigcup_{g \in G} O_g^+$ there exists a $g \in G$ such that $q \in O_g^+$. Therefore (4.2) yields $p = hL \in \mathrm{Pr}^{-1}(q) \subset \mathrm{Pr}^{-1}(O_g^+) = O_g$ and $p' = hu^- L \in O_g$—that is,

$$p = hL, \quad p' = hu^- L \in O_g.$$

Now, let us express the vector $w_a \in T_q(G/Q^-)$ as $w_a = \sum_{i=1}^n \lambda_a^i (\partial/\partial z_g^i)_q$ for $a = 1, \dots, r$. Then (4.3), (4.4) and $(d\,\mathrm{Pr})_p v_a = w_a$, $(d\,\mathrm{Pr})_{p'} v_a' = w_a$ enable us to show that $v_a = \sum_{i=1}^n \lambda_a^i (\partial/\partial x_g^i)_p$, $v_a' = \sum_{i=1}^n \lambda_a^i (\partial/\partial x_g^i)_{p'}$ for all $1 \le a \le r$. Consequently we have

$$\text{(4.9)} \quad \begin{cases} \alpha_p(v_1, \dots, v_r) = \displaystyle\sum_{i,j,\dots,k=1}^n \lambda_1^i \lambda_2^j \cdots \lambda_r^k \alpha\left(\left(\frac{\partial}{\partial x_g^i}\right)_p, \left(\frac{\partial}{\partial x_g^j}\right)_p, \dots, \left(\frac{\partial}{\partial x_g^k}\right)_p\right) \\ \qquad\qquad\qquad = \displaystyle\sum_{i,j,\dots,k=1}^n \lambda_1^i \lambda_2^j \cdots \lambda_r^k \alpha_{ij\cdots k}(p); \\ \alpha_{p'}(v_1', \dots, v_r') = \displaystyle\sum_{i,j,\dots,k=1}^n \lambda_1^i \lambda_2^j \cdots \lambda_r^k \alpha_{ij\cdots k}(p'). \end{cases}$$

Here the definition of $(O_g, \psi_g = (x_g^1, \dots, x_g^n, y_g^1, \dots, y_g^n))$, together with $u^- \in U^-$, assures that

$$x_g^i(p) = x_g^i(hL) = x_g^i(hu^- L) = x_g^i(p') \quad \text{for all } 1 \le i \le n.$$

For this reason one can conclude (4.8) by $\alpha \in \Omega^r(G/L)$, Corollary 2.14-(2) and (4.9). Accordingly (4.7) is well-defined.

Since (4.7) is well-defined, $\partial/\partial x_g^i \in T^+(G/L)$ and $\partial/\partial z_g^i = d\,\mathrm{Pr}(\partial/\partial x_g^i)$, one can assert that on O_g^+,

$$\text{(4.10)} \qquad (\zeta_r(\alpha))\left(\frac{\partial}{\partial z_g^{i_1}}, \frac{\partial}{\partial z_g^{i_2}}, \dots, \frac{\partial}{\partial z_g^{i_r}}\right) = \alpha\left(\frac{\partial}{\partial x_g^{i_1}}, \frac{\partial}{\partial x_g^{i_2}}, \dots, \frac{\partial}{\partial x_g^{i_r}}\right) \circ \gamma_g.$$

Here γ_g is a local cross-section on O_g^+ (namely, $\mathrm{Pr} \circ \gamma_g = \mathrm{id}$ on O_g^+) defined as follows:

$$\gamma_g : O_g^+ \to O_g, \quad gu^+ Q^- \mapsto gu^+ L.$$

From (4.10) we deduce that $\zeta_r(\alpha)$ is of class C^∞ and $\zeta_r(\alpha) \in \mathcal{D}^r(G/Q^-)$. It is immediate from (4.7) that $\zeta_r : \Omega^r(G/L) \to \mathcal{D}^r(G/Q^-)$, $\alpha \mapsto \zeta_r(\alpha)$, is a linear mapping. Furthermore, it holds that for any $\tilde{\alpha} \in \Omega^r(G/L)$, $\tilde{p} \in G/L$ and $\tilde{v}_1, \dots, \tilde{v}_r \in T^+_{\tilde{p}}(G/L)$,

$$\tilde{\alpha}_{\tilde{p}}(\tilde{v}_1, \dots, \tilde{v}_r) \overset{(4.7)}{=} (\zeta_r(\tilde{\alpha}))_{\mathrm{Pr}(\tilde{p})}((d\,\mathrm{Pr})_{\tilde{p}}\tilde{v}_1, \dots, (d\,\mathrm{Pr})_{\tilde{p}}\tilde{v}_r) = (\mathrm{Pr}^*(\zeta_r(\tilde{\alpha})))_{\tilde{p}}(\tilde{v}_1, \dots, \tilde{v}_r).$$

This yields (1) $\mathrm{Pr}^* \circ \zeta_r = \mathrm{id}$, since $\tilde{\alpha}, \mathrm{Pr}^*(\zeta_r(\tilde{\alpha})) \in \mathscr{A}^{(r,0)}(G/L)$, Remark 2.11-(iii) and $T_{\tilde{p}}(G/L) = T^+_{\tilde{p}}(G/L) \oplus T^-_{\tilde{p}}(G/L)$.

From now on, we are going to prove (2) $\zeta_r \circ \mathrm{Pr}^* = \mathrm{id}$. Take any $\xi \in \mathcal{D}^r(G/Q^-)$ and suppose it to be expressed as

$$\xi = \sum_{i_1 < \cdots < i_r} \xi_{i_1 \cdots i_r}\, dz_g^{i_1} \wedge \cdots \wedge dz_g^{i_r}, \quad \xi_{i_1 \cdots i_r} = \xi_{i_1 \cdots i_r}(z_g^1, \dots, z_g^n)$$

on $(O_g^+, (z_g^1, \dots, z_g^n))$. Then, (4.3) implies that

$$\mathrm{Pr}^*\,\xi = \sum_{i_1 < \cdots < i_r} (\xi_{i_1 \cdots i_r} \circ \mathrm{Pr})\, dx_g^{i_1} \wedge \cdots \wedge dx_g^{i_r}$$

on $(O_g, (x_g^1, \dots, x_g^n, y_g^1, \dots, y_g^n))$, and so it follows from Lemma 4.5, (4.10) and $\mathrm{Pr} \circ \gamma_g = \mathrm{id}$ that

$$\begin{aligned}\zeta_r(\mathrm{Pr}^*\,\xi) &= \sum_{i_1 < \cdots < i_r} ((\xi_{i_1 \cdots i_r} \circ \mathrm{Pr}) \circ \gamma_g)\, dz_g^{i_1} \wedge \cdots \wedge dz_g^{i_r} \\ &= \sum_{i_1 < \cdots < i_r} \xi_{i_1 \cdots i_r}\, dz_g^{i_1} \wedge \cdots \wedge dz_g^{i_r} = \xi\end{aligned}$$

on O_g^+. Therefore (2) holds also. (1) and (2) imply that the linear mapping $\zeta_r : \Omega^r(G/L) \to \mathcal{D}^r(G/Q^-)$, $\alpha \mapsto \zeta_r(\alpha)$, is isomorphic.

The rest of proof is to conclude (3) $d \circ \zeta_r = \zeta_{r+1} \circ \partial$. By Corollary 2.14-(2), any $\alpha \in \Omega^r(G/L)$ is expressed as

$$\alpha = \sum_{i_1 < \cdots < i_r} \alpha_{i_1 \cdots i_r}\, dx_g^{i_1} \wedge \cdots \wedge dx_g^{i_r}, \quad \alpha_{i_1 \cdots i_r} = \alpha_{i_1 \cdots i_r}(x_g^1, \dots, x_g^n)$$

on $(O_g, (x_g^1, \dots, x_g^n, y_g^1, \dots, y_g^n))$. On the one hand; (4.10) implies that $\zeta_r(\alpha) = \sum_{i_1 < \cdots < i_r} (\alpha_{i_1 \cdots i_r} \circ \gamma_g)\, dz_g^{i_1} \wedge \cdots \wedge dz_g^{i_r}$, and therefore

$$d(\zeta_r(\alpha)) = \sum_{k=1}^{n} \sum_{i_1 < \cdots < i_r} \frac{\partial(\alpha_{i_1 \cdots i_r} \circ \gamma_g)}{\partial z_g^k}\, dz_g^k \wedge dz_g^{i_1} \wedge \cdots \wedge dz_g^{i_r} \tag{4.11}$$

on $(O_g^+, (z_g^1, \dots, z_g^n))$. On the other hand; since $\partial\alpha = \sum_{k=1}^n \sum_{i_1<\cdots<i_r} (\partial\alpha_{i_1\cdots i_r}/\partial x_g^k)\, dx_g^k \wedge dx_g^{i_1} \wedge \cdots \wedge dx_g^{i_r}$ and (4.10) we see that

$$\zeta_{r+1}(\partial\alpha) = \sum_{k=1}^n \sum_{i_1<\cdots<i_r} \left(\frac{\partial\alpha_{i_1\cdots i_r}}{\partial x_g^k} \circ \gamma_g\right) dz_g^k \wedge dz_g^{i_1} \wedge \cdots \wedge dz_g^{i_r}. \tag{4.12}$$

Here, the derivation of composite function, $\partial\alpha_{i_1\cdots i_r}/\partial y_g^j = 0$, $x_g^j = z_g^j \circ \mathrm{Pr}$ and $\mathrm{Pr} \circ \gamma_g = \mathrm{id}$ yield

$$\begin{aligned}\frac{\partial(\alpha_{i_1\cdots i_r} \circ \gamma_g)}{\partial z_g^k}(q') &= \sum_{j=1}^n \frac{\partial\alpha_{i_1\cdots i_r}}{\partial x_g^j}(\gamma_g(q')) \cdot \frac{\partial(x_g^j \circ \gamma_g)}{\partial z_g^k}(q') = \sum_{j=1}^n \frac{\partial\alpha_{i_1\cdots i_r}}{\partial x_g^j}(\gamma_g(q')) \cdot \delta_k^j \\ &= \frac{\partial\alpha_{i_1\cdots i_r}}{\partial x_g^k}(\gamma_g(q'))\end{aligned}$$

for all $q' \in O_g^+$. For this reason, (4.11) coincides with (4.12). Thus (3) follows. □

We are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. For $r \in \mathbf{Z}_{\geq 0}$ we set

$$\begin{cases} Z^r(G/Q^-) := \{\xi \in \mathcal{D}^r(G/Q^-) \mid d\xi = 0\}, \\ B^r(G/Q^-) := \{d\xi \mid \xi \in \mathcal{D}^{r-1}(G/Q^-)\} \ (r > 0), \quad B^0(G/Q^-) := \{0\}. \end{cases}$$

About the linear isomorphism $\zeta_r : \Omega^r(G/L) \to \mathcal{D}^r(G/Q^-)$ in Proposition 4.6 it follows that $\zeta_r(\mathscr{Z}^r(G/L)) = Z^r(G/Q^-)$, $\zeta_r(\mathscr{B}^r(G/L)) = B^r(G/Q^-)$ by virtue of (2.15) and Proposition 4.6-(3). Accordingly, $\mathscr{H}^r(G/L) = \mathscr{Z}^r(G/L)/\mathscr{B}^r(G/L)$ is linear isomorphic to $H^r(G/Q^-) = Z^r(G/Q^-)/B^r(G/Q^-)$ via $[\alpha] \mapsto [\zeta_r(\alpha)]$ for every $r \in \mathbf{Z}_{\geq 0}$. □

4.2. An Appendix: A Circular Cylinder and a Hyperboloid of One Sheet are Diffeomorphic, But Not Paraholomorphically Diffeomorphic. Let us give an example.

Example 4.13. Let $G = SL(1+n, \mathbf{R})$ and

$$S = \begin{pmatrix} n & \mathbf{0} \\ {}^t\mathbf{0} & -I_n \end{pmatrix},$$

where $\mathbf{0} = (0, \dots, 0) \in \mathbf{R}^n$ and I_n stands for the identity matrix of degree n. Define a Cartan involution θ of G by $\theta(g) := {}^tg^{-1}$ for $g \in G$. Then $\theta_*(S) = -S$ and S is a hyperbolic element of $\mathfrak{g} = \mathfrak{sl}(1+n, \mathbf{R})$. Moreover, it turns out that

$$C_G(S) = \left\{ \begin{pmatrix} a & \mathbf{0} \\ {}^t\mathbf{0} & B_n \end{pmatrix} \middle| \, a \in GL(1, \mathbf{R}),\ B_n \in GL(n, \mathbf{R}),\ a \det B_n = 1 \right\}$$
$$= S(GL(1, \mathbf{R}) \times GL(n, \mathbf{R}))$$

and $G^\theta = K = SO(1+n)$. Note that $C_G(S)$ consists of two connected components, and set

$$L = C_G(S)_0 \quad \text{or} \quad C_G(S).$$

With this setting we conclude that

$$\mathfrak{l} = \left\{ \begin{pmatrix} z & \mathbf{0} \\ {}^t\mathbf{0} & W_n \end{pmatrix} \middle| \, z \in \mathbf{R},\ W_n \in \mathfrak{gl}(n, \mathbf{R}),\ z + \mathrm{tr}(W_n) = 0 \right\},$$

$$\mathfrak{u}^+ = \mathfrak{g}^{1+n} = \left\{ \begin{pmatrix} 0 & \boldsymbol{u} \\ {}^t\mathbf{0} & O_n \end{pmatrix} \middle| \, \boldsymbol{u} \in \mathbf{R}^n \right\},$$

$$\mathfrak{u}^- = \mathfrak{g}^{-(1+n)} = \left\{ \begin{pmatrix} 0 & \mathbf{0} \\ {}^t\boldsymbol{v} & O_n \end{pmatrix} \middle| \, \boldsymbol{v} \in \mathbf{R}^n \right\}.$$

- In case of $L = C_G(S)_0$, one has

$$L = \left\{ \begin{pmatrix} \lambda & \mathbf{0} \\ {}^t\mathbf{0} & B_n \end{pmatrix} \middle| \, \lambda > 0,\ B_n \in GL(n, \mathbf{R}),\ \lambda \det B_n = 1 \right\}$$
$$= S(GL(1, \mathbf{R}) \times GL(n, \mathbf{R}))_0,$$

$$Q^- = \left\{ \begin{pmatrix} 1 & \mathbf{0} \\ {}^t\boldsymbol{v} & I_n \end{pmatrix} x \middle| \, \boldsymbol{v} \in \mathbf{R}^n,\ x \in L \right\},$$

$$K \cap Q^- = K \cap L = \left\{ \begin{pmatrix} 1 & \mathbf{0} \\ {}^t\mathbf{0} & X_n \end{pmatrix} \middle| \, X_n \in SO(n) \right\} = SO(n).$$

Accordingly $G/Q^- = K/(K \cap L) = SO(1+n)/SO(n)$ is an n-dimensional sphere, and therefore $\dim_{\mathbf{R}} H^r(G/Q^-) = 1$ if $r = 0$ or n, and $\dim_{\mathbf{R}} H^r(G/Q^-) = 0$ if $0 < r < n$. Hence Theorem 4.1 and two Propositions 2.20-(iv) and 3.22 tell us that

$$\dim_{\mathbf{R}} \mathscr{H}^r(G/L) = \dim_{\mathbf{R}} \overline{\mathscr{H}}^r(G/L) = \begin{cases} 1 & \text{if } r = 0 \text{ or } n, \\ 0 & \text{if } 0 < r < n \end{cases}$$

for the invariant paracomplex structure I_G of $G/L = SL(1+n, \mathbf{R})/S(GL(1, \mathbf{R}) \times GL(n, \mathbf{R}))_0$ given in (3.18).

· In case of $L = C_G(S)$, we see that $G/L = SL(1+n, \mathbf{R})/S(GL(1, \mathbf{R}) \times GL(n, \mathbf{R}))$, $G/Q^- = K/(K \cap L) = SO(1+n)/S(O(1) \times O(n))$ is an n-dimensional real projective space, and

$$\dim_{\mathbf{R}} \mathscr{H}^r(G/L) = \dim_{\mathbf{R}} \overline{\mathscr{H}}^r(G/L) = \begin{cases} 1 & \text{if } r = 0 \text{ or } n, \\ 0 & \text{if } 0 < r < n \end{cases} \quad (n = 2k+1);$$

$$\begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{if } 0 < r \leq n \end{cases} \quad (n = 2k).$$

On the one hand; Example 4.13 implies that

$$\dim_{\mathbf{R}} \mathscr{H}^1(F^2) = \dim_{\mathbf{R}} \overline{\mathscr{H}}^1(F^2) = 1$$

for the invariant paracomplex structure $I_{SL(2,\mathbf{R})}$ of $F^2 = SL(2, \mathbf{R})/S(GL(1, \mathbf{R}) \times GL(1, \mathbf{R}))$ given in (3.18). On the other hand; Example 2.18 implies that

$$\dim_{\mathbf{R}} \mathscr{H}^1(S^1 \times \mathbf{R}) = 1, \quad \dim_{\mathbf{R}} \overline{\mathscr{H}}^1(S^1 \times \mathbf{R}) = 0$$

for the paracomplex structure $I_{S^1 \times \mathbf{R}}$ of $S^1 \times \mathbf{R}$ given in Remark 2.4. Accordingly, it follows from Propositions 2.19 and 2.20 that the hyperboloid $(F^2, I_{SL(2,\mathbf{R})})$ of one sheet and the circular cylinder $(S^1 \times \mathbf{R}, I_{S^1 \times \mathbf{R}})$ are neither paraholomorphically nor anti-paraholomorphically diffeomorphic to each other.

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