

A FORMULA FOR THE HURWITZ ZETA FUNCTION, RIEMANN'S FUNCTIONAL EQUATION AND CERTAIN INTEGRAL REPRESENTATIONS

By

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Abstract. There is a well-known formula for the Hurwitz zeta function which implies the functional equation for the Riemann zeta function. We give a new proof of that formula and recover certain integral representations for the Hurwitz and Riemann zeta functions.

1. Introduction

The Riemann zeta function $\zeta(s)$ is the unique meromorphic function on \mathbf{C} , which, for $\operatorname{Re}(s) > 1$, coincides with the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is well-known ([7], [9], [14], [34], [36]) that $\zeta(s)$ is analytic on $\mathbf{C} \setminus \{1\}$, has a simple pole at $s = 1$ with residue 1 and is given by the formula ([7], Theorem 2.4)

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2})} \left(\frac{1}{s(s-1)} + \frac{1}{2} \int_1^{\infty} (x^{s/2-1} + x^{-(s+1)/2}) (\vartheta(x) - 1) dx \right),$$

where $\vartheta(x)$ is the Jacobi theta function given by the formula

$$\vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}.$$

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Riemann ([28]) proved that $\zeta(s)$ satisfies the following functional equation:

$$\zeta(s) = 2\Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) (2\pi)^{s-1} \zeta(1-s). \quad (1)$$

One of the purposes of this paper is to give a new proof of the above functional equation. It is known ([14], [22], [27], [37]) that Euler ([16]) had discovered a version of Formula (1) for integer values of s , based on Abel summation for divergent series and certain reflection formulas that he had discovered for the zeta function, the gamma function and the Dirichlet beta function. Riemann gave two complete proofs of the functional equation in [28]: one based on contour integrals and one using the \mathfrak{g} -function and its Mellin transform. Another proof is based on the Riemann-Siegel formula ([14]). There are seven classical proofs of Formula (1) in [36]. A particularly noteworthy proof along with important generalizations was given in Tate's thesis ([35]) on Fourier Analysis in number fields. Several other proofs and generalizations obtained by various techniques exist in the literature: making no claim whatsoever of providing a complete list of suitable references, we confine ourselves to mentioning [23] for a proof using Lipschitz summation, [29] for another proof using Mellin transforms, [2], [39] for a generalization to Lerch zeta functions, [6] for a generalization to Dirichlet L -functions and [8], [18] for a generalization to automorphic L -functions and multi-variable settings.

We also recall that the Hurwitz zeta function $\zeta(s; x)$ is the unique meromorphic function of s which, for $x > 0$ and for $\operatorname{Re}(s) > 1$ coincides with the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

It is well-known ([9], [14], [19], [34], [36]) that, for each fixed $x > 0$, the function $\zeta(s; x)$ is analytic with respect to s on $\mathbf{C} \setminus \{1\}$ and has a simple pole at $s = 1$ with residue 1. Note that $\zeta(s; 1) = \zeta(s)$.

Moreover,

$$\zeta(s; x) = 2\Gamma(1-s) \sum_{k=1}^{\infty} \frac{\sin(2k\pi x + \frac{\pi}{2}s)}{(2k\pi)^{1-s}}, \quad (2)$$

for $0 < x \leq 1$ and $\operatorname{Re}(s) < 0$. Note that, for $x = 1$, Formula (2) becomes Formula (1).

In this paper we give a new proof of Formula (2) and thereby also a new proof of Formula (1). In the process, we deduce certain integral representations for the Hurwitz and Riemann zeta functions. The approach is based on calculations involving the Laplace-Mellin transforms of certain functions together with an inductive argument involving the range of the variable x in Formula 2. An effort has been made to keep the arguments as elementary and self-contained as possible.

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2. Auxiliary results

In this section, we recall some results about improper Riemann integrals that will be needed in the sequel. Since improper Riemann integrable functions are not always Lebesgue integrable (e.g. Example 2, page 275 in [1]), we have chosen to avoid the use of standard results of Lebesgue integration, like the Tonelli-Hobson theorem (Theorem 15.8 in [1]), in justifying certain steps in our proofs. We have opted instead to collect certain facts about improper Riemann integrals here, giving appropriate references where needed.

The first fact that will be needed is a Fubini-type statement for improper Riemann integrals which is a consequence of uniform convergence and seems to be particularly useful in cases of absence of absolute integrability. It is a slight restatement of a theorem given by Loya in [26]. We include it here for the sake of completeness:

THEOREM 2.1. *Let $a, b \in \mathbf{R}$ and $f(x, y)$ a continuous complex-valued function on $(a, \infty) \times (b, \infty)$ which satisfies the following three conditions:*

(1) *The improper Riemann integral*

$$\int_b^\infty f(x, y) dy$$

exists and converges uniformly for x in compact subintervals $[c, d]$ of (a, ∞) ,

(2) *the improper Riemann integral*

$$\int_a^\infty f(x, y) dx$$

exists and converges uniformly for y in compact subintervals $[c, d]$ of (b, ∞) ,

- (3) there exists a real-valued function $g(x)$ on (a, ∞) such that, for all compact subintervals $[c, d]$ of (b, ∞) , we have

$$\left| \int_c^d f(x, y) dy \right| \leq g(x) \quad \text{and} \quad \int_a^\infty g(x) dx \text{ exists.}$$

Then the improper iterated Riemann integrals

$$\int_a^\infty \int_b^\infty f(x, y) dy dx \quad \text{and} \quad \int_b^\infty \int_a^\infty f(x, y) dx dy$$

exist and are equal.

REMARK 2.2. The above integrals may be improper of mixed type, i.e. they may also be improper at a or at b .

The following consequence of Theorem 2.1 will be important:

COROLLARY 2.3. Let $a > 0$, $\lambda > 0$ and $0 < \operatorname{Re}(s) < 1$. Then

$$\int_0^\infty \frac{\sin(\lambda(u-a))}{u^s} du = \lambda^{s-1} \Gamma(1-s) \cos\left(a\lambda + \frac{\pi}{2}s\right).$$

PROOF.

$$\begin{aligned} \int_0^\infty \frac{\sin(\lambda(u-a))}{u^s} du &= \cos(a\lambda) \int_0^\infty \frac{\sin(\lambda u)}{u^s} du - \sin(a\lambda) \int_0^\infty \frac{\cos(\lambda u)}{u^s} du \\ &= \lambda^{s-1} \cos(a\lambda) \int_0^\infty \frac{\sin x}{x^s} dx - \lambda^{s-1} \sin(a\lambda) \int_0^\infty \frac{\cos x}{x^s} dx. \end{aligned}$$

The last two integrals appearing above are Mellin transforms (for the parameter $t = 1 - s$) of the functions $\sin x$ and $\cos x$ and are known as generalized Fresnel integrals. Their evaluations are well-known: they are listed in [5], pages 68 and 10, proven in [9], page 92, using contour integration, and also proven (for real values of s) in [26]. Taking also into account the reflection formula for the Gamma function ([9], page 90), we get that, for $0 < \operatorname{Re}(s) < 1$, the above combination of generalized Fresnel integrals equals

$$\begin{aligned} & \lambda^{s-1} \cos(a\lambda)\Gamma(1-s) \cos\left(\frac{\pi}{2}s\right) - \lambda^{s-1} \sin(a\lambda)\Gamma(1-s) \sin\left(\frac{\pi}{2}s\right) \\ &= \lambda^{s-1}\Gamma(1-s) \cos\left(a\lambda + \frac{\pi}{2}s\right). \quad \square \end{aligned}$$

The second fact that will be needed is a Dominated Convergence Theorem for improper Riemann integrals (following from Theorem 25.21, page 359 in [4]). Note that, in the presence of interesting counterexamples ([4], Exercises 25.U and 25.T), special care is needed to ensure that the limit function is improper Riemann integrable:

THEOREM 2.4. *Let $f_n(x)$ be a sequence of complex-valued functions which converges pointwise to a function $f(x)$ on (a, ∞) . Assume that the functions f_n and f are Riemann integrable over $[c, d]$, for all compact subintervals $[c, d]$ of (a, ∞) . Suppose also that there exists a real-valued function $g(x)$ on (a, ∞) such that*

$$\int_a^\infty g(x) dx \text{ exists and } |f_n(x)| \leq g(x), \quad \text{for all } x > a \text{ and for all } n \in \mathbf{N}.$$

Then the functions f_n and f are improper Riemann integrable on (a, ∞) and

$$\lim_{n \rightarrow \infty} \int_a^\infty f_n(x) dx = \int_a^\infty f(x) dx.$$

REMARK 2.5. The above integrals may be improper of mixed type, i.e. they may also be improper at the point a .

The third fact that will be needed is a Beppo Levi theorem for improper Riemann integrals (following from Theorem 2.3.7 in [30], page 126; see also [1], Theorems 10.25 and 10.33):

THEOREM 2.6. *Let $f_n(x)$ be a sequence of complex-valued functions such that the series*

$$\sum_{n=0}^{\infty} f_n(x)$$

converges pointwise to a function $f(x)$ on $(0, \infty)$. Assume that the functions f_n and f are improper Riemann integrable on $(0, \infty)$ and that the series

$$\sum_{n=0}^{\infty} \int_0^{\infty} |f_n(x)| dx$$

converges. Then

$$\int_0^{\infty} f(x) dx = \sum_{n=0}^{\infty} \int_0^{\infty} f_n(x) dx.$$

3. Laplace-Mellin transforms and integral formulas

If $\varphi(x)$ is a piecewise continuous real-valued function on $(0, \infty)$, its Laplace-Mellin transform $\hat{\varphi}_a(s)$ is a mixture of its Mellin transform and its Laplace transform. Specifically, for $a > 0$, define

$$\hat{\varphi}_a(s) = \int_0^{\infty} e^{-ax} x^{s-1} \varphi(x) dx.$$

The region of convergence of $\hat{\varphi}_a(s)$ depends on the asymptotic behaviour of $\varphi(x)$ near 0 and near ∞ . The general theory of the analytic properties of this transform is fully presented in [21].

The remainder of this section is devoted to calculations involving the Laplace-Mellin transform of the function $\varphi : (0, \infty) \rightarrow \mathbf{R}$ given by the formula

$$\varphi(x) = \frac{1}{x^2 + \lambda^2},$$

where $\lambda \in (0, \infty)$. We start with an easy lemma:

LEMMA 3.1. *Let $b > 0$, $r \in \mathbf{N}$ and $\operatorname{Re}(s) < 1$. Then*

$$\int_0^b \frac{(b-u)^r}{u^s} du = \frac{r! b^{r+1-s}}{(1-s)(2-s) \cdots (r+1-s)}.$$

PROOF. The claim is clearly true for $r = 0$. Assume that it is true for r . Using integration by parts,

$$\int_0^b \frac{(b-u)^{r+1}}{u^s} du = \frac{(b-u)^{r+1} u^{1-s}}{1-s} \Big|_{u=0}^b + \frac{r+1}{1-s} \int_0^b \frac{(b-u)^r}{u^{s-1}} du,$$

and the claim follows by induction. □

PROPOSITION 3.2. *Let $a > 0$, $\lambda > 0$ and $\operatorname{Re}(s) > 0$. Then*

$$\hat{\varphi}_a(s) = \int_0^\infty e^{-ax} \frac{x^{s-1}}{x^2 + \lambda^2} dx = \frac{\Gamma(s)}{\lambda} \int_0^\infty \frac{\sin(\lambda u)}{(a+u)^s} du.$$

PROOF. First note that, since $\varphi(x)$ is bounded near ∞ and analytic near 0, Lemma 1.3 in [21] implies that the Laplace-Mellin transform $\hat{\varphi}_a(s)$ converges for $\operatorname{Re}(s) > 0$. Now it is trivial to show that

$$\int_0^\infty e^{-xu} \sin(\lambda u) du = \frac{\lambda}{x^2 + \lambda^2}, \quad \text{for } x > 0.$$

Therefore,

$$\lambda \int_0^\infty e^{-ax} \frac{x^{s-1}}{x^2 + \lambda^2} dx = \int_0^\infty e^{-ax} x^{s-1} \int_0^\infty e^{-xu} \sin(\lambda u) du dx.$$

Consider the complex-valued function

$$f(x, u) = e^{-ax} x^{s-1} e^{-xu} \sin(\lambda u) \quad \text{on } (0, \infty) \times (0, \infty).$$

We will check that it satisfies the hypotheses of Theorem 2.1. We have

$$\int_0^\infty f(x, u) du = e^{-ax} x^{s-1} \frac{\lambda}{x^2 + \lambda^2}, \quad \int_0^\infty f(x, u) dx = \frac{\Gamma(s) \sin(\lambda u)}{(a+u)^s}.$$

Also, let $I = [c, d]$ be a compact subinterval of $(0, \infty)$. Let $\sigma = \operatorname{Re}(s)$. Choose a positive integer n such that $n \geq \sigma + 1$. If $y > z > 0$, then

$$\begin{aligned} \left| \int_z^y f(x, u) dx \right| &\leq \int_z^y e^{-(a+u)x} x^{\sigma-1} dx \leq \int_z^y \frac{n! x^{\sigma-1}}{((a+u)x)^n} dx \\ &\leq \frac{n!}{(a+c)^n (n-\sigma) z^{n-\sigma}}. \end{aligned}$$

Therefore, given $\varepsilon > 0$, we can always find $t(\varepsilon)$ large enough so that for all $u \in I$ and for any $y > z > t(\varepsilon)$ the inequality

$$\left| \int_z^y f(x, u) dx \right| < \varepsilon$$

holds. In other words, Cauchy's criterion for uniform convergence ([4], page 352) of the improper Riemann integral

$$\int_0^\infty f(x, u) dx$$

on the interval I is satisfied.

Similarly, for $y > z > 0$, we have

$$\begin{aligned} \left| \int_z^y f(x, u) du \right| &\leq e^{-ax} x^{\sigma-1} \int_z^y e^{-xu} du \leq e^{-ax} x^{\sigma-1} \int_z^y \frac{n!}{(xu)^n} du \\ &\leq \frac{n!}{(n-1)c^{n-\sigma+1} z^{n-1}}. \end{aligned}$$

Therefore, given $\varepsilon > 0$, we can always find $t(\varepsilon)$ large enough so that for all $x \in I$ and for any $y > z > t(\varepsilon)$ the inequality

$$\left| \int_z^y f(x, u) du \right| < \varepsilon$$

holds. Again by Cauchy's criterion for uniform convergence, it follows that the improper Riemann integral

$$\int_0^\infty f(x, u) du$$

converges uniformly on the interval I .

Finally, if $y > z > 0$, then

$$\begin{aligned} &\left| \int_z^y f(x, u) du \right| \\ &= e^{-ax} x^{\sigma-1} \left| \int_z^y e^{-xu} \sin(\lambda u) du \right| \\ &= e^{-ax} x^{\sigma-1} \left| \frac{-e^{-xy}(x \sin(\lambda y) + \lambda \cos(\lambda y)) + e^{-xz}(x \sin(\lambda z) + \lambda \cos(\lambda z))}{x^2 + \lambda^2} \right| \\ &\leq 2e^{-ax} x^{\sigma-1} \frac{x + \lambda}{x^2 + \lambda^2} \leq 2e^{-ax} x^{\sigma-1} \frac{x + \lambda}{\lambda^2}. \end{aligned}$$

If $g(x) = 2e^{-ax} x^{\sigma-1} \frac{x+\lambda}{\lambda^2}$, then

$$\int_0^\infty g(x) dx = \frac{2}{\lambda^2} \left(\frac{\Gamma(\sigma+1)}{a^{\sigma+1}} + \frac{\lambda \Gamma(\sigma)}{a^\sigma} \right).$$

Therefore, an application of Theorem 2.1 gives

$$\begin{aligned} \int_0^\infty e^{-ax} \frac{x^{s-1}}{x^2 + \lambda^2} dx &= \frac{1}{\lambda} \int_0^\infty e^{-ax} x^{s-1} \int_0^\infty e^{-xu} \sin(\lambda u) dudx \\ &= \frac{1}{\lambda} \int_0^\infty \sin(\lambda u) \int_0^\infty e^{-(a+u)x} x^{s-1} dx du = \frac{\Gamma(s)}{\lambda} \int_0^\infty \frac{\sin(\lambda u)}{(a+u)^s} du. \quad \square \end{aligned}$$

PROPOSITION 3.3. *Let $a > 0$, $\lambda > 0$ and $0 < \operatorname{Re}(s) < 1$. Then*

$$\begin{aligned} \hat{\varphi}_a(s) &= \int_0^\infty e^{-ax} \frac{x^{s-1}}{x^2 + \lambda^2} dx \\ &= \frac{\Gamma(s)\Gamma(1-s)}{\lambda^{2-s}} \cos\left(a\lambda + \frac{\pi}{2}s\right) \\ &\quad + \frac{\Gamma(s-1)}{a^{s-1}\lambda} \sum_{m=1}^\infty (-1)^m \frac{(a\lambda)^{2m-1}}{(2-s)(3-s)\cdots(2m-s)}. \end{aligned}$$

PROOF. By Proposition 3.2, it suffices to prove that

$$\frac{\Gamma(s)}{\lambda} \int_0^\infty \frac{\sin(\lambda u)}{(a+u)^s} du$$

equals

$$\frac{\Gamma(s)\Gamma(1-s)}{\lambda^{2-s}} \cos\left(a\lambda + \frac{\pi}{2}s\right) + \frac{\Gamma(s-1)}{a^{s-1}\lambda} \sum_{m=1}^\infty (-1)^m \frac{(a\lambda)^{2m-1}}{(2-s)(3-s)\cdots(2m-s)}.$$

Let M be a positive real number. Since s is not an integer, repeated application of integration by parts gives

$$\begin{aligned} \int_0^M \frac{\sin(\lambda u)}{(a+u)^s} du &= \left(\cos(\lambda u) \sum_{m=1}^{n-1} \frac{(-1)^m \lambda^{2m-1}}{(s-1)(s-2)\cdots(s-2m)(a+u)^{s-2m}} \right) \Big|_0^M \\ &\quad + \left(\sin(\lambda u) \sum_{m=1}^n \frac{(-1)^m \lambda^{2m-2}}{(s-1)(s-2)\cdots(s-2m+1)(a+u)^{s-2m+1}} \right) \Big|_0^M \\ &\quad + \frac{(-1)^{n-1} \lambda^{2n-1}}{(s-1)(s-2)\cdots(s-2n+1)} \int_0^M \frac{\cos(\lambda u)}{(a+u)^{s-2n+1}} du. \end{aligned}$$

The limit of the integral on the left-hand side as $M \rightarrow \infty$ exists. We may therefore substitute M by $M_k = 2\pi k/\lambda$ and then take the limit as $k \rightarrow \infty$. Since $\cos(\lambda M_k) = 1$ and $\sin(\lambda M_k) = 0$, we have

$$\begin{aligned} &\int_0^{M_k} \frac{\sin(\lambda u)}{(a+u)^s} du \\ &= \frac{(-1)^{n-1} \lambda^{2n-1}}{(s-1)(s-2)\cdots(s-2n+1)} \int_0^{M_k} \frac{\cos(\lambda u)}{(a+u)^{s-2n+1}} du \\ &\quad + \sum_{m=1}^{n-1} (-1)^m \frac{\lambda^{2m-1}}{(s-1)(s-2)\cdots(s-2m)} \left(\frac{1}{(a+M_k)^{s-2m}} - \frac{1}{a^{s-2m}} \right). \quad (3) \end{aligned}$$

Note that for $2n > \sigma + 1$, we have

$$\begin{aligned} & \left| \frac{(-1)^{n-1} \lambda^{2n-1}}{(s-1)(s-2) \cdots (s-2n+1)} \int_0^{M_k} \frac{\cos(\lambda u)}{(a+u)^{s-2n+1}} du \right| \\ & \leq \frac{|\lambda|^{2n-1}}{|(s-1)(s-2) \cdots (s-2n+1)|} \int_0^{M_k} (a+u)^{2n-\sigma-1} du \\ & \leq \frac{|\lambda|^{2n-1} (a+M_k)^{2n-\sigma}}{|(s-1)(s-2) \cdots (s-2n+1)|}. \end{aligned}$$

Let

$$d_n = \frac{|\lambda|^{2n-1} (a+M_k)^{2n-\sigma}}{|(s-1)(s-2) \cdots (s-2n+1)|}.$$

Clearly,

$$\lim_{n \rightarrow \infty} \frac{d_{n+1}}{d_n} = 0,$$

hence the sequence d_n also converges to 0. Letting $n \rightarrow \infty$ in Formula (3), we get

$$\begin{aligned} \int_0^{M_k} \frac{\sin(\lambda u)}{(a+u)^s} du &= \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^{2m-1}}{(s-1) \cdots (s-2m)} \left(\frac{1}{(a+M_k)^{s-2m}} - \frac{1}{a^{s-2m}} \right) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^{2m-1}}{(s-1) \cdots (s-2m)(a+M_k)^{s-2m}} \\ &\quad - \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^{2m-1}}{(s-1) \cdots (s-2m)a^{s-2m}}. \end{aligned} \tag{4}$$

By Lemma 3.1 for $r = 2m - 1$ and $b = a + M_k$, Formula (4) gives

$$\begin{aligned} \int_0^{M_k} \frac{\sin(\lambda u)}{(a+u)^s} du &= \sum_{m=1}^{\infty} \int_0^{a+M_k} \frac{(-1)^m (\lambda(a+M_k-u))^{2m-1}}{u^s (2m-1)!} du \\ &\quad - \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^{2m-1}}{(s-1) \cdots (s-2m)a^{s-2m}}. \end{aligned}$$

By uniform convergence of the Maclaurin series for $\sin x$ on compact intervals, we get

$$\int_0^{M_k} \frac{\sin(\lambda u)}{(a+u)^s} du = \int_0^{a+M_k} \frac{\sin(\lambda(u-a-M_k))}{u^s} du - \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^{2m-1}}{(s-1) \cdots (s-2m) a^{s-2m}}.$$

Since $\lambda M_k = 2\pi k$, we get

$$\int_0^{M_k} \frac{\sin(\lambda u)}{(a+u)^s} du = \int_0^{a+M_k} \frac{\sin(\lambda(u-a))}{u^s} du - \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^{2m-1}}{(s-1) \cdots (s-2m) a^{s-2m}}.$$

Letting $k \rightarrow \infty$, we get

$$\int_0^{\infty} \frac{\sin(\lambda u)}{(a+u)^s} du = \int_0^{\infty} \frac{\sin(\lambda(u-a))}{u^s} du - \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^{2m-1}}{(s-1) \cdots (s-2m) a^{s-2m}}.$$

Multiplying both sides by $\Gamma(s)/\lambda$ and using Proposition 3.2 and Corollary 2.3 completes the proof. □

Recall the definition of Lommel’s functions of two variables ([5], page 372):

$$V_s(w, z) = \cos\left(\frac{w}{2} + \frac{z^2}{2w} + \frac{s\pi}{2}\right) + \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{2-s+2m} J_{2-s+2m}(z),$$

where $J_\nu(z)$ denote the Bessel functions of the first kind ([5], page 370):

$$J_\nu(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{k! \Gamma(\nu+k+1)}.$$

A straightforward calculation shows that Proposition 3.3 can be restated in terms of specific values of $V_s(w, z)$. In other words, we have a new and elementary proof of the following classical Proposition (listed in [5], page 138, and proven in [38], page 548, Formula (4), via contour integration):

PROPOSITION 3.4. *Let $a > 0$, $\lambda > 0$ and $0 < \text{Re}(s) < 1$. Then*

$$\hat{\varphi}_a(s) = \int_0^{\infty} e^{-ax} \frac{x^{s-1}}{x^2 + \lambda^2} dx = \lambda^{s-2} \frac{\pi}{\sin(s\pi)} V_s(2a\lambda, 0).$$

REMARK 3.5. Not surprisingly, for $|a\lambda| < 1$, the infinite sum appearing on the right-hand side of the equality in Proposition 3.3 can also be expressed in

terms of confluent hypergeometric series, namely it equals

$$\frac{{}_1F_1(1; 2-s; -ia\lambda) - {}_1F_1(1; 2-s; ia\lambda)}{2i}.$$

REMARK 3.6. For $\operatorname{Re}(s) < 1$, the infinite sum appearing on the right-hand side of the equality in Proposition 3.3 also equals

$$\frac{s-1}{(a\lambda)^{1-s}} \int_0^{a\lambda} \frac{\sin(a\lambda - x)}{x^s} dx.$$

This follows by combining the Maclaurin series of $\sin x$ with the formula in Lemma 3.1.

4. The Hurwitz zeta function in terms of a Laplace-Mellin transform

We first discuss some well-known facts about the partial fraction decompositions of some trigonometric functions. Consider the function

$$C(x) = \operatorname{csch}\left(\frac{x}{2}\right) = \frac{2}{e^{x/2} - e^{-x/2}}, \quad \text{for } x \neq 0.$$

Using a real version of the Poisson summation formula as in [1], page 334, one obtains a partial fraction decomposition for the hyperbolic cotangent function:

$$\operatorname{coth}(x) = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 + (k\pi)^2}, \quad \text{for } x \neq 0.$$

Although it is not necessary for our discussion, we mention that the well-known formula, obtained by methods of contour integration, for the usual cotangent function (see [17], page 188, or [31], page 391), combined with the equality $\operatorname{coth}(z) = i \cot(iz)$, gives a similar partial fraction decomposition for the hyperbolic cotangent function when x is complex.

Since for $x \in \mathbf{R} \setminus \{0\}$ we have

$$\begin{aligned} \operatorname{csch}(x) &= \operatorname{coth}\left(\frac{x}{2}\right) - \operatorname{coth}(x) = \left(\frac{2}{x} + 4 \sum_{k \text{ even}} \frac{x}{x^2 + (k\pi)^2}\right) - \left(\frac{1}{x} + 2 \sum_{k=1}^{\infty} \frac{x}{x^2 + (k\pi)^2}\right) \\ &= \frac{1}{x} + 2 \sum_{k \text{ even}} \frac{x}{x^2 + (k\pi)^2} - 2 \sum_{k \text{ odd}} \frac{x}{x^2 + (k\pi)^2} = \frac{1}{x} + 2 \sum_{k=1}^{\infty} (-1)^k \frac{x}{x^2 + (k\pi)^2}, \end{aligned}$$

it follows that

$$C(x) = \frac{2}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{4x}{x^2 + (2k\pi)^2}, \quad \text{for } x \neq 0. \tag{5}$$

The following statement is well-known ([13], page 396). It has been generalized for certain theta series in [21]. A similar generalization has been given by Coppo and Candelpergher in [11]. We also refer the reader to the paper by Kölbig ([24]) for connections between the Hurwitz zeta function and Laplace-Mellin transforms of logarithmic functions.

We give a proof of the statement below for the sake of completeness:

PROPOSITION 4.1.

$$\hat{C}_a(s) = 2\Gamma(s)\zeta\left(s; a + \frac{1}{2}\right), \quad \text{for } \text{Re}(s) > 1 \text{ and } a > 0.$$

PROOF. Since $C(x) \sim \frac{2}{x}$ as $x \rightarrow 0+$, Lemma 1.3 of [21] implies that $\hat{C}_a(s)$ converges for $\text{Re}(s) > 1$ and $a > 0$. Now, for $x > 0$, we have $e^{-x} < 1$, therefore

$$C(x) = \frac{2e^{-x/2}}{1 - e^{-x}} = 2 \sum_{k=0}^{\infty} e^{-(k+1/2)x}.$$

Hence,

$$\hat{C}_a(s) = 2 \int_0^{\infty} \left(\sum_{k=0}^{\infty} x^{s-1} e^{-(a+k+1/2)x} \right) dx.$$

Let $\sigma = \text{Re}(s)$. For $n \geq 0$, the functions

$$g_n(x) = 2x^{s-1} e^{-(a+n+1/2)x}$$

are improper Riemann integrable on $(0, \infty)$ with

$$\int_0^{\infty} g_n(x) dx = \frac{2\Gamma(s)}{(a+n+\frac{1}{2})^s}, \quad \int_0^{\infty} |g_n(x)| dx = \frac{2\Gamma(\sigma)}{(a+n+\frac{1}{2})^\sigma}$$

and the series

$$\sum_{n=0}^{\infty} g_n(x)$$

converges pointwise to the function $x^{s-1}e^{-ax}C(x)$ on $(0, \infty)$. Since

$$\sum_{n=0}^{\infty} \int_0^{\infty} |g_n(x)| dx = 2\Gamma(\sigma)\zeta\left(\sigma; a + \frac{1}{2}\right) < \infty,$$

the claim follows from Theorem 2.6. □

REMARK 4.2. We also note that, by Formula (5),

$$\hat{C}_a(s) = 2 \int_0^{\infty} \left(e^{-ax}x^{s-2} + 2 \sum_{k=1}^{\infty} (-1)^k \frac{e^{-ax}x^s}{x^2 + (2k\pi)^2} \right) dx.$$

Now, for $n \geq 1$, the functions

$$f_n(x) = e^{-ax}x^{s-2} + 2 \sum_{k=1}^n (-1)^k \frac{e^{-ax}x^s}{x^2 + (2k\pi)^2}$$

are Riemann integrable on every compact subinterval $[c, d]$ of $(0, \infty)$ and the sequence $f_n(x)$ converges pointwise to the Riemann integrable function $e^{-ax}x^{s-1}C(x)$ on $[c, d]$. Also, for all $x > 0$ and all $n \geq 1$, we have

$$|f_n(x)| \leq e^{-ax}x^{\sigma-2} + 2e^{-ax}x^{\sigma} \sum_{k=1}^n \frac{1}{4k^2\pi^2} = e^{-ax}x^{\sigma-2} + \frac{1}{12}e^{-ax}x^{\sigma}.$$

Setting

$$g(x) = e^{-ax}x^{\sigma-2} + \frac{1}{12}e^{-ax}x^{\sigma},$$

we see that, since $\sigma > 1$ and $a > 0$, the improper Riemann integral $\int_0^{\infty} g(x) dx$ exists and equals

$$\frac{\Gamma(\sigma-1)}{a^{\sigma-1}} + \frac{\Gamma(\sigma+1)}{12a^{\sigma+1}}.$$

Therefore, by Theorem 2.4, we get

$$\hat{C}_a(s) = 2 \int_0^{\infty} e^{-ax}x^{s-2} dx + 4 \sum_{k=1}^{\infty} (-1)^k \int_0^{\infty} \frac{e^{-ax}x^s}{x^2 + (2k\pi)^2} dx.$$

Since $\Gamma(s) = (s - 1)\Gamma(s - 1)$, it follows that

$$\hat{C}_a(s) = \frac{2\Gamma(s)}{a^{s-1}(s-1)} + 4 \sum_{k=1}^{\infty} (-1)^k \int_0^{\infty} \frac{e^{-ax} x^s}{x^2 + (2k\pi)^2} dx. \tag{6}$$

5. Proof of a special case of formula (2)

We first prove Formula (2) for $\frac{1}{2} < x < 1$ and $-1 < \text{Re}(s) < 0$. Setting $x = a + (1/2)$, this is equivalent to showing that

$$\zeta\left(s; a + \frac{1}{2}\right) = 2\Gamma(1 - s) \sum_{k=1}^{\infty} (-1)^k \frac{\sin(2k\pi a + \frac{\pi}{2}s)}{(2k\pi)^{1-s}},$$

for $0 < a < 1/2$ and $-1 < \text{Re}(s) < 0$.

We will make use of the following well-known example in Fourier series expansions ([1], page 337):

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(2k\pi u)}{k\pi} = \begin{cases} u & \text{if } 0 < u < \frac{1}{2} \\ u - l & \text{if } \frac{2l-1}{2} < u < \frac{2l+1}{2}, l = 1, 2, \dots \end{cases} \tag{7}$$

We know that for $\text{Re}(s) > 1$ and $a > 0$, Proposition 4.1 and Formula (6) give

$$\zeta\left(s; a + \frac{1}{2}\right) - \frac{1}{(s-1)a^{s-1}} = \frac{2}{\Gamma(s)} \sum_{k=1}^{\infty} (-1)^k \int_0^{\infty} \frac{e^{-ax} x^s}{x^2 + (2k\pi)^2} dx. \tag{8}$$

At first glance, the left-hand side of Formula (8) is an analytic function on the region given by $\text{Re}(s) > 1$, while the right-hand side is an analytic function on the region given by $\text{Re}(s) > -1$. Given the fact that the Hurwitz zeta function extends to a meromorphic function on \mathbf{C} having a unique and simple pole at $s = 1$ with residue 1 and since

$$\lim_{s \rightarrow 1} \left(\frac{1}{s-1} - \frac{1}{(s-1)a^{s-1}} \right) = \log(a),$$

it follows that Formula (8) is in fact an equality of analytic functions on the region given by $\text{Re}(s) > -1$. Let us now also assume that $\text{Re}(s) < 0$. Then Proposition 3.3 together with Remark 3.6 (with s replaced by $s + 1$ and λ replaced by $2k\pi$) imply that

$$\begin{aligned}
\zeta\left(s; a + \frac{1}{2}\right) - \frac{1}{(s-1)a^{s-1}} &= \sum_{k=1}^{\infty} (-1)^k \frac{2s\Gamma(-s)}{(2k\pi)^{1-s}} \cos\left(2k\pi a + \frac{\pi}{2}(s+1)\right) \\
&\quad + \sum_{k=1}^{\infty} (-1)^k \frac{2s}{(2k\pi)^{1-s}} \int_0^{2k\pi a} \frac{\sin(2k\pi a - x)}{x^{s+1}} dx \\
&= -2s\Gamma(-s) \sum_{k=1}^{\infty} (-1)^k \frac{\sin\left(2k\pi a + \frac{\pi}{2}s\right)}{(2k\pi)^{1-s}} \\
&\quad + \sum_{k=1}^{\infty} s \left(\frac{2\pi}{a}\right)^s \int_0^{2\pi} (-1)^k \frac{\sin\left(2k\pi\left(a - \frac{u}{2\pi}a\right)\right)}{k\pi u^{s+1}} du. \quad (9)
\end{aligned}$$

For $k \geq 1$, consider the function h_k on $(0, \infty)$ defined by

$$h_k(u) = \begin{cases} (-1)^k \frac{\sin\left(2k\pi\left(a - \frac{u}{2\pi}a\right)\right)}{k\pi u^{s+1}} & \text{if } 0 < u < 2\pi \\ 0 & \text{if } u \geq 2\pi \end{cases}$$

Also, for $n \geq 1$, let f_n be the function on $(0, \infty)$ defined by

$$f_n(u) = \sum_{k=1}^n h_k(u).$$

Since $a < 1/2$, we have $0 < a - (ua)/(2\pi) < 1/2$ for $0 < u < 2\pi$. Therefore, by Formula (7), it follows that the sequence of functions (f_n) converges to the function f on $(0, \infty)$ given by

$$f(u) = \lim_{n \rightarrow \infty} f_n(u) = \begin{cases} \frac{-a + \frac{u}{2\pi}a}{u^{s+1}} & \text{if } 0 < u < 2\pi \\ 0 & \text{if } u \geq 2\pi \end{cases}$$

Clearly, the functions f_n and f are integrable over all compact subintervals of $(0, \infty)$. Also,

$$\begin{aligned}
\sum_{k=1}^n (-1)^k \sin\left(2k\pi\left(a - \frac{u}{2\pi}a\right)\right) &= \sum_{k=1}^n \sin\left(2k\pi\left(a - \frac{u}{2\pi}a\right) + k\pi\right) \\
&= \sum_{k=1}^n \sin(k(2\pi a - ua + \pi)).
\end{aligned}$$

For $0 < u < 2\pi$, we have $0 < 2\pi a - ua + \pi < \pi$, so, by a well-known summation formula ([4], page 400), we have

$$\begin{aligned} \left| \sum_{k=1}^n (-1)^k \sin\left(2k\pi\left(a - \frac{u}{2\pi}a\right)\right) \right| &\leq \frac{1}{\sin\left(\pi a - \frac{u}{2}a + \frac{\pi}{2}\right)} \\ &= \frac{1}{\cos\left(\pi a - \frac{u}{2}a\right)} < \frac{1}{\cos(\pi a)}. \end{aligned}$$

By Abel's summation formula ([1], page 194), it follows that

$$\begin{aligned} \left| \sum_{k=1}^n (-1)^k \frac{\sin(2k\pi(a - \frac{u}{2\pi}a))}{k} \right| &\leq \frac{1}{\cos(\pi a)} \frac{1}{n+1} + \sum_{k=1}^n \frac{1}{\cos(\pi a)} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \frac{1}{\cos(\pi a)}. \end{aligned}$$

Therefore, the inequality

$$|f_n(u)| \leq \frac{1}{\pi \cos(\pi a) u^{s+1}}$$

holds for all $u \in (0, 2\pi)$. Obviously, it trivially holds for $u \geq 2\pi$ also. Consider the function given by

$$g(u) = \begin{cases} \frac{1}{\pi \cos(\pi a) u^{s+1}} & \text{if } 0 < u < 2\pi \\ 0 & \text{if } u \geq 2\pi \end{cases}$$

Since $0 < \operatorname{Re}(s+1) < 1$, the integral

$$\int_0^\infty g(u) \, du$$

exists. Therefore, by Theorem 2.4, Formula (9) gives

$$\begin{aligned} &\zeta\left(s; a + \frac{1}{2}\right) - \frac{1}{(s-1)a^{s-1}} \\ &= -2s\Gamma(-s) \sum_{k=1}^\infty (-1)^k \frac{\sin(2k\pi a + \frac{\pi}{2}s)}{(2k\pi)^{1-s}} + \lim_{n \rightarrow \infty} s \left(\frac{2\pi}{a}\right)^s \int_0^\infty f_n(u) \, du \\ &= -2s\Gamma(-s) \sum_{k=1}^\infty (-1)^k \frac{\sin(2k\pi a + \frac{\pi}{2}s)}{(2k\pi)^{1-s}} + s \left(\frac{2\pi}{a}\right)^s \int_0^\infty f(u) \, du \\ &= -2s\Gamma(-s) \sum_{k=1}^\infty (-1)^k \frac{\sin(2k\pi a + \frac{\pi}{2}s)}{(2k\pi)^{1-s}} + s \left(\frac{2\pi}{a}\right)^s \frac{a}{(2\pi)^s s(1-s)}, \end{aligned}$$

hence

$$\begin{aligned}\zeta\left(s; a + \frac{1}{2}\right) &= -2s\Gamma(-s) \sum_{k=1}^{\infty} (-1)^k \frac{\sin(2k\pi a + \frac{\pi}{2}s)}{(2k\pi)^{1-s}} \\ &= 2\Gamma(1-s) \sum_{k=1}^{\infty} (-1)^k \frac{\sin(2k\pi a + \frac{\pi}{2}s)}{(2k\pi)^{1-s}},\end{aligned}$$

for $0 < a < 1/2$ and $-1 < \operatorname{Re}(s) < 0$.

6. Proof of the general case of formula (2)

The following duplication formula for the Hurwitz zeta function is well-known ([9], page 77) and easy to prove:

$$\zeta(s; x) + \zeta\left(s; x + \frac{1}{2}\right) = 2^s \zeta(s; 2x), \quad \text{for } x > 0 \text{ and } s \in \mathbb{C} \setminus \{1\}. \quad (10)$$

Also, the following formula is an easy consequence of absolute convergence:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\sin(2k\pi x + \frac{\pi}{2}s)}{(2k\pi)^{1-s}} + \sum_{k=1}^{\infty} \frac{\sin(2k\pi(x + \frac{1}{2}) + \frac{\pi}{2}s)}{(2k\pi)^{1-s}} \\ = 2^s \sum_{k=1}^{\infty} \frac{\sin(4k\pi x + \frac{\pi}{2}s)}{(2k\pi)^{1-s}}, \quad \text{for } x > 0 \text{ and } \operatorname{Re}(s) < 0.\end{aligned} \quad (11)$$

We now prove that, for a positive integer m , Formula (2) holds for $2^{-m} < x < 2^{-(m-1)}$ and for $-1 < \operatorname{Re}(s) < 0$:

We use induction on m . For $m = 1$, the claim has been proven in the previous section. Assume that it holds for some $m \geq 1$. Let $2^{-(m+1)} < x < 2^{-m}$ and $-1 < \operatorname{Re}(s) < 0$. Since $2^{-m} < 2x < 2^{-(m-1)}$ and $1/2 < x + (1/2) < 1$, the induction hypothesis together with Formulas (10) and (11) show that Formula (2) also holds for x .

Furthermore, by the Weierstrass M -test, both series of functions

$$\sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{(2k\pi)^{1-s}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{(2k\pi)^{1-s}}$$

are uniformly convergent for $\operatorname{Re}(s) < 0$, hence the series

$$\sum_{k=1}^{\infty} \frac{\sin(2k\pi x + \frac{\pi}{2}s)}{(2k\pi)^{1-s}} = \cos\left(\frac{\pi}{2}s\right) \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{(2k\pi)^{1-s}} + \sin\left(\frac{\pi}{2}s\right) \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{(2k\pi)^{1-s}}$$

is a continuous function of x . Also, the obvious relation

$$\frac{\partial \zeta(s; x)}{\partial x} = -s\zeta(s + 1; x), \quad \text{for } \operatorname{Re}(s) > 1$$

is valid on $\mathbb{C} \setminus \{1\}$ by analytic continuation and shows in particular that $\zeta(s; x)$ is a continuous function of x . Therefore, for a positive integer m , Formula (2) also holds for $x = 2^{-(m-1)}$ and for $-1 < \operatorname{Re}(s) < 0$. Therefore, Formula (2) has been proven for $0 < x \leq 1$ and for $-1 < \operatorname{Re}(s) < 0$. Finally, by analytic continuation once again, we see that Formula (2) remains valid for $0 < x \leq 1$ and for $\operatorname{Re}(s) < 0$, and this completes the proof.

7. Integral representations

The literature on integral expressions involving the zeta function and related functions is vast. We refer the reader to [3] for a wealth of such information. In this section, we deduce some expressions of this type.

Since, for $k \geq 1$, we have

$$\int_0^\infty \sin\left(\frac{xu}{2}\right) e^{-k\pi u} du = \frac{2x}{x^2 + 4k^2\pi^2},$$

Formula (5) gives

$$C(x) - \frac{2}{x} = 2 \sum_{k=1}^\infty \int_0^\infty (-1)^k e^{-k\pi u} \sin\left(\frac{x}{2}u\right) du.$$

The sequence of functions (f_n) on $(0, \infty)$ defined by

$$f_n(u) = 2 \sum_{k=1}^n (-1)^k e^{-k\pi u} \sin\left(\frac{x}{2}u\right) = -2 \sin\left(\frac{x}{2}u\right) \frac{1 - (-e^{-\pi u})^n}{1 + e^{\pi u}}$$

converge to the function f given by

$$f(u) = -2 \sin\left(\frac{x}{2}u\right) \frac{1}{1 + e^{\pi u}}.$$

The functions f and f_n are Riemann integrable over all compact subintervals of $(0, \infty)$ and we also have $|f_n(u)| \leq 4/(1 + e^{\pi u})$ on $(0, \infty)$. Since

$$\int_0^\infty \frac{4}{1 + e^{\pi u}} du$$

exists, Theorem 2.4 applies and gives the following integral expression for $C(x)$:

COROLLARY 7.1.

$$C(x) = \frac{2}{x} + \int_0^\infty \sin\left(\frac{xu}{2}\right) \left(-1 + \tanh\left(\frac{\pi u}{2}\right)\right) du, \quad \text{for } x > 0.$$

Note that this corrects a misprint in [5] (page 88, Table 2.9 of Fourier sine transforms, entry (4)).

Applying the Laplace-Mellin transform to both sides of the equality in Corollary 7.1 and using Proposition 4.1 gives

$$\begin{aligned} \zeta\left(s; a + \frac{1}{2}\right) - \frac{a^{1-s}}{s-1} \\ = \frac{1}{2\Gamma(s)} \int_0^\infty e^{-ax} x^{s-1} \int_0^\infty \sin\left(\frac{x}{2}u\right) \left(-1 + \tanh\left(\frac{\pi}{2}u\right)\right) dudx, \\ \text{for } \operatorname{Re}(s) > 1 \text{ and for } a > 0. \end{aligned} \tag{12}$$

The integrand in the latter double integral is the function

$$f(x, u) = \frac{-2e^{-ax}x^{s-1} \sin\left(\frac{xu}{2}\right)}{1 + e^{\pi u}} \quad \text{on } (0, \infty) \times (0, \infty),$$

where $\operatorname{Re}(s) > 1$ and $a > 0$. We will check that it satisfies the hypotheses of Theorem 2.1. By Corollary 7.1, we have

$$\int_0^\infty f(x, u) du = e^{-ax}x^{s-1} \left(C(x) - \frac{2}{x}\right).$$

By considering the Fourier sine transform of $e^{-ax}x^{s-1}$ ([5], page 72, Formula (7)), we get

$$\int_0^\infty f(x, u) dx = \frac{-2\Gamma(s) \left(a^2 + \frac{u^2}{4}\right)^{-s/2} \sin\left(s \arctan\left(\frac{u}{2a}\right)\right)}{1 + e^{\pi u}}.$$

Let $I = [c, d]$ be a compact subinterval of $(0, \infty)$. Let $\sigma = \operatorname{Re}(s)$. Choose a positive integer n such that $n \geq \sigma + 1$. If $y > z > 0$, then

$$\begin{aligned} \left| \int_z^y f(x, u) dx \right| &\leq \frac{2}{1 + e^{\pi u}} \int_z^y e^{-ax} x^{\sigma-1} dx \leq \int_z^y \frac{n! x^{\sigma-1}}{(ax)^n} dx \\ &= \frac{n!}{(n - \sigma) a^n z^{n-\sigma}}. \end{aligned}$$

Therefore, given $\varepsilon > 0$, we can find $t(\varepsilon)$ large enough so that for all $u \in I$ and for any $y > z > t(\varepsilon)$ the inequality

$$\left| \int_z^y f(x, u) dx \right| < \varepsilon$$

holds, i.e. Cauchy's criterion for uniform convergence of the improper Riemann integral

$$\int_0^\infty f(x, u) dx$$

on the interval I is satisfied.

Similarly, for $y > z > 0$, we have

$$\left| \int_z^y f(x, u) du \right| \leq 2e^{-ax} x^{\sigma-1} \int_z^y \frac{1}{1 + e^{\pi u}} du \leq 2x^{\sigma-1} \int_z^y e^{-\pi u} du \leq \frac{2d^{\sigma-1}}{\pi e^{\pi z}}.$$

Therefore, given $\varepsilon > 0$, we can find $t(\varepsilon)$ large enough so that for all $x \in I$ and for any $y > z > t(\varepsilon)$ the inequality

$$\left| \int_z^y f(x, u) du \right| < \varepsilon$$

holds, i.e. Cauchy's criterion for uniform convergence of the improper Riemann integral

$$\int_0^\infty f(x, u) du$$

on the interval I is satisfied.

Finally, if $y > z > 0$, then

$$\begin{aligned} \left| \int_z^y f(x, u) du \right| &\leq 2e^{-ax} x^{\sigma-1} \int_z^y \frac{1}{1 + e^{\pi u}} du \leq 2e^{-ax} x^{\sigma-1} \int_z^y e^{-\pi u} du \\ &= \frac{2e^{-ax} x^{\sigma-1} (e^{-\pi z} - e^{\pi y})}{\pi} \leq \frac{2}{\pi} e^{-ax} x^{\sigma-1}. \end{aligned}$$

If $g(x) = \frac{2}{\pi} e^{-ax} x^{\sigma-1}$, then

$$\int_0^\infty g(x) dx = \frac{2\Gamma(\sigma)}{\pi a^\sigma}.$$

Therefore, Theorem 2.1 applies and Formula (12) becomes

$$\begin{aligned} \zeta\left(s; a + \frac{1}{2}\right) - \frac{a^{1-s}}{s-1} \\ = \frac{1}{2\Gamma(s)} \int_0^\infty \left(-1 + \tanh\left(\frac{\pi}{2}u\right)\right) \int_0^\infty e^{-ax} x^{s-1} \sin\left(\frac{x}{2}u\right) dx du, \\ \text{for } \operatorname{Re}(s) > 1 \text{ and for } a > 0. \end{aligned} \quad (13)$$

Again by [5], page 72, Formula (7), the inside integral equals

$$\Gamma(s) \left(a^2 + \frac{u^2}{4}\right)^{-s/2} \sin\left(s \arctan\left(\frac{u}{2a}\right)\right), \quad \text{for } \operatorname{Re}(s) > -1 \text{ and for } a > 0.$$

Therefore, the change of variables $u = 2a \tan \theta$ gives

$$\begin{aligned} \zeta\left(s; a + \frac{1}{2}\right) = \frac{a^{1-s}}{s-1} + a^{1-s} \int_0^{\pi/2} (\cos \theta)^{s-2} \sin(s\theta) (\tanh(a\pi \tan \theta) - 1) d\theta, \\ \text{for } \operatorname{Re}(s) > 1 \text{ and for } a > 0. \end{aligned} \quad (14)$$

We claim that Formula (14) is valid more generally for $s \neq 1$ and for $a > 0$. By analytic continuation of the Hurwitz zeta function on $\mathbf{C} \setminus \{1\}$, it suffices to show that the integral in Formula (14) exists for all $s \in \mathbf{C}$. Let $\sigma = \operatorname{Re}(s)$. The integrand equals

$$\frac{-2(\cos \theta)^{s-2} \sin(s\theta)}{1 + e^{2a\pi \tan \theta}}.$$

Its limit as $\theta \rightarrow b$ exists for all $0 \leq b < \pi/2$, so it remains to show that

$$\lim_{\theta \rightarrow (\pi/2)^-} \frac{(\cos \theta)^{s-2}}{1 + e^{2\pi a \tan \theta}}$$

exists. Setting $\theta = (\pi/2) - x$, the latter limit becomes

$$\lim_{x \rightarrow 0^+} \frac{(\sin x)^{s-2}}{1 + e^{2\pi a \cot x}}.$$

We will show that the latter limit equals 0. Choose a positive integer n with the property that $\sigma - 2 + n \geq 0$. Then

$$\left| \frac{(\sin x)^{s-2}}{1 + e^{2\pi a \cot x}} \right| \leq \frac{n! (\sin x)^{\sigma-2}}{(2\pi a \cot x)^n} = \frac{n!}{(2\pi a)^n} \frac{(\sin x)^{\sigma-2+n}}{(\cos x)^n},$$

which tends to 0 as $x \rightarrow 0+$. We have therefore proved the following integral expression for the Hurwitz zeta function:

COROLLARY 7.2. *For $s \in \mathbb{C} \setminus \{1\}$ and for $a > 0$, we have*

$$\zeta\left(s; a + \frac{1}{2}\right) = \frac{a^{1-s}}{s-1} + a^{1-s} \int_0^{\pi/2} (\cos \theta)^{s-2} \sin(s\theta) (\tanh(a\pi \tan \theta) - 1) d\theta.$$

It turns out that the formula in Corollary 7.2 is equivalent to Formula (23), page 160 in the book [32] by Srivastava and Choi, where the latter formula is considered known and given without proof (but may have been derived using Plana’s summation formula). It is not difficult to see that the formula in Corollary 7.2 and Formula (23) in [32] are linked via a change of variables and integration by parts.

COROLLARY 7.3. *For $\operatorname{Re}(s) > 1$ and for $a > 0$, we have*

$$\zeta\left(s; a + \frac{1}{2}\right) = a^{1-s} \int_0^{\pi/2} (\cos \theta)^{s-2} \sin(s\theta) \tanh(a\pi \tan \theta) d\theta.$$

PROOF. Since

$$\frac{\partial((\cos \theta)^{s-1} \cos((s-1)\theta))}{\partial \theta} = (1-s)(\cos \theta)^{s-2} \sin(s\theta),$$

it follows that

$$\int_0^{\pi/2} (\cos \theta)^{s-2} \sin(s\theta) d\theta = \frac{1}{1-s} (\cos \theta)^{s-1} \cos((s-1)\theta) \Big|_0^{\pi/2} = \frac{1}{s-1},$$

because $\operatorname{Re}(s) > 1$. The claim now follows from Corollary 7.2. □

REMARK 7.4. The formulas in Corollaries 7.2 and 7.3 resemble similar formulas attributed to Lindelöf ([25] and also [3], Formula 25.11.29) and to Hermite ([15], Formula (1.5)), but, as far as the authors can tell, no obvious direct correlation seems to exist.

If we set $a = 1/2$ in Corollaries 7.2 and 7.3, we recover the following integral expressions for the Riemann zeta function, which are due to Jensen ([20]) (see also [32], Formula (41), page 171, and [3], Formula 25.5.12):

COROLLARY 7.5. For $s \in \mathbf{C} \setminus \{1\}$, we have:

$$\zeta(s) = \frac{2^{s-1}}{s-1} + 2^{s-1} \int_0^{\pi/2} (\cos \theta)^{s-2} \sin(s\theta) \left(\tanh\left(\frac{\pi}{2} \tan \theta\right) - 1 \right) d\theta.$$

In particular, for $\operatorname{Re}(s) > 1$, we have

$$\zeta(s) = 2^{s-1} \int_0^{\pi/2} (\cos \theta)^{s-2} \sin(s\theta) \tanh\left(\frac{\pi}{2} \tan \theta\right) d\theta.$$

For the special case $s = n$, where n is a positive integer with $n \geq 2$, comparison of the imaginary parts of the two sides of the equality

$$e^{2i\theta}(1 + e^{2i\theta})^{n-2} = \sum_{k=1}^{n-1} \binom{n-2}{k-1} e^{2ki\theta}$$

combined with Corollary 7.5 gives

COROLLARY 7.6. For each positive integer n with $n \geq 2$, we have

$$\zeta(n) = 2 \sum_{k=1}^{n-1} \binom{n-2}{k-1} \int_0^{\pi/2} \sin(2k\theta) \tanh\left(\frac{\pi}{2} \tan \theta\right) d\theta.$$

Other representations for $\zeta(n)$ in terms of trigonometric integrals have been given by Srivastava, Glasser and Adamchik in [33] and by Cvijović and Klinowski in [12]. We also refer the reader to [10] for representations of $\zeta(n)$ and $\zeta(n; x)$ which do not involve trigonometric integrals.

References

- [1] Apostol, T., *Mathematical Analysis*, Addison-Wesley, 1974.
- [2] Apostol, T., On the Lerch zeta function, *Pacific Journal of Mathematics* **1** (1951), 161–167.
- [3] Apostol, T., Zeta and related functions, NIST Digital Library of Mathematical Functions, 601–616.
- [4] Bartle, R., *Elements of Real Analysis*, John Wiley & Sons, 1967.
- [5] Bateman, H., *Tables of Integral Transforms*, Vol. I, Bateman Manuscript Project, California Institute of Technology, Edited by A. Erdélyi, McGraw Hill Book Company, 1954.
- [6] Berndt, B., A new proof of the functional equation for Dirichlet L -functions, *Proceedings of the American Mathematical Society* **37** (1973), 355–357.
- [7] Borwein, P., Choi, S., Rooney, B. and Weirathmueller, A., *The Riemann Hypothesis*, Canadian Mathematical Society Books in Mathematics, Springer, 2008.
- [8] Bump, D., *Automorphic Forms and Representations*, Cambridge University Press, 1998.

- [9] Cohen, H., *Number Theory, Volume II: Analytic and Modern Tools*, Springer, 2000.
- [10] Coppo, M.-A., Nouvelles expressions des formules de Hasse et de Hermite pour la fonction Zêta d' Hurwitz, *Expositiones Mathematicae* **27** (2009), 79–86.
- [11] Coppo, M.-A. and Candelpergher, B., The Arakawa-Kaneko zeta function, *The Ramanujan Journal* **22** (2010), 153–162.
- [12] Cvijović, D. and Klinowski, J., Integral Representations of the Riemann zeta function for odd-integer arguments, *Journal of Computational and Applied Mathematics* **142** (2002), 435–439.
- [13] Debnath, L. and Bhatta, D., *Integral Transforms and Their Applications*, CRC Press, Taylor and Francis Group, 2015.
- [14] Edwards, H., *Riemann's Zeta Function*, Academic Press, 1974.
- [15] Espinosa, O. and Moll, V., On Some Integrals Involving the Hurwitz Zeta Function: Part 1, *The Ramanujan Journal* **6** (2002), 159–188.
- [16] Euler, L., Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques, *Memoires de l' Academie de Science de Berlin* **17** (1768), 83–106.
- [17] Freitag, E. and Busam, R., *Complex Analysis*, Springer, 2005.
- [18] Friedberg, S. (Managing Editor), Bump, D., Goldfeld, D. and Hoffstein J. (Editors), *Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory*, *Proceedings of Symposia in Pure Mathematics*, American Mathematical Society, **75**, 2005.
- [19] Hurwitz, A., Einige Eigenschaften der Dirichletschen Funktionen $F(s) = \sum \left(\frac{a}{n}\right) \frac{1}{n^s}$, die bei der Bestimmung der Klassenzahlen binären quadratischer Formen auftreten, *Zeitschrift für Mathematik und Physik* **27** (1882), 86–101.
- [20] Jensen, J., Sur la fonction $\zeta(s)$ de Riemann, *Comptes Rendus de l' Académie des Sciences Paris* **104** (1887), 1156–1159.
- [21] Jorgenson, J. and Lang, S., *Basic Analysis of Regularized Series and Products*, *Lecture Notes in Mathematics*, Springer, 1993.
- [22] Kaneko, M., Kurokawa, N. and Wakayama, M., A variation of Euler's approach to values of the Riemann zeta function, *Kyushu Journal of Mathematics* **57** (2003), 175–192.
- [23] Knopp, M. and Robins, S., Easy proofs of Riemann's functional equation for $\zeta(s)$ and of Lipschitz summation, *Proceedings of the American Mathematical Society* **129** (2001), 1915–1922.
- [24] Kölbig, K., On the Integral $\int_0^\infty e^{-\mu t} t^{v-1} \log^m t dt$, *Mathematics of Computation* **41** (1983), 171–182.
- [25] Lindelöf, E., *Le calcul des résidus et ses applications à la théorie des fonctions*, Gauthier-Villars, 1905.
- [26] Loya, P., Dirichlet and Fresnel Integrals via Iterated Integration, *Mathematics Magazine* **78** (2005), 63–67.
- [27] Ossicini, A., An alternative form of the functional equation for Riemann's zeta function, *Atti del Seminario Matematico e Fisico dell' Università di Modena e Reggio Emilia* **56** (2008–09), 95–111.
- [28] Riemann, B., Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsberichte der Berliner Akademie* (1859), 10 pp.
- [29] Rooney, P., Another Proof of the Functional Equation for the Riemann Zeta Function, *Journal of Mathematical Analysis and Applications*, **185** (1994), 223–228.
- [30] Roussos, I., *Improper Riemann Integrals*, CRC Press, Taylor & Francis, 2014.
- [31] Simon, B., *Basic Complex Analysis, A Comprehensive Course in Analysis, Part 2A*, American Mathematical Society, 2015.
- [32] Srivastava, H. M. and Choi, J., *Series Associated with the Zeta and Related Functions*, Elsevier, 2012.
- [33] Srivastava, H. M., Glasser, M. and Adamchik, V., Some definite integrals associated with the Riemann zeta function, *Zeitschrift für Analysis und ihre Anwendungen*, **19** (2000), 831–846.

- [34] Steuding, J., An Introduction to the Theory of L -functions, A course given at the Universidad Autónoma de Madrid, 2005/06.
- [35] Tate, J., Fourier Analysis in Number Fields and Hecke's Zeta-Functions, Doctoral Dissertation, Princeton University, 1950.
- [36] Titchmarsh, E., The Theory of the Riemann Zeta-Function, Oxford University Press, 1986.
- [37] Varadarajan, V. S., Euler Through Time: A New Look at Old Themes, American Mathematical Society, 2006.
- [38] Watson, G., A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1944.
- [39] Weil, A., Elliptic Functions according to Eisenstein and Kronecker, *Ergebnisse der Mathematik und ihre Grenzgebiete*, Vol. **88**, Springer, 1976.

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