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A NOTE ON THE PRODUCT OF INDEPENDENT RANDOM VARIABLES WITH REGULARLY VARYING TAILS

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Abstract. The tail probability of the product of independent random variables is discussed. The class of random variables with regularly varying tails is known to be closed under multiplication and some explicit examples are given by Kifer-Varadhan [2]. We extend them.

1. Introduction

Let X and Y be independent random variables with regularly varying tail probabilities. We study the asymptotics of $P(XY > x)$ (the tail probability of the product random variable) as $x \to \infty$. The problem is motivated by Kifer-Varadhan [2], which will be explained a little later. Here, by *regularly varying function* with *index* $\rho \in \mathbf{R}$ we mean a measurable positive function $f(x)$ defined on an interval $[A, \infty)$ with the property that

$$\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\rho} \quad (\forall \lambda > 0). \tag{1.1}$$

Throughout R_ρ will denote the class of all regularly varying functions with index ρ. When $\rho = 0$, we say *slowly varying* rather than regularly varying. So $f \in R_\rho$ if and only if $f(x) = x^{\rho} L(x)$ for some slowly varying L. It is well known that the convergence in (1.1) is automatically uniform on every compact λ-set in $(0, \infty)$ (see e.g. [1, page 6]). Typical examples of slowly varying functions are $c(\log x)^{\delta_1}(\log \log x)^{\delta_2}$ where $c > 0$ and $\delta_1, \delta_2 \in \mathbf{R}$. Another example is $\exp\{(\log x)^{\beta}\}$ $(0 < \beta < 1)$.

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We now return to the problem we stated at the beginning. In what follows we assume, for simplicity, that $X \geq 0$ and $Y \geq 0$ unless stated explicitly. General cases may easily be reduced to this case because

$$(1.2) \qquad P(XY > x) = P(X^+Y^+ > x) + P(X^-Y^- > x) \quad (x > 0).$$

A basic result on our problem is Theorem A below obtained by Embrechts and Goldie [3]. Throughout $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$ and $f(x) = o(g(x))$ denotes $\lim_{x\to\infty} f(x)/g(x) = 0$.

THEOREM A (Theorem 3 and Corollary of [3]). *Let $X \geq 0$ and $Y \geq 0$ be independent random variables satisfying the following with some $\alpha > 0$.*

(1) $P(X > x) \sim x^{-\alpha}L_1(x) \in R_{-\alpha}$

(2) *Either* $P(Y > x) \sim x^{-\alpha}L_2(x) \in R_{-\alpha}$ *or* $P(Y > x) = o(P(X > x))$

Then,

$$P(XY > x) \sim x^{-\alpha}L_3(x) \in R_{-\alpha}$$

for some $L_3 (\in R_0)$.

In other words, the domain of attraction of the α-stable law is closed under convolution (when $0 < \alpha < 2$).

An easy corollary of Theorem A is the following: Let $\{X_j\}_{j=1}^n$ be independent random variables such that $P(X_j > x) \in R_{-\alpha_j}$. Then

$$P(X_1X_2\cdots X_n > x) \in R_{-\alpha}, \quad \text{where } \alpha = \min_j \alpha_j.$$

However, Theorem A above is a theoretical result and does not mention any explicit formulas for L_3 in terms of L_1 and L_2 (even in the simplest case where L_1 and L_2 are constants!). In fact it seems hard to find a simple formula that is applicable to the general case, but if we confine ourselves to a certain subclass, which should include the normal domain of attraction of α-stable laws, there is an explicit formula obtained by [2] we mentioned above. It studies the case where $\{X_j\}_j$ are independent random variables such that, for some $\alpha_j > 0$ and $k_j \geq 0$,

$$(1.3) \qquad P(X_j > x) \sim c_j x^{-\alpha_j}(\log x)^{k_j}, \quad j = 1, 2, \dots$$

In [2, Proposition 3.1] it is proved, after a lot of calculi, that if $\alpha_1 < \alpha_2$, then

$$(1.4) \qquad P(X_1X_2 > x) \sim E[X_2^{\alpha_1}]c_1x^{-\alpha_1}(\log x)^{k_1}$$

and, if $\alpha_1 = \alpha_2$, then

$$P(X_1X_2 > x) \sim \alpha_1 B(k_1+1, k_2+1)c_1c_2x^{-\alpha_1}(\log x)^{k_1+k_2+1}, \tag{1.5}$$

where $B(p,q)$ is the beta function. Using these two formulas [2] studied the tails of random variables of the form

$$Z = F(X_1, \dots, X_n)$$

where $F(x_1, \dots, x_n)$ is a polynomial, and it is proved that the tail behavior of Z is also of the form (1.3). A typical example is the following: Let $X_1, \dots, X_n$ be independent identically distributed (i.i.d.) nonnegative random variables such that

$$P(X_1 > x) \sim cx^{-\alpha}$$

for some $c, \alpha > 0$. Then,

$$P(X_1X_2\cdots X_n > x) \sim \frac{\alpha^{n-1}c^n}{(n-1)!}x^{-\alpha}(\log x)^{n-1}. \tag{1.6}$$

This shows that the class (1.3) is the minimal one when we consider our problem.

Now in the present article we try to relax the condition (1.3) to generalize (1.4) and (1.5). One of the aim of this generalization is to clarify the mechanism and simplify the proofs of [2] above.

Our results are as follows. The proofs will be given in Section 3.

Proposition 1.1. *Let $X \geq 0$ and $Y \geq 0$ be independent random variables such that $P(X > x) \in R_{-\alpha}$ for some $\alpha > 0$.*

(i)

$$\liminf_{x\to\infty} \frac{P(XY > x)}{P(X > x)} \geq E[Y^\alpha]. \tag{1.7}$$

(ii) *If $E[Y^\alpha] = \infty$, then*

$$P(X > x) = o(P(XY > x)).$$

(iii) *If $E[Y^{\alpha+\epsilon}] < \infty$ $(\exists \epsilon > 0)$, then*

$$\lim_{x\to\infty} \frac{P(XY > x)}{P(X > x)} = E[Y^\alpha]. \tag{1.8}$$

Note that the condition of (iii) is satisfied if $P(Y > x) \in R_{-\beta}$ for some $\beta > \alpha$, So (iii) includes (1.4).

In fact Proposition 1.1 is almost clear and we do not claim that this result is new, although the proof will be given in Section 3 to make sure. See also Shimura [4] for the extreme case $\alpha = 0$.

Our generalization of (1.5) is the following.

Theorem 1.1. *Let $X \geq 0$ and $Y \geq 0$ be independent random variables with the following tail probabilities; for some $\alpha, \beta, \gamma > 0$,*

$$P(X > x) \sim x^{-\alpha}\varphi(\log x) \quad \text{with } \varphi(x) \in R_{\beta-1};$$

$$P(Y > x) \sim x^{-\alpha}\psi(\log x) \quad \text{with } \psi(x) \in R_{\gamma-1}.$$

Then,

$$P(XY > x) \sim \alpha B(\beta, \gamma) x^{-\alpha}\varphi(\log x)\psi(\log x) \log x. \tag{1.9}$$

Remark 1.1. Here, the condition $\beta, \gamma > 0$ implies $E[X^\alpha] = E[Y^\alpha] = \infty$, for $E[X^\alpha] = \alpha \int_0^\infty x^{\alpha-1} P[X > x]\, dx$. So Theorem 1.1 corresponds to the case (ii) of Proposition 1.1.

We considered the case where $X, Y \geq 0$. The general case can be reduced to this case using (1.2). Namely, if X, Y are not necessarily nonnegative and satisfy

$$\lim_{x\to\infty} \frac{P(\pm X > x)}{x^{-\alpha}\varphi(\log x)} = a_\pm, \quad \lim_{x\to\infty} \frac{P(\pm Y > x)}{x^{-\alpha}\psi(\log x)} = b_\pm,$$

where $a_\pm \geq 0$, $b_\pm \geq 0$. Then,

$$P(XY > x) \sim \alpha B(\beta, \gamma) x^{-\alpha} C\varphi(\log x)\psi(\log x) \log x,$$

where $C = a_+b_+ + a_-b_-$. (Even in the case where some (or all) of $\{a_\pm, b_\pm\}$ vanish, the assertion remains valid with a suitable convention.)

Corollary 1.1. *Let $X_1, \dots, X_n$ be independent nonnegative random variables such that*

$$P(X_j > x) \sim x^{-\alpha}\varphi_j(\log x) \quad \varphi \in R_{\beta_j - 1} \ (j = 1, \dots, n)$$

for some $\alpha, \beta_j > 0$. Then

$$P(X_1 X_2 \cdots X_n > x) \sim \alpha^{n-1} D(\beta_1, \dots, \beta_n) x^{-\alpha} (\log x)^{n-1} \prod_{j=1}^n \varphi_j(\log x),$$

where

$$D(\beta_1, \dots, \beta_n) = \prod_{j=1}^{n} \Gamma(\beta_j) \Big/ \Gamma\left(\sum_{j=1}^{n} \beta_j\right).$$

In Theorem 1.1, the condition that $\beta, \gamma > 0$ is crucial, because the beta function $B(\beta, \gamma)$ appears in (1.9). But it might be of interest to study the critical case $\gamma = 0$, which case is not discussed by [2] except for the following example.

Example 1.1 ([2, Example 1]). If

$$P(X > x) \sim c_1 x^{-\alpha}, \quad P(Y > x) \sim c_2 x^{-\alpha}/\log x,$$

then

$$P(XY > x) \sim \alpha c_1 c_2 x^{-\alpha} \log(\log x).$$

Our generalization of this example is the following: For $\psi \in R_{\gamma-1}$, let

$$\Psi(x) := \int_1^x \psi(u)\, du. \quad x \ge 1.$$

It is well known that, if $\gamma > 0$, then

$$\Psi(x) \sim (1/\gamma) x \psi(x) \in R_\gamma \tag{1.10}$$

and this remains valid in the extreme case $\gamma = 0$ in the sense that $\Phi \in R_0$ and $x\varphi(x) = o(\Psi(x))$. See e.g. page 26 of [1] for the proofs, if necessary. These facts will be used later. Our generalization of Example 1.1 is

Theorem 1.2. *In Theorem* 1.1 *replace the condition* $\gamma > 0$ *by* $\gamma = 0$. *Then,*

$$P(XY > x) \sim \alpha x^{-\alpha} \varphi(\log x) \Psi(\log x) \tag{1.11}$$

provided that $\Psi(\infty) = \infty$.

Example 1.1 is the special case where $\varphi(x) = c_1$, $\psi(x) = c_2/x$ so that $\Psi(x) = c_2 \log x$.

The assertion of Theorem 1.2 is not surprising because, when $\gamma > 0$, it holds that $\psi(x)x \sim \gamma\Psi(x)$ as we mentioned in (1.10). So (1.9) may be rewritten as

$$P(XY > x) \sim \alpha \frac{\Gamma(\beta)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma)} x^{-\alpha} \varphi(\log x) \Psi(\log x).$$

Therefore, letting $\gamma \to +0$ we naturally expect (1.11) to hold. The case where $\beta = \gamma = 0$ can also be treated in a similar way. For example;

EXAMPLE 1.2. If

$$P(X > x) \sim c_1 x^{-\alpha}/\log x, \quad P(Y > x) \sim c_2 x^{-\alpha}/\log x,$$

then

$$P(XY > x) \sim 2\alpha c_1 c_2 x^{-\alpha} (\log x)^{-1} \log(\log x).$$

REMARK 1.2. In Theorem 1.2, we may not remove the condition $\Psi(\infty) = \infty$, (which condition is equivalent to $E[Y^\alpha] = \infty$). Indeed, if $\Psi(\infty) < \infty$ and hence $E[Y^\alpha] < \infty$, then we may possibly have a formula like (1.8), where the constant depends not on the tail probability but on $E[Y^\alpha]$.

2. Preliminaries: Abelian theorem for convolution

In this section we prepare some results on convolutions. Throughout this section $\varphi(x)$ and $\psi(x)$ are locally bounded measurable positive functions defined on the half line $[0, \infty)$ and $\varphi * \psi(x)$ denotes the convolution;

$$\varphi * \psi(x) = \int_0^x \varphi(x-u)\psi(u)\, du \quad (x \ge 0).$$

PROPOSITION 2.1. *If $\varphi \in R_{\beta-1}$ $(\beta > 0)$ and $\psi \in R_{\gamma-1}$ $(\gamma > 0)$, then*

$$\varphi * \psi(x) \sim B(\beta, \gamma) x \varphi(x) \psi(x).$$

PROOF. Laplace transform is useful to handle convolutions. So when $\varphi * \psi(x)$ is monotone, the assertion follows immediately from Karamata's Tauberian theorem for Laplace transform. However, we shall not adopt this idea because we do not want to assume the monotonicity of $\varphi * \psi(x)$.

By a simple change of variables we see

$$\frac{1}{x\varphi(x)\psi(x)} \int_0^x \varphi(x-u)\psi(u)\, du = \int_0^1 \frac{\varphi((1-u)x)}{\varphi(x)} \frac{\psi(ux)}{\psi(x)}\, du.$$

As for the integrand we have

$$\lim_{x\to\infty} \frac{\varphi((1-u)x)}{\varphi(x)} \frac{\psi(ux)}{\psi(x)} = (1-u)^{\beta-1} u^{\gamma-1} \quad (0 < u < 1)$$

by the assumption $\varphi \in R_{\beta-1}$, $\psi \in R_{\gamma-1}$. So the assertion is almost clear. This may look a heuristic argument, but it is routine to justify it using well-known Potter's theorem (see [1, page 25]) combined with the dominated convergence theorem as follows: Choose $\varepsilon > 0$ small enough so that $\beta - \varepsilon > 0$, $\gamma - \varepsilon > 0$. Then, by Potter's theorem, there exist $C > 0$ and $A > 0$ such that

$$\frac{\varphi(y)}{\varphi(x)} \le C \max\{(y/x)^{\beta-\varepsilon-1}, (y/x)^{\beta+\varepsilon-1}\},$$

$$\frac{\psi(y)}{\psi(x)} \le C \max\{(y/x)^{\gamma-\varepsilon-1}, (y/x)^{\gamma+\varepsilon-1}\}$$

for all $x, y > A$. So

$$\frac{\varphi((1-u)x)}{\varphi(x)} \frac{\psi(ux)}{\psi(x)} \le C(1-u)^{\beta-\varepsilon-1} u^{\gamma-\varepsilon-1} \quad (A/x \le \forall u \le 1-(A/x)).$$

Therefore, by the dominated convergence theorem,

$$\lim_{x\to\infty} \int_{A/x}^{1-(A/x)} \frac{\varphi((1-u)x)}{\varphi(x)} \frac{\psi(ux)}{\psi(x)} \, du = \int_0^1 (1-u)^{\beta-1} u^{\gamma-1} \, du.$$

So it remains only to show that

$$\lim_{x\to\infty} \int_0^{A/x} \frac{\varphi((1-u)x)}{\varphi(x)} \frac{\psi(ux)}{\psi(x)} \, du = 0 \tag{2.1}$$

and

$$\lim_{x\to\infty} \int_{1-(A/x)}^1 \frac{\varphi((1-u)x)}{\varphi(x)} \frac{\psi(ux)}{\psi(x)} \, du = 0. \tag{2.2}$$

Let us see (2.1). Since

$$\frac{\varphi((1-u)x)}{\varphi(x)}$$

converges uniformly in u around 0, it suffices to show

$$\lim_{x\to\infty} \int_0^{A/x} \frac{\psi(ux)}{\psi(x)} \, du = 0,$$

or, equivalently,

$$\lim_{x\to\infty} \frac{1}{x\psi(x)} \int_0^A \psi(u) \, du = 0.$$

But this is trivial because $\psi \in R_{\beta-1}$ $(\beta > 0)$ implies $x\psi(x) \to \infty$. Similarly, we see (2.2) by changing the role of φ and ψ. □

We next study the critical case where $\gamma = 0$ (but $\beta > 0$). In this case, $\Psi(x) := \int_0^x \psi(u)\,du$ is slowly varying and it holds that $\psi(x) = o(x^{-1}\Psi(x))$ as we mentioned before Theorem 1.2.

The most typical example is, if $\psi(x) \sim 1/x$, then $\Psi(x) \sim \log x$. More generally, if $\psi(x) \sim x^{-1}(\log x)^{\delta-1}$, then $\Psi(x) \sim \frac{1}{\delta}(\log x)^{\delta}$ if $\delta > 0$.

Proposition 2.2. *Suppose that $\varphi \in R_{\beta-1}$ $(\beta > 0)$ and $\psi \in R_{-1}$. Then,*

$$\varphi * \psi(x) \sim \varphi(x)\Psi(x)$$

Proof. The proof is essentially the same as that of Proposition 2.1: Recall

$$\frac{1}{\varphi(x)\Psi(x)}\int_0^x \varphi(x-u)\psi(u)\,du = \int_0^1 \frac{\varphi((1-u)x)}{\varphi(x)}\,\frac{x\psi(ux)}{\Psi(x)}\,du, \tag{2.3}$$

and let us evaluate the right-hand side.

For every $0 < \varepsilon < 1$, we have, as before,

$$\lim_{x\to\infty}\int_\varepsilon^1 \frac{\varphi((1-u)x)}{\varphi(x)}\,\frac{\psi(ux)}{\psi(x)}\,du = \int_\varepsilon^1 (1-u)^{\beta-1}u^{-1}\,du.$$

Since, as we mentioned before, $\psi(x) = o(x^{-1}\Psi(x))$, replacing $\psi(x)$ in the denominator by $x^{-1}\Psi(x)$, we see

$$\lim_{x\to\infty}\int_\varepsilon^1 \frac{\varphi((1-u)x)}{\varphi(x)}\,\frac{x\psi(ux)}{\Psi(x)}\,du = 0.$$

So the main part of (2.3) is

$$I_\varepsilon(x) := \int_0^\varepsilon \frac{\varphi((1-u)x)}{\varphi(x)}\,\frac{x\psi(ux)}{\Psi(x)}\,du, \tag{2.4}$$

for small $\varepsilon > 0$.

Now notice that

$$\frac{\varphi((1-u)x)}{\varphi(x)} \to (1-u)^{\beta-1} \quad (x \to \infty)$$

holds uniformly in $u \in [0, \varepsilon]$. So, when an $\varepsilon'(>0)$ is given arbitrarily, we can choose small $\varepsilon > 0$ so that

$$1 - \varepsilon' < \frac{\varphi((1-u)x)}{\varphi(x)} < 1 + \varepsilon'$$

holds uniformly in $u \in [0, \varepsilon]$ for all large x. So $I_\varepsilon(x)$ in (2.4) may arbitrarily be approximated by

$$J_\varepsilon(x) := \int_0^\varepsilon \frac{x\psi(ux)}{\Psi(x)}\, du = \frac{1}{\Psi(x)} \Psi(\varepsilon x).$$

Since $\Psi(x)$ is known to be slowly varying by the general theory as we mentioned before, the right-hand side converges to 1 as $x \to \infty$. □

The critical case $\beta = \gamma = 0$ is:

Proposition 2.3. *Let* $\varphi, \psi \in R_{-1}$ *and define* $\Phi(x) = \int_0^x \varphi(u)\, du$, $\Psi(x) = \int_0^x \psi(u)\, du$, *so that* $\Phi, \Psi \in R_0$. *Then,*

$$\varphi * \psi(x) = (1 + o(1))\varphi(x)\Psi(x) + (1 + o(1))\Phi(x)\psi(x).$$

Proof. As in the proof of Proposition 2.2, we have

$$\int_0^{x/2} \varphi(x-u)\psi(u)\, du \sim \varphi(x)\Psi(x).$$

Since

$$\int_{x/2}^1 \varphi(x-u)\psi(u)\, du = \int_0^{x/2} \psi(x-u)\varphi(u)\, du,$$

we similarly have

$$\int_{x/2}^1 \varphi(x-u)\psi(u)\, du \sim \psi(x)\Phi(x).$$

Summing these two we have the assertion. □

Example 2.1. Suppose that $\varphi(x) \sim x^{\beta-1}(\log x)^{\delta_1 - 1}$ and $\psi(x) \sim x^{\gamma-1}(\log x)^{\delta_2 - 1}$ for some $\delta_1, \delta_2 \in \mathbf{R}$.

(i) If $\beta > 0$, $\gamma > 0$, then

$$\varphi * \psi(x) \sim B(\beta, \gamma) x^{\beta+\gamma-1}(\log x)^{\delta_1+\delta_2-2}.$$

(ii) If $\beta > 0$, $\gamma = 0$ and $\delta_2 > 0$, then $\Psi(x) \sim (1/\delta_2)(\log x)^{\delta_2}$ and hence,

$$\varphi * \psi(x) \sim \frac{1}{\delta_2} x^{\beta-1}(\log x)^{\delta_1+\delta_2-1}.$$

(iii) If $\beta = \gamma = 0$ and $\delta_1, \delta_2 > 0$, then $\Phi(x) \sim (1/\delta_1)(\log x)^{\delta_1}$ and $\Psi(x) \sim (1/\delta_2)(\log x)^{\delta_2}$. Hence,

$$\varphi * \psi(x) \sim \left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right) x^{-1}(\log x)^{\delta_1+\delta_2-1}.$$

PROPOSITION 2.4. *Suppose that* $\varphi \in R_{\beta-1}$ $(\beta \geq 0)$ *and* $\psi \in R_{\gamma-1}$ $(\gamma \geq 0)$. *Also assume that* ψ *has locally bounded variation on* $[0, \infty)$ *and is absolutely continuous on* (A, ∞) $(\exists A \geq 0)$ *with the derivative* $\psi'(x) = o(\psi(x))$. *We put* $\Phi(x) = \int_0^x \varphi(u)\, du$, $\Psi(x) = \int_0^x \psi(u)\, du$ *and, when* $\gamma = 0$, *we further assume that* $\Psi(+\infty) = \infty$.

Now, for $\alpha > 0$, *let* $\varphi_\alpha(x) = e^{-\alpha x}\varphi(x)$, $\psi_\alpha(x) = e^{-\alpha x}\psi(x)$ *and define*

$$f(x) = -\int_0^x \varphi_\alpha(x-u)\, d\psi_\alpha(u).$$

Then:

(i) *When* $\beta > 0$, $\gamma > 0$,

$$f(x) \sim \alpha e^{-\alpha x} B(\beta, \gamma)\varphi(x)\psi(x)x, \quad x \geq 0.$$

(ii) *When* $\beta > 0$, $\gamma = 0$,

$$f(x) \sim \alpha e^{-\alpha x}\varphi(x)\Psi(x).$$

(iii) *When* $\beta = \gamma = 0$,

$$f(x) = \alpha e^{-\alpha x}\{(1 + o(1))\varphi(x)\Psi(x) + (1 + o(1))\Phi(x)\psi(x)\}.$$

The meaning of $f(x)$ above is the following: Let $X^*, Y^* \geq 0$ be independent random variables such that $P(X^* > x) = \varphi_\alpha(x)$, $P(Y^* > x) = \psi_\alpha(x)$, then $P(X^* + Y^* > x) = f(x)$.

PROOF. Since

$$d\psi_\alpha(x) = -\alpha e^{-ax}\psi(x)\, dx + e^{-ax}\, d\psi(x),$$

we see

$$f(x) = \alpha e^{-\alpha x}(\varphi * \psi)(x) - e^{-\alpha x}\int_0^x \varphi(x-u)\, d\psi(u).$$

But $\varphi * \psi(x)$ is already discussed in Propositions 2.1–2.3. So it remains only to show that the second term in the right-hand side is negligible; i.e.,

$$\int_0^x \varphi(x-u)\, d\psi(u) = o(\varphi * \psi(x)). \tag{2.5}$$

For every given $\epsilon > 0$, there exists an $M(> A)$ such that $|\psi'(x)| < \epsilon\psi(x)$ for all $x > M$. So we have

$$\left|\int_M^x \varphi(x-u)\psi'(u)\, du\right| \le \epsilon \int_M^x \varphi(x-u)\psi(u)\, du \le \epsilon\varphi * \psi(x). \tag{2.6}$$

In order to handle the remainder, note that

$$\lim_{x\to\infty} \int_0^M \frac{\varphi(x-u)}{\varphi(x)}\, d\psi(u) = \int_0^M \lim_{x\to\infty} \frac{\varphi(x-u)}{\varphi(x)}\, d\psi(u) = \int_0^M 1\, d\psi(u).$$

(The first equality is allowed because the convergence of the integrand is uniform in $u \in [0, M]$.) So

$$\int_0^M \varphi(x-u)\, d\psi(u) = O(\varphi(x)). \tag{2.7}$$

Here, notice that $\varphi(x) = o(\varphi * \psi(x))$, which follows from Propositions 2.1–2.3, because, when $\gamma > 0$, it holds $x\psi(x) \to \infty$, and when $\gamma = 0$, we assumed $\Psi(+\infty) = \infty$. Therefore, combining (2.6) and (2.7) we have (2.5). □

3. Proofs

Proof of Proposition 1.1. By the assumption $P(X > x) \in R_{-\alpha}$, we have

$$\lim_{x\to\infty} \frac{P(X > x/u)}{P(X > x)} = u^\alpha \quad (u > 0).$$

So, formally,

$$\begin{aligned} \lim_{x\to\infty} \frac{P(XY > x)}{P(X > x)} &= \lim_{x\to\infty} \int_0^\infty \frac{P(X > x/u)}{P(X > x)} P(Y \in du) \\ &= \int_0^\infty u^\alpha P(Y \in du) = E[Y^\alpha]. \end{aligned} \tag{3.1}$$

In general this cannot be justified without additional conditions. But Fatou's lemma is applicable and therefore, we have

$$\liminf_{x\to\infty} \frac{P(XY > x)}{P(X > x)} \ge E[Y^\alpha],$$

which proves (i) and (ii). We next prove (iii). In this case the justification of (3.1) is routine:

By Potter's theorem there exist $C, M > 0$ such that

$$P(X > y)/P(X > x) \le C \max\{(y/x)^{-\alpha-\varepsilon}, (y/x)^{-\alpha+\varepsilon}\} \quad (\forall x, y > M).$$

So if $x > M$, then

$$\frac{P(X > x/u)}{P(X > x)} \le C \max\{u^{\alpha-\varepsilon}, u^{\alpha+\varepsilon}\} \quad (\forall u \in (0, x/M]).$$

By the assumption $E[Y^{\alpha+\varepsilon}] < \infty$,

$$\int_0^\infty \max\{u^{\alpha-\varepsilon}, u^{\alpha+\varepsilon}\} P(Y \in du) < \infty.$$

Therefore, we can apply the dominated convergence theorem to obtain

$$\lim_{x\to\infty} \int_0^{x/M} \frac{P(X > x/u)}{P(X > x)} P(Y \in du) = \int_0^\infty u^\alpha P(Y \in du).$$

So, it remains only to show

$$\lim_{x\to\infty} \int_{x/M}^\infty \frac{P(X > x/u)}{P(X > x)} P(Y \in du) = 0. \tag{3.2}$$

To see this note that

$$\int_{x/M}^\infty \frac{P(X > x/u)}{P(X > x)} P(Y \in du) \le \int_{x/M}^\infty \frac{1}{P(X > x)} P(Y \in du) = \frac{P(Y > x/M)}{P(X > x)}.$$

To evaluate the extreme right-hand side, we use Chebyshev's inequality,

$$P(Y > x/M) \le (x/M)^{-(\alpha+\varepsilon)} E[Y^{\alpha+\varepsilon}].$$

So

$$\int_{x/M}^\infty \frac{P(X > x/u)}{P(X > x)} P(Y \in du) \le \frac{M^{\alpha+\varepsilon} E[Y^{\alpha+\varepsilon}]}{x^{\alpha+\varepsilon} P(X > x)}.$$

Since $P(X > x) \in R_{-\alpha}$ by assumption, it holds $x^{\alpha+\varepsilon} P(X > x) \to \infty$ and therefore we have (3.2). □

Thus Proposition 1.1 is proved. An easy corollary is

COROLLARY 3.1. *Let $X, Y_1, Y_2 \geq 0$ be random variables such that X is independent of Y_i $(i = 1, 2)$. Then,*

$$P(Y_1 > x) \sim P(Y_2 > x)$$

implies

$$P(XY_1 > x) \sim P(XY_2 > x),$$

if $P(X > x) \in R_{-\alpha}$ $(\alpha > 0)$ and if $E[Y_1^\alpha] = \infty$.

PROOF. By Proposition 1.1(ii), we have $P(X > x) = o(P(XY_i > x))$ $(i = 1, 2)$. So, for every fixed $A > 0$,

$$P(XY_i > x, Y_i \leq A) \leq P(X > x/A) \sim A^\alpha P(X > x) = o(P(XY_i > x)).$$

Therefore,

$$P(XY_i > x) \sim P(XY_i > x, Y_i > A) = \int_0^\infty P(Y_i > \max(x/u, A))P(X \in du),$$

for any $A > 0$, and therefore the assertion is clear. □

We now proceed to the proofs of Theorems 1.1 and 1.2.

Since $E[X^\alpha] = E[Y^\alpha] = \infty$ (see Remarks 1.1 and 1.2), we may assume that $X, Y > 1$ without loss of generality thanks to Corollary 3.1. So letting $X^* = \log X$ and $Y^* = \log Y$, we see that Theorem 1.1 is equivalent to

THEOREM 3.1. *Let X^* and Y^* be nonnegative independent random variables such that, for some $\alpha > 0$,*

$$P(X^* > x) = e^{-\alpha x}\varphi(x) \quad \text{with } \varphi \in R_{\beta-1} \ (\beta > 0);$$

$$P(Y^* > x) = e^{-\alpha x}\psi(x) \quad \text{with } \psi \in R_{\gamma-1} \ (\gamma > 0).$$

Then,

$$P(X^* + Y^* > x) \sim \alpha B(\beta, \gamma)e^{-\alpha x}\varphi(x)\psi(x)x. \tag{3.3}$$

(The convolution of two Gamma distributions is again a Gamma distribution. Theorem 3.1 is its asymptotic version.)

Now let us prove Theorem 3.1. Its assertion (and hence that of Theorem 1.1) is almost clear by Proposition 2.4, where $f(x)$ corresponds to $P(X^* + Y^* > x)$.

The only problem is that $\psi(x)$ does not necessarily satisfy the condition that it is absolutely continuous and $\psi'(x) = o(\psi(x))$. However, as we remarked in Corollary 3.1, only the asymptotics of $\psi(x)$ is essential and the smoothness is not at all important. Therefore, it is sufficient to see that there is a random variable $Z^* \geq 0$ such that $P(Z^* > x) = e^{-\alpha x}\psi_0(x)$ where $\psi(x) \sim \psi_0(x)$ and $\psi_0'(x) = o(\psi_0(x))$. But this is almost trivial if we recall the canonical representation of slowly varying functions (see [1, page 12]); i.e., $\psi \in R_\gamma$ has the representation

$$\psi(x) = x^\gamma c(x) \exp \int_1^x \frac{\varepsilon(u)}{u}\, du$$

where $c(x) \to c > 0$ and $\varepsilon(x) \to 0$. So define

$$\psi_0(x) = c x^\gamma \exp \int_1^x \frac{\varepsilon(u)}{u}\, du$$

for large x. Then $\psi_0(x) \sim \psi(x)$ and $\psi_0'(x) = o(\psi_0(x))$. Since it is easy to see that $x^{-\alpha}\psi_0(x)$ is decreasing for large x and tends to 0, the function $F(x) := 1 - x^{-\alpha}\psi_0(x)$ becomes a distribution function if we modify it on a finite interval, if necessary. Then we have a desired one.

Similarly, Theorem 1.2 follows from Proposition 2.4 (ii). Also Example 1.2 may be shown by Proposition 2.4 (iii).

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