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ON AN ANALOG OF THE ARAKAWA-KANEKO ZETA FUNCTION AND RELATIONS OF SOME MULTIPLE ZETA VALUES

By

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Abstract. T. Ito defined an analog of the Arakawa-Kaneko zeta function to obtain relations among Mordell-Tornheim multiple zeta values. In this paper, we develop two things related to an analog of the Arakawa-Kaneko zeta function. One is to find an analog of the Arakawa-Kaneko zeta function of Miyagawa-type (defined by T. Miyagawa) and to obtain relations among Miyagawa-type multiple zeta values. The other is to find a class of zeta functions to which Ito's zeta functions of the case of general index are related.

1. Introduction

The Arakawa-Kaneko zeta function is the following function introduced in [2].

Definition 1 (The Arakawa-Kaneko zeta function). For $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{N}^n$ and $s \in \mathbf{C}$ with $\Re(s) > 1 - n$, the Arakawa-Kaneko zeta function is defined by

$$\xi(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \operatorname{Li}_{\mathbf{k}}(1 - e^{-t})\, dt,$$

where $\operatorname{Li}_{\mathbf{k}}(z)$ is the multi-polylogarithm defined by

$$\operatorname{Li}_{\mathbf{k}}(z) = \sum_{0 < m_1 < m_2 < \cdots < m_n} \frac{z^{m_n}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}} \quad (|z| < 1).$$

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The Arakawa-Kaneko zeta function has a connection with Euler-Zagier multiple zeta values (EZ-type MZVs for brevity) and poly-Bernoulli numbers (see [2]). Here, EZ-type MZVs are the special values of the following functions.

Definition 2 (The Euler-Zagier multiple zeta function (EZ-type MZF)). For $\mathbf{s} = (s_1, \dots, s_n) \in \mathbf{C}^n$, the Euler-Zagier multiple zeta function is defined by

$$\zeta(\mathbf{s}) = \sum_{0<m_1<m_2<\cdots<m_n} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_n^{s_n}}.$$

This series converges absolutely when

$$\sum_{i=0}^{k} \Re(s_{n-i}) > k+1$$

for any k with $0 \le k \le n-1$ (see [6]) and can be continued meromorphically to the whole $\mathbf{C}^n$ space (see [1]). The values of EZ-type MZFs at $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{N}^n$ with $k_n \ge 2$ are called Euler-Zagier multiple zeta values (EZ-type MZVs).

In this paper, we focus on the fact that the properties of the Arakawa-Kaneko zeta function lead to certain relations among EZ-type MZVs. Regarding this, there is the work of Ito [3]. Ito introduced the following function as an analog of the Arakawa-Kaneko zeta function:

Definition 3 (The Ito zeta function). For $k_1, \dots, k_r \in \mathbf{N}$ and $s \in \mathbf{C}$ with $\Re(s) > 1-r$, we define

$$\xi_{MT}(k_1, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \prod_{i=1}^{r} \mathrm{Li}_{k_i}(1 - e^{-t})\, dt. \tag{1}$$

We call the function (1) the Ito zeta function in this paper. We also introduce MT-type MZVs which are the special values of the following functions.

Definition 4 (The Mordell-Tornheim multiple zeta function (MT-type MZF)). For $s_1, \dots, s_{r+1} \in \mathbf{C}$, the Mordell-Tornheim multiple zeta function is defined by

$$\zeta_{MT}(s_1, \dots, s_r; s_{r+1}) = \sum_{m_1=1, \dots, m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^{s_{r+1}}}.$$

This series converges absolutely when

$$\sum_{l=1}^{j} \Re(s_{k_l}) + \Re(s_{r+1}) > j$$

with $1 \leq k_1 < k_2 < \cdots < k_j \leq r$ for any $j = 1, 2, \dots, r$ (see [8]) and can be continued meromorphically to the whole $\mathbf{C}^n$ space (see [5]). The values of MT-type MZF at non-negative integer points in the domain of convergence are called Mordell-Tornheim multiple zeta values (MT-type MZVs). Ito used his zeta function to obtain certain relations among MT-type MZVs. Therefore, the Ito zeta function is an analog of the Arakawa-Kaneko zeta function of MT-type.

There is a generalization of the Ito zeta function, which was given by Ito himself as follows.

Definition 5 (The generalized Ito zeta function $(r = 1)$). For $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{N}^n$, $k_{n+1} \in \mathbf{Z}_{\geq 0}$ and $s \in \mathbf{C}$ with $\Re(s) > 1 - n$, we define

$$\xi_{MT}((\mathbf{k}; k_{n+1}); s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} \mathrm{Li}_{\mathbf{k}; k_{n+1}}(1 - e^{-t})\, dt, \tag{2}$$

where

$$\mathrm{Li}_{\mathbf{k}; k_{n+1}}(z) = \sum_{m_1=1, \dots, m_n=1}^{\infty} \frac{z^{\sum_{j=1}^{n} m_j}}{m_1^{k_1} \cdots m_n^{k_n} (\sum_{j=1}^{n} m_j)^{k_{n+1}}} \quad (|z| < 1).$$

Ito considered a version of the function (2), in which $\mathrm{Li}_{\mathbf{k}; k_{n+1}}(1 - e^{-t})$ is replaced by a product of r quantities of the form $\mathrm{Li}_{\mathbf{k}; k_{n+1}}(1 - e^{-t})$ ([3, Definition 13]). Therefore, we call the function (2) the generalized Ito zeta function $(r = 1)$.

On the other hand, there is also a generalization of MT-type MZF, which was given by Miyagawa as follows.

Definition 6 (The Miyagawa multiple zeta function (Miyagawa-type MZF)). For $s_1, \dots, s_{r+1} \in \mathbf{C}$, we define

$$\begin{aligned} \hat{\zeta}_{MT,j,r}(s_1, \dots, s_j; s_{j+1}, \dots, s_{r+1}) \\ = \sum_{m_1=1, \dots, m_r=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_j^{s_j} \prod_{u=j+1}^{r+1} (\sum_{v=1}^{u-1} m_v)^{s_u}}. \end{aligned} \tag{3}$$

This function was introduced by Miyagawa [7]. Moreover, he showed that the function (3) can be continued meromorphically to the whole $\mathbf{C}^{r+1}$ space. We call the function (3) the Miyagawa multiple zeta function (Miyagawa-type MZF). Moreover, we call the values of Miyagawa-type MZFs at non-negative integer points in the domain of convergence Miyagawa multiple zeta values (Miyagawa-type MZVs). In the present paper, we use the function (2) to obtain certain relations among Miyagawa-type MZVs. Ito's method uses functional relations between functions (1) and MT-type MZFs (Theorem 4), and our method uses functional relations between functions (2) and Miyagawa-type MZFs (Theorem 6). However, Theorem 4 and Theorem 6 only give functional relations in the case when all the indices k_i are 2 in the function (1) and the function (2). In this paper, we also study functional relations for the function (1) and the function (2) with general indices. For this purpose, we introduce the following new class of multiple zeta functions.

Definition 7 (The generalized Mordell-Tornheim multiple zeta function (GMT-type MZF)). For $\mathbf{s}_i = (s_{i,1}, \dots, s_{i,n_i}) \in \mathbf{C}^{n_i}$ $(1 \le i \le r+1,\ n_i \in \mathbf{N})$, we define

$$(4)\quad \zeta_{MT}(\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_r; \mathbf{s}_{r+1})$$

$$= \sum_{\substack{0<m_{1,1}<m_{1,2}<\cdots<m_{1,n_1}\\ \vdots \\ 0<m_{r,1}<m_{r,2}<\cdots<m_{r,n_r}}} \sum_{m_{r+1,1}=1,\dots,m_{r+1,n_{r+1}-1}=1}^{\infty} \frac{1}{\prod_{i=1}^{r}\prod_{j=1}^{n_i} m_{i,j}^{s_{i,j}} \prod_{u=1}^{n_{r+1}} (\sum_{v=1}^{r} m_{v,n_v} + \sum_{w=1}^{u-1} m_{r+1,w})^{s_{r+1,u}}}$$

$$= \sum_{\substack{0<m_{1,1},0<m_{1,2},\dots,0<m_{1,n_1}\\ \vdots \\ 0<m_{r,1},0<m_{r,2},\dots,0<m_{r,n_r}}} \sum_{m_{r+1,1}=1,\dots,m_{r+1,n_{r+1}-1}=1}^{\infty} \frac{1}{\prod_{i=1}^{r}\prod_{k=1}^{n_i} (\sum_{j=1}^{k} m_{i,j})^{s_{i,k}} \prod_{u=1}^{n_{r+1}} (\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{i,j} + \sum_{w=1}^{u-1} m_{r+1,w})^{s_{r+1,u}}}.$$

We call this function the generalized Mordell-Tornheim multiple zeta function (GMT-type MZF) and call the values of GMT-type MZFs at non-negative integer points in the domain of convergence generalized Mordell-Tornheim multiple zeta values (GMT-type MZVs). As a consequence of the present study, we can obtain relations among special values of functions (4).

Regarding relations among those functions, the known results and the results shown in the present paper are summarized as follows.

ξ-function	special value	functional relation	
Arakawa-Kaneko zeta function	EZ-type (Theorem 1)	EZ-type (Theorem 2)	
Ito zeta function	MT-type (Theorem 3)	MT-type (Theorem 8)	if $k_i \leq 2$
		GMT-type $(n_{r+1} = 1)$ (Theorem 8)	general
Generalized Ito zeta function $(r = 1)$	Miyagawa-type (Theorem 5)	Miyagawa-type (Theorem 10)	if $k_i \leq 2$ $(1 \leq i \leq n)$
		GMT-type (Theorem 10)	general

Remark 1. Theorem 4 expresses the relationship between Ito zeta functions with $k_i = 2$ and MT-type MZFs, and Theorem 6 expresses the relationship between generalized Ito zeta functions $(r = 1)$ with $k_i = 2$ $(1 \leq i \leq n)$ and Miyagawa-type MZFs.

In Section 2, we provide some notations, lemmas and known results which we need in later sections. In Section 3, we discuss the function (2) and prove Theorem 5 and Theorem 6. As a consequence, we can obtain relations among Miyagawa-type MZVs. In Section 4, we discuss the function (4) and prove several propositions on the function (4). In Section 5, using functions (4), we generalize Theorem 4 and 6 to Theorem 8 and 10, respectively. As a consequence, we can obtain relations among GMT-type MZVs.

2. Preliminaries

In this section, we provide some notations, lemmas and known results which we need later. In this paper, unless otherwise noted, k, n and r denote positive integers and s is a complex number also when these have subscripts. Moreover, $\{k\}^n$ denotes n repetitions of k. For example, $(1,2,2,3) = (1,\{2\}^2,3)$. For $\mathbf{k} =$

$(k_1, \dots, k_n)$, we define $\mathbf{k}_\pm = (k_1, \dots, k_{n-1}, k_n \pm 1)$ and ${}_\pm\mathbf{k} = (k_1 \pm 1, k_2, \dots, k_n)$. Let $\mathbf{k}^*$ denote the dual index of $\mathbf{k}$.

LEMMA 1 ([2, Lemma 1] and [3, Lemma 4]). *For* $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{N}^n$, *the following formulas hold*:

(i)

$$\frac{d}{dz} \operatorname{Li}_{\mathbf{k}}(z) = \begin{cases} \frac{1}{z} \operatorname{Li}_{\mathbf{k}_-}(z) & (\text{if } k_n > 1), \\ \frac{1}{1-z} \operatorname{Li}_{k_1, \dots, k_{n-1}}(z) & (\text{if } k_n = 1), \end{cases}$$

(ii)

$$\frac{d}{dz} \operatorname{Li}_{\mathbf{k}; k_{n+1}}(z) = \begin{cases} \frac{1}{z} \operatorname{Li}_{\mathbf{k}; k_{n+1}-1}(z) & (\text{if } k_{n+1} > 1), \\ \frac{1}{z} \prod_{i=1}^n \operatorname{Li}_{k_i}(z) & (\text{if } k_{n+1} = 1). \end{cases}$$

By Lemma 1, we have

$$\frac{d}{du} \operatorname{Li}_{\mathbf{k}_+}(1 - e^{-t-u}) = \frac{1}{e^{t+u} - 1} \operatorname{Li}_{\mathbf{k}}(1 - e^{-t-u}),$$

$$\frac{d}{du} \operatorname{Li}_{\mathbf{k}; 1}(1 - e^{-t-u}) = \frac{1}{e^{t+u} - 1} \prod_{i=1}^n \operatorname{Li}_{k_i}(1 - e^{-t-u}),$$

$$\frac{d}{du} \operatorname{Li}_{\mathbf{k}; k_{n+1}+1}(1 - e^{-t-u}) = \frac{1}{e^{t+u} - 1} \operatorname{Li}_{\mathbf{k}; k_{n+1}}(1 - e^{-t-u}).$$

Therefore, we obtain the following corollary.

COROLLARY 1. *The following formulas hold*:

$$\int_0^\infty \frac{1}{e^{t+u} - 1} \operatorname{Li}_{\mathbf{k}}(1 - e^{-t-u})\, du = \zeta(\mathbf{k}_+) - \operatorname{Li}_{\mathbf{k}_+}(1 - e^{-t}),$$

$$\int_0^\infty \frac{1}{e^{t+u} - 1} \prod_{i=1}^n \operatorname{Li}_{k_i}(1 - e^{-t-u})\, du = \zeta_{MT}(k_1, \dots, k_n; 1) - \operatorname{Li}_{\mathbf{k}; 1}(1 - e^{-t}),$$

$$\int_0^\infty \frac{1}{e^{t+u} - 1} \operatorname{Li}_{\mathbf{k}; k_{n+1}}(1 - e^{-t-u})\, du = \zeta_{MT}(k_1, \dots, k_n; k_{n+1} + 1) - \operatorname{Li}_{\mathbf{k}; k_{n+1}+1}(1 - e^{-t}).$$

LEMMA 2. *For a matrix* $A = (a_{i,j})_{1 \le i \le n, 1 \le j \le r}$, $a_{i,j} \in \mathbf{R}_{\ge 0}$ *and* $\mathbf{s} = (s_1, \dots, s_n) \in \mathbf{C}^n$ *we define*

$$(5)\quad \zeta(\mathbf{s}; A)$$

$$= \sum_{m_1=1,\ldots,m_r=1}^{\infty} (a_{1,1}m_1 + \cdots + a_{1,r}m_r)^{-s_1} \cdots (a_{n,1}m_1 + \cdots + a_{n,r}m_r)^{-s_n}$$

$$= \sum_{m_1=1,\ldots,m_r=1}^{\infty} \prod_{i=1}^{n} \frac{1}{(A\begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix})_i^{s_i}},$$

where $()_i$ *represents the i-th element of a vertical vector. For* $\Re(s_i) > 0$ $(1 \le i \le n)$, *if* $\zeta(\mathbf{s}; A)$ *converges absolutely then*

$$(6)\qquad \zeta(\mathbf{s}; A) = \frac{1}{\prod_{i=1}^{n} \Gamma(s_i)} \int_0^{\infty} \cdots \int_0^{\infty} \left(\prod_{i=1}^{r} \frac{1}{\exp({}^tA\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix})_i - 1} \right) \prod_{i=1}^{n} t_i^{s_i - 1}\, dt_i,$$

where ${}^tA$ *represents the transposed matrix of* A.

Remark 2. The function $\zeta(\mathbf{s}; A)$ was introduced by Matsumoto [5]. Moreover, he showed that the function $\zeta(\mathbf{s}; A)$ can be continued meromorphically to the whole $\mathbf{C}^n$ space.

Remark 3. If there exists i satisfying $\Re(s_i) \le 0$, then the integral on the right hand side of (6) diverges.

Proof of Lemma 2. By using $\Gamma(s) = m^s \int_0^{\infty} t^{s-1} e^{-mt}\, dt$ $(\Re(s) > 0)$, we have

$$\zeta(\mathbf{s}; A)$$

$$= \sum_{m_1=1,\ldots,m_r=1}^{\infty} \frac{1}{\prod_{i=1}^{n} \Gamma(s_i)}$$

$$\times \int_0^{\infty} \cdots \int_0^{\infty} e^{-t_1(a_{1,1}m_1 + \cdots + a_{1,r}m_r)} \cdots e^{-t_n(a_{n,1}m_1 + \cdots + a_{n,r}m_r)} \prod_{i=1}^{n} t_i^{s_i - 1}\, dt_i$$

$$= \sum_{m_1=1,\ldots,m_r=1}^{\infty} \frac{1}{\prod_{i=1}^{n} \Gamma(s_i)}$$

$$\times\int_0^\infty\cdots\int_0^\infty e^{-(t_1a_{1,1}+\cdots+t_na_{n,1})m_1}\cdots e^{-(t_1a_{1,r}+\cdots+t_na_{n,r})m_r}\prod_{i=1}^n t_i^{s_i-1}\,dt_i$$

$$=\frac{1}{\prod_{i=1}^n\Gamma(s_i)}\int_0^\infty\cdots\int_0^\infty\frac{1}{e^{t_1a_{1,1}+\cdots+t_na_{n,1}}-1}\cdots\frac{1}{e^{t_1a_{1,r}+\cdots+t_na_{n,r}}-1}\prod_{i=1}^n t_i^{s_i-1}\,dt_i.$$

The last equality holds since $\zeta(\mathbf{s};A)$ converges absolutely. □

The following results on the Arakawa-Kaneko zeta function and the Ito zeta function are known.

THEOREM 1 ([4, Theorem 2.5]). *For* $\mathbf{k}=(k_1,\dots,k_n)\in\mathbf{N}^n$, *we write* $|\mathbf{k}|=k_1+\cdots+k_n$ *and call it the weight of* $\mathbf{k}$, *and* $d(\mathbf{k})=n$, *the depth of* $\mathbf{k}$. *Moreover, we write*

$$b(\mathbf{k};\mathbf{j})=\prod_{i=1}^n\binom{k_i+j_i-1}{j_i}.$$

For any $m\in\mathbf{N}$, *we have*

$$\xi(\mathbf{k};m)=\sum_{|\mathbf{j}|=m-1,\,d(\mathbf{j})=d((\mathbf{k}_+)^*)}b((\mathbf{k}_+)^*;\mathbf{j})\zeta((\mathbf{k}_+)^*+\mathbf{j}),$$

where the sum is over all $\mathbf{j}=(j_1,\dots,j_n)\in\mathbf{Z}_{\geq 0}^n$ *satisfying* $|\mathbf{j}|=m-1$, $d(\mathbf{j})=d((\mathbf{k}_+)^*)$.

THEOREM 2 ([4, Theorem 3.6]). *The Arakawa-Kankeo zeta function* $\xi(\mathbf{k};s)$ *can be written in terms of EZ-type MZFs as*

$$\xi(\mathbf{k};s)=\sum_{\mathbf{k}',j\geq 0}c_{\mathbf{k}}(\mathbf{k}';j)\binom{s+j-1}{j}\zeta(\mathbf{k}',j+s).$$

Here, the sum is over indices $\mathbf{k}'$ *and integers* $j\geq 0$ *satisfying* $|\mathbf{k}'|+j\leq|\mathbf{k}|$, *and* $c_{\mathbf{k}}(\mathbf{k}';j)$ *is a* $\mathbf{Q}$*-linear combination of EZ-type MZVs of weight* $|\mathbf{k}|-|\mathbf{k}'|-j$.

THEOREM 3 ([3, Proposition 5]). *For* $m\in\mathbf{Z}_{\geq 0}$,

$$\xi_{MT}(k_1,\dots,k_r;m+1)=\frac{1}{m!}\zeta_{MT}(k_1,\dots,k_r,\{1\}^m;1).$$

Theorem 4 ([3, Theorem 8]). *For* $r \in \mathbf{N}$ *and* $s \in \mathbf{C}$,

$$\sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^j \zeta(2)^{r-1-j} \Gamma(s) \xi_{MT}(\{2\}^j; s)$$

$$= \sum_{j=0}^{r-1} \binom{r-1}{j} \Gamma(s+j) \zeta_{MT}(0, \{2\}^{r-1-j}, \{1\}^j; j+s).$$

Remark 4. Ito obtained relations among MT-type MZVs by putting $s = m+1$ in Theorem 4 and using Theorem 3 for $\xi_{MT}(\{2\}^j; m+1)$.

3. An analog of the Arakawa-Kaneko zeta function of Miyagawa-type

Miyagawa [7] defined the multiple zeta funcion (3). We write the Miyagawa-type MZF as follows.

Definition 8. For $\mathbf{s}_{r+1} = (s_{r+1,1}, \dots, s_{r+1,n_{r+1}}) \in \mathbf{C}^{n_{r+1}}$, we write

$$\zeta_{MT}(s_1, \dots, s_r; \mathbf{s}_{r+1})$$

$$= \sum_{0<m_1,\dots,0<m_r} \sum_{m_{r+1,1}=1,\dots,m_{r+1,n_{r+1}-1}=1}^{\infty} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} \prod_{u=1}^{n_{r+1}} (\sum_{v=1}^{r} m_v + \sum_{w=1}^{u-1} m_{r+1,w})^{s_{r+1,u}}}.$$

Proposition 1. *The Miyagawa-type MZF* $\zeta_{MT}(s_1, s_2, \dots, s_r; \mathbf{s}_{r+1})$ *converges absolutely when*

$$\sum_{i=0}^{k} \Re(s_{r+1,n_{r+1}-i}) > k+1$$

for any $k = 1, \dots, n_{r+1} - 2$ *and*

$$\sum_{l=1}^{j} \Re(s_{k_l}) + \sum_{i=0}^{n_{r+1}-1} \Re(s_{r+1,n_{r+1}-i}) - n_{r+1} + 1 > j$$

with $1 \le k_1 < k_2 < \cdots < k_j \le r$ *for any* $j = 1, 2, \dots, r$ *are all satisfied.*

PROOF. The series $\sum_{n=1}^{\infty}(N+n)^{-\sigma}$ $(N>0)$ converges only when $\sigma>1$, and

$$\sum_{n=1}^{\infty}\frac{1}{(N+n)^{\sigma}}\leq\frac{1}{(\sigma-1)N^{\sigma-1}}.$$

Let

$$\mathbf{s}^{(k)}=\left(\Re(s_{r+1,1}),\dots,\Re(s_{r+1,n_{r+1}-k-1}),\sum_{i=0}^{k}\Re(s_{r+1,n_{r+1}-i})-k\right).$$

Then we have

$$\begin{aligned}
&\zeta_{MT}(\Re(s_1),\Re(s_2),\dots,\Re(s_r);\Re(s_{r+1,1}),\dots,\Re(s_{r+1,n_{r+1}}))\\
&\quad\ll\zeta_{MT}(\Re(s_1),\Re(s_2),\dots,\Re(s_r);\mathbf{s}^{(1)})\\
&\quad\ll\zeta_{MT}(\Re(s_1),\Re(s_2),\dots,\Re(s_r);\mathbf{s}^{(2)})\\
&\quad\ll\cdots\\
&\quad\ll\zeta_{MT}\left(\Re(s_1),\Re(s_2),\dots,\Re(s_r);\sum_{i=0}^{n_{r+1}-1}\Re(s_{r+1,n_{r+1}-i})-n_{r+1}+1\right),
\end{aligned}$$

where the implicit constants of $\ll$ depend on $\mathbf{s}_{r+1}=(s_{r+1,1},\dots,s_{r+1,n_{r+1}})$. By absolute convergence of the MT-type MZF, the assertion of this proposition follows. □

REMARK 5. By Lemma 2, we have

$$\begin{aligned}
(7)\qquad&\zeta_{MT}(0,s_2,\dots,s_r;\mathbf{s}_{r+1})\\
&=\frac{1}{\prod_{i=2}^{r}\Gamma(s_i)\prod_{j=1}^{n_{r+1}}\Gamma(s_{r+1,j})}\int_0^{\infty}\cdots\int_0^{\infty}\prod_{i=2}^{r}\frac{t_i^{s_i-1}\,dt_i}{e^{t_i+t_{r+1,1}+\cdots+t_{r+1,n_{r+1}}}-1}\\
&\quad\times\prod_{u=1}^{n_{r+1}}\frac{t_{r+1,u}^{s_{r+1,u}-1}\,dt_{r+1,u}}{e^{t_{r+1,u}+\cdots+t_{r+1,n_{r+1}}}-1}.
\end{aligned}$$

This identity is a specialization of identity (9) below. We can obtain the formula (7) as a consequence of identity (9).

In this section, we discuss the function (2), and obtain certain relations among Miyagawa-type MZVs. Namely we may regard the argument in this section as a Miyagawa-type analog of Ito's work.

In the two subsections in this section, we show a relationship between special values of the function (2) and Miyagawa-type MZVs, and a relationship between functions (2) and Miyagawa-type MZFs. In conclusion, we obtain certain relations among Miyagawa-type MZVs. Therefore, we may regard the function (2) as an analog of the Arakawa-Kaneko zeta function of Miyagawa-type.

3.1. Special values.

The special values of the function (2) can be written in terms of Miyagawa-type MZVs as follows.

THEOREM 5. *For* $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{N}^n$, $k_{n+1} \in \mathbf{Z}_{\geq 0}$ *and* $m \in \mathbf{Z}_{\geq 0}$, *we have*

$$\begin{aligned}&\xi_{MT}((\mathbf{k}; k_{n+1}); m+1)\\&\quad = \sum_{a_1+\cdots+a_{k_{n+1}+1}=m} \frac{1}{a_{k_{n+1}+1}!}\\&\qquad \times \zeta_{MT}(\{1\}^{a_{k_{n+1}+1}}, k_1, \dots, k_n; {}_{-}((a_1+1, \dots, a_{k_{n+1}}+1, 2)^*)),\end{aligned}$$

where the sum is over all $a_1, \dots, a_{k_{n+1}+1} \in \mathbf{Z}_{\geq 0}$ *satisfying* $a_1 + \cdots + a_{k_{n+1}+1} = m$.

To prove this theorem, we need the following lemma.

LEMMA 3. *Let* $\mathbf{k}_{r+1} = (k_{r+1,1}, \dots, k_{r+1,n_{r+1}}) \in \mathbf{N}^{n_{r+1}}$ ($n_{r+1} = 1$ or $k_{r+1,n_{r+1}} \geq 2$) *and* $(({}_{+}\mathbf{k}_{r+1})^*)_{-} = (l_1, \dots, l_d)$. *We have*

$$\begin{aligned}&\zeta_{MT}(k_1, \dots, k_r; \mathbf{k}_{r+1})\\&\quad = \frac{1}{\prod_{i=1}^{d} \Gamma(l_i)} \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^{d} \frac{t_i^{l_i-1}}{e^{t_i+\cdots+t_d}-1} \right) \prod_{i=1}^{r} \mathrm{Li}_{k_i}(1-e^{-t_d})\, dt_1 \cdots dt_d.\end{aligned}$$

PROOF. Let $z \in \mathbf{R}_{>0}$ and $\mathbf{k}_{r+1} = (b_0, \{1\}^{a_1-1}, b_1+1, \dots, \{1\}^{a_h-1}, b_h+1)$ with $b_0, a_i, b_i \in \mathbf{N}$ $(1 \leq i \leq h)$. By using the same method as in the proof of equation (23) in Kaneko-Tsumura [4], we have

$$(8)\qquad \sum_{0<m_1,\ldots,0<m_r}\ \sum_{m_{r+1,1}=1,\ldots,m_{r+1,n_{r+1}-1}=1}^{\infty} \frac{(1-e^{-z})^{\sum_{v=1}^{r} m_v+\sum_{w=1}^{n_{r+1}-1} m_{r+1,w}}}{m_1^{k_1}\cdots m_r^{k_r}\prod_{u=1}^{n_{r+1}}(\sum_{v=1}^{r} m_v+\sum_{w=1}^{u-1} m_{r+1,w})^{k_{r+1,u}}}$$
$$= \int_{0<t_1<\cdots<t_{b_0+\cdots+b_h}<z}$$
$$\times \left(\prod_{i=1}^{b_h}\frac{1}{e^{t_{b_0+\cdots+b_{h-1}+i}}-1}\right)\frac{1}{a_h!}(t_{b_0+\cdots+b_{h-1}+1}-t_{b_0+\cdots+b_{h-1}})^{a_h}$$
$$\times \left(\prod_{i=1}^{b_{h-1}}\frac{1}{e^{t_{b_0+\cdots+b_{h-2}+i}}-1}\right)\frac{1}{a_{h-1}!}(t_{b_0+\cdots+b_{h-2}+1}-t_{b_0+\cdots+b_{h-2}})^{a_{h-1}}\cdots$$
$$\times \left(\prod_{i=1}^{b_1}\frac{1}{e^{t_{b_0+i}}-1}\right)\frac{1}{a_1!}(t_{b_0+1}-t_{b_0})^{a_1}$$
$$\times \left(\prod_{i=1}^{b_0}\frac{1}{e^{t_i}-1}\right)\prod_{i=1}^{r}\mathrm{Li}_{k_i}(1-e^{-t_1})\,dt_1\cdots dt_{b_0+\cdots+b_h}.$$

Since

$$((_{+}\mathbf{k}_{r+1})^*)_- = (\{1\}^{b_h-1}, a_h+1, \ldots, \{1\}^{b_1-1}, a_1+1, \{1\}^{b_0}) = (l_1, \ldots, l_d),$$

by taking the limit as z tends to infinity, we obtain

$$\zeta_{MT}(k_1, \ldots, k_r; \mathbf{k}_{r+1})$$
$$= \int_{0<t_1<\cdots<t_{b_0+\cdots+b_h}<\infty}$$
$$\times \left(\prod_{i=1}^{b_h}\frac{1}{e^{t_{b_0+\cdots+b_{h-1}+i}}-1}\right)\frac{1}{a_h!}(t_{b_0+\cdots+b_{h-1}+1}-t_{b_0+\cdots+b_{h-1}})^{a_h}$$
$$\times \left(\prod_{i=1}^{b_{h-1}}\frac{1}{e^{t_{b_0+\cdots+b_{h-2}+i}}-1}\right)\frac{1}{a_{h-1}!}(t_{b_0+\cdots+b_{h-2}+1}-t_{b_0+\cdots+b_{h-2}})^{a_{h-1}}\cdots$$
$$\times \left(\prod_{i=1}^{b_1}\frac{1}{e^{t_{b_0+i}}-1}\right)\frac{1}{a_1!}(t_{b_0+1}-t_{b_0})^{a_1}$$

$$\times\left(\prod_{i=1}^{b_0}\frac{1}{e^{t_i}-1}\right)\prod_{i=1}^{r}\mathrm{Li}_{k_i}(1-e^{-t_1})\,dt_1\cdots dt_{b_0+\cdots+b_h}$$

$$=\frac{1}{a_1!}\cdots\frac{1}{a_h!}\int_0^\infty\cdots\int_0^\infty\left(\prod_{i=1}^{b_h}\frac{1}{e^{t_{b_0+\cdots+b_{h-1}+i}+\cdots+t_1}-1}\right)t_{b_0+\cdots+b_{h-1}+1}^{a_h}$$

$$\times\left(\prod_{i=1}^{b_{h-1}}\frac{1}{e^{t_{b_0+\cdots+b_{h-2}+i}+\cdots+t_1}-1}\right)t_{b_0+\cdots+b_{h-2}+1}^{a_{h-1}}\cdots$$

$$\times\left(\prod_{i=1}^{b_1}\frac{1}{e^{t_{b_0+i}+\cdots+t_1}-1}\right)t_{b_0+1}^{a_1}$$

$$\times\left(\prod_{i=1}^{b_0}\frac{1}{e^{t_i+\cdots+t_1}-1}\right)\prod_{i=1}^{r}\mathrm{Li}_{k_i}(1-e^{-t_1})\,dt_1\cdots dt_{b_0+\cdots+b_h}$$

$$=\frac{1}{\prod_{i=1}^{d}\Gamma(l_i)}\int_0^\infty\cdots\int_0^\infty\left(\prod_{i=1}^{d}\frac{t_i^{l_{d+1-i}-1}}{e^{t_i+\cdots+t_1}-1}\right)\prod_{i=1}^{r}\mathrm{Li}_{k_i}(1-e^{-t_1})\,dt_1\cdots dt_d$$

$$=\frac{1}{\prod_{i=1}^{d}\Gamma(l_i)}\int_0^\infty\cdots\int_0^\infty\left(\prod_{i=1}^{d}\frac{t_i^{l_i-1}}{e^{t_i+\cdots+t_d}-1}\right)\prod_{i=1}^{r}\mathrm{Li}_{k_i}(1-e^{-t_d})\,dt_1\cdots dt_d.$$

This completes the proof. □

Proof of Theorem 5. For $\Re(s)>0$, by using the case $n_{r+1}=1$ of the equation (8), we have

$$\Gamma(s)\xi_{MT}((\mathbf{k};k_{n+1});s)$$

$$=\int_0^\infty\frac{t^{s-1}}{e^t-1}\int_{0<t_1<\cdots<t_{k_{n+1}}<t}\left(\prod_{i=1}^{k_{n+1}}\frac{1}{e^{t_i}-1}\right)\prod_{i=1}^{n}\mathrm{Li}_{k_i}(1-e^{-t_1})\,dt_1\cdots dt_{k_{n+1}}dt$$

$$=\int_0^\infty\cdots\int_0^\infty\frac{(t_{k_{n+1}+1}+\cdots+t_1)^{s-1}}{e^{t_{k_{n+1}+1}+\cdots+t_1}-1}\left(\prod_{i=1}^{k_{n+1}}\frac{1}{e^{t_i+\cdots+t_1}-1}\right)$$

$$\times\prod_{i=1}^{n}\mathrm{Li}_{k_i}(1-e^{-t_1})\,dt_1\cdots dt_{k_{n+1}}dt_{k_{n+1}+1}$$

$$= \int_0^\infty \cdots \int_0^\infty \frac{(t_1 + \cdots + t_{k_{n+1}+1})^{s-1}}{e^{t_1+\cdots+t_{k_{n+1}+1}} - 1} \left(\prod_{i=2}^{k_{n+1}+1} \frac{1}{e^{t_i+\cdots+t_{k_{n+1}+1}} - 1} \right)$$

$$\times \prod_{i=1}^{n} \mathrm{Li}_{k_i}(1 - e^{-t_{k_{n+1}+1}})\, dt_1 \cdots dt_{k_{n+1}} dt_{k_{n+1}+1}.$$

By putting $s = m + 1$ in this equation and using the formula $\mathrm{Li}_1(1 - e^{-t}) = t$ and Lemma 3, we have

$$m!\xi_{MT}((\mathbf{k}; k_{n+1}); m + 1)$$

$$= \int_0^\infty \cdots \int_0^\infty (t_1 + \cdots + t_{k_{n+1}+1})^m \left(\prod_{i=1}^{k_{n+1}+1} \frac{1}{e^{t_i+\cdots+t_{k_{n+1}+1}} - 1} \right)$$

$$\times \prod_{i=1}^{n} \mathrm{Li}_{k_i}(1 - e^{-t_{k_{n+1}+1}})\, dt_1 \cdots dt_{k_{n+1}} dt_{k_{n+1}+1}$$

$$= \sum_{a_1+\cdots+a_{k_{n+1}+1}=m} \frac{m!}{a_1! \cdots a_{k_{n+1}+1}!}$$

$$\times \int_0^\infty \cdots \int_0^\infty t_1^{a_1} \cdots t_{k_{n+1}}^{a_{k_{n+1}}} (\mathrm{Li}_1(1 - e^{-t_{k_{n+1}+1}}))^{a_{k_{n+1}+1}}$$

$$\times \left(\prod_{i=1}^{k_{n+1}+1} \frac{1}{e^{t_i+\cdots+t_{k_{n+1}+1}} - 1} \right) \prod_{i=1}^{n} \mathrm{Li}_{k_i}(1 - e^{-t_{k_{n+1}+1}})\, dt_1 \cdots dt_{k_{n+1}} dt_{k_{n+1}+1}$$

$$= \sum_{a_1+\cdots+a_{k_{n+1}+1}=m} \frac{m!}{a_{k_{n+1}+1}!}$$

$$\times \zeta_{MT}(\{1\}^{a_{k_{n+1}+1}}, k_1, \ldots, k_n; {}_{-}((a_1 + 1, \ldots, a_{k_{n+1}} + 1, 2)^*)).$$

This completes the proof. □

REMARK 6. We can also prove Theorem 5 by using the Yamamoto-integral defined by Yamamoto [9]. This method is more intuitive. Here, we use the notation in [9]. Since

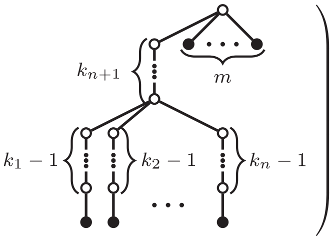
$$m!\xi_{MT}((\mathbf{k}; k_{n+1}); m + 1) = \int_0^\infty \frac{(\mathrm{Li}_1(1 - e^{-t}))^m}{e^t - 1} \mathrm{Li}_{\mathbf{k}; k_{n+1}}(1 - e^{-t})\, dt$$

$$= \int_0^1 \frac{(\mathrm{Li}_1(t))^m}{t} \mathrm{Li}_{\mathbf{k}; k_{n+1}}(t)\, dt,$$

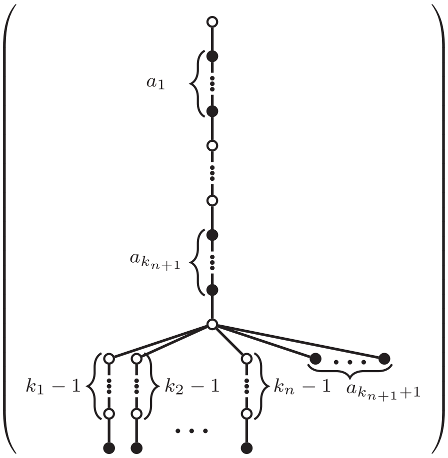
by using the Yamamoto-integral, we have

$$m!\xi_{MT}((\mathbf{k};k_{n+1});m+1)$$

$$= I\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) = \sum_{a_1+\cdots+a_{k_{n+1}+1}=m} \frac{m!}{a_{k_{n+1}+1}!} I\left(\vphantom{\int}\right.$$

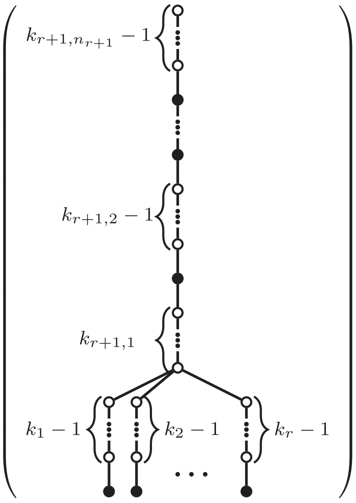


$$\left.\vphantom{\int}\right).$$

The second equality is obtained by ordering m black vertices and $1+k_{n+1}$ white vertices. Here, by Lemma 1, the special values of the Miyagawa-type MZF is written as

$$\begin{aligned}
&\zeta_{MT}(k_1,\dots,k_r;\mathbf{k}_{r+1})\\
&= \int_{0<t_1<\cdots<t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}}}<1} \left(\prod_{i=2}^{k_{r+1,n_{r+1}}} \frac{1}{t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}-1}+i}}\right)\\
&\quad\times \frac{1}{1-t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}-1}+1}} \left(\prod_{i=2}^{k_{r+1,n_{r+1}-1}} \frac{1}{t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}-2}+i}}\right)\cdots\\
&\quad\times \frac{1}{1-t_{k_{r+1,1}+k_{r+1,2}+1}} \left(\prod_{i=2}^{k_{r+1,2}} \frac{1}{t_{k_{r+1,1}+i}}\right)\\
&\quad\times \frac{1}{1-t_{k_{r+1,1}+1}} \left(\prod_{i=1}^{k_{r+1,1}} \frac{1}{t_i}\right) \prod_{i=1}^{r} \mathrm{Li}_{k_i}(t_1)\, dt_1 \cdots dt_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}}}
\end{aligned}$$

$$= I$$

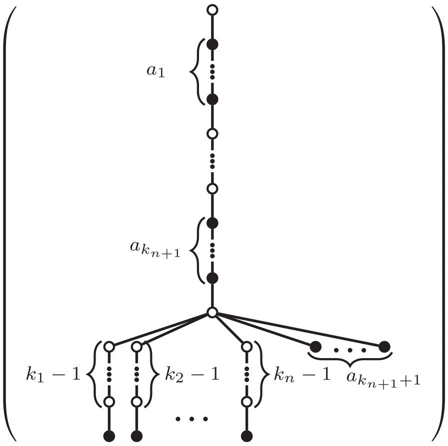


.

Therefore, we obtain

$$\zeta_{MT}(\{1\}^{a_{k_{n+1}+1}}, k_1, \dots, k_n; \_((a_1+1, \dots, a_{k_{n+1}}+1, 2)^*))$$

$$= I$$



.

Therefore, we obtain Theorem 5.

3.2. Functional relations. Functions (2) has the relationship with Miyagawa-type MZFs as follows.

THEOREM 6. *For* $l, k \in \mathbf{N}$, $s \in \mathbf{C}$, *we have*

$$\zeta(2)^l\zeta(\{1\}^k,s)+\sum_{j=1}^{l}\binom{l}{j}\zeta(2)^{l-j}(-1)^j\left(\sum_{i=1}^{k}(-1)^{i-1}\zeta_{MT}(\{2\}^j;i)\zeta(\{1\}^{k-i},s)\right.$$
$$\left.+(-1)^k\xi_{MT}((\{2\}^j;k);s)\right)$$
$$=\sum_{a+b_1+\cdots+b_{k+1}=l}\frac{l!}{a!}\binom{s+b_{k+1}-1}{b_{k+1}}$$
$$\times\zeta_{MT}(0,\{2\}^a,\{1\}^{l-a};b_1+1,\dots,b_k+1,b_{k+1}+s),$$

where the sum on the right hand side is over all $a\in\mathbf{Z}_{\geq 0}$ *and* $b_i\in\mathbf{Z}_{\geq 0}$ *satisfying* $a+b_1+\cdots+b_{k+1}=l$.

Remark 7. If we understand the sum in i as in equal to 0 when $k=0$, then Theorem 6 holds also when $k=0$ and coincides with Theorem 4.

Proof of Theorem 6. For $s\in\mathbf{C}$ with $\Re(s)>1$, let

$$J=\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}\left(\prod_{i=1}^{l}\frac{u_i+t_1+\cdots+t_{k+1}}{e^{u_i+t_1+\cdots+t_{k+1}}-1}\right)$$
$$\times\frac{1}{e^{t_1+\cdots+t_{k+1}}-1}\,\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,du_1\cdots du_l dt_1\cdots dt_{k+1}.$$

We calculate J in two different ways.

The first calculation is to integrate directly by using Corollary 1. By integrating with respect to $u_1,\dots,u_l$, we have

$$J=\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}(\zeta(2)-\mathrm{Li}_2(1-e^{-(t_1+\cdots+t_{k+1})}))^l$$
$$\times\frac{1}{e^{t_1+\cdots+t_{k+1}}-1}\,\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_1\cdots dt_{k+1}$$
$$=\Gamma(s)\zeta(2)^l\zeta(\{1\}^k,s)$$
$$+\sum_{j=1}^{l}\binom{l}{j}\zeta(2)^{l-j}(-1)^j\int_0^\infty\cdots\int_0^\infty t_{k+1}^{s-1}(\mathrm{Li}_2(1-e^{-(t_1+\cdots+t_{k+1})}))^j$$
$$\times\frac{1}{e^{t_1+\cdots+t_{k+1}}-1}\,\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_1\cdots dt_{k+1}.$$

We compute the above integral by parts in order of $t_1, \dots, t_k$ to obtain

$$
\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty t_{k+1}^{s-1}(\mathrm{Li}_2(1-e^{-(t_1+\cdots+t_{k+1})}))^j \\
&\qquad \times \frac{1}{e^{t_1+\cdots+t_{k+1}}-1}\frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_1\cdots dt_{k+1} \\
&= \int_0^\infty \cdots \int_0^\infty t_{k+1}^{s-1}(\zeta_{MT}(\{2\}^j;1) - \mathrm{Li}_{\{2\}^j;1}(1-e^{-(t_2+\cdots+t_{k+1})})) \\
&\qquad \times \frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\frac{1}{e^{t_3+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_2\cdots dt_{k+1} \\
&= \Gamma(s)\zeta_{MT}(\{2\}^j;1)\zeta(\{1\}^{k-1},s) \\
&\qquad - \int_0^\infty \cdots \int_0^\infty t_{k+1}^{s-1}\,\mathrm{Li}_{\{2\}^j;1}(1-e^{-(t_2+\cdots+t_{k+1})}) \\
&\qquad \times \frac{1}{e^{t_2+\cdots+t_{k+1}}-1}\frac{1}{e^{t_3+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_2\cdots dt_{k+1} \\
&= \Gamma(s)\zeta_{MT}(\{2\}^j;1)\zeta(\{1\}^{k-1},s) - \Gamma(s)\zeta_{MT}(\{2\}^j;2)\zeta(\{1\}^{k-2},s) \\
&\qquad + \int_0^\infty \cdots \int_0^\infty t_{k+1}^{s-1}\,\mathrm{Li}_{\{2\}^j;2}(1-e^{-(t_3+\cdots+t_{k+1})}) \\
&\qquad \times \frac{1}{e^{t_3+\cdots+t_{k+1}}-1}\frac{1}{e^{t_4+\cdots+t_{k+1}}-1}\cdots\frac{1}{e^{t_{k+1}}-1}\,dt_3\cdots dt_{k+1} \\
&= \cdots \\
&= \Gamma(s)\sum_{i=1}^{k}(-1)^{i-1}\zeta_{MT}(\{2\}^j;i)\zeta(\{1\}^{k-i},s) \\
&\qquad + (-1)^k\int_0^\infty t_{k+1}^{s-1}\,\mathrm{Li}_{\{2\}^j;k}(1-e^{-t_{k+1}})\frac{1}{e^{t_{k+1}}-1}\,dt_{k+1}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
J &= \Gamma(s)\zeta(2)^l\zeta(\{1\}^k,s) + \Gamma(s)\sum_{j=1}^{l}\binom{l}{j}\zeta(2)^{l-j}(-1)^j \\
&\quad \times \left(\sum_{i=1}^{k}(-1)^{i-1}\zeta_{MT}(\{2\}^j;i)\zeta(\{1\}^{k-i},s) + (-1)^k\xi_{MT}((\{2\}^j;k);s)\right).
\end{aligned}
$$

The second calculation is to use the polynomial expansion and the equation (7). By symmetry of $u_1, \dots, u_l$, we have

$$\begin{aligned} J &= \sum_{a+b_1+\cdots+b_{k+1}=l} \frac{l!}{a!b_1!\cdots b_{k+1}!} \int_0^\infty \cdots \int_0^\infty t_{k+1}^{s-1} u_1 \cdots u_a t_1^{b_1} \cdots t_{k+1}^{b_{k+1}} \\ &\quad \times \left(\prod_{i=1}^{l} \frac{1}{e^{u_i+t_1+\cdots+t_{k+1}} - 1} \right) \left(\prod_{i=1}^{k+1} \frac{1}{e^{t_i+\cdots+t_{k+1}} - 1} \right) du_1 \cdots du_l dt_1 \cdots dt_{k+1} \\ &= \sum_{a+b_1+\cdots+b_{k+1}=l} \frac{l!\Gamma(b_{k+1}+s)}{a!b_{k+1}!} \zeta(0, \{2\}^a, \{1\}^{l-a}; 1+b_1, \dots, 1+b_k, b_{k+1}+s). \end{aligned}$$

By the first and second calculations, we obtain the desired identity for $\Re(s) > 1$. By the analytic continuation of the EZ-type MZF, the generalized Ito zeta function ([3, Theorem 14]) and the Miyagawa-type MZF, we obtain the stated theorem. □

Note that J was calculated by Arakawa and Kaneko when $l = 1$, and Ito when $k = 0$ (see the proof of [2, Theorem 6.(ii)] and [3, Theorem 8]). Therefore, we may regard this proof as a fusion of the method of Arakawa and Kaneko, and the method of Ito.

By putting $s = m+1$ with $m \in \mathbf{N}$ in Theorem 6 and using Theorem 5 for $\xi_{MT}((\{2\}^j; k); s)$, we can obtain the following relations among Miyagawa-type MZVs.

Theorem 7. *For $l, k, m \in \mathbf{N}$, we have*

$$\begin{aligned} &\zeta(2)^l \zeta(\{1\}^k, m+1) \\ &\quad + \sum_{j=1}^{l} \binom{l}{j} \zeta(2)^{l-j} (-1)^j \left(\sum_{i=1}^{k} (-1)^{i-1} \zeta_{MT}(\{2\}^j; i) \zeta(\{1\}^{k-i}, m+1) \right. \\ &\quad \left. + (-1)^k \sum_{a_1+\cdots+a_{k+1}=m} \frac{1}{a_{k+1}!} \zeta_{MT}(\{1\}^{a_{k+1}}, \{2\}^j; {}_{-}((a_1+1, \dots, a_k+1, 2)^*)) \right) \\ &= \sum_{a+b_1+\cdots+b_{k+1}=l} \frac{l!}{a!} \binom{m+b_{k+1}}{b_{k+1}} \\ &\quad \times \zeta_{MT}(0, \{2\}^a, \{1\}^{l-a}; b_1+1, \dots, b_k+1, b_{k+1}+m+1). \end{aligned}$$

EXAMPLE 1. Let $l=1$, $k=1$, $m=1$. Then we obtain

$$\begin{aligned}&\zeta(2)\zeta(1,2)-\zeta_{MT}(2;1)\zeta(2)+\zeta_{MT}(2;1,2)+\zeta_{MT}(1,2;2)\\&\quad=\zeta_{MT}(0,2;1,2)+\zeta_{MT}(0,1;2,2)+2\zeta_{MT}(0,1;1,3).\end{aligned}$$

When $l=1$, $k=1$, $m=2$ and $l=1$, $k=2$, $m=1$. Then we obtain

$$\begin{aligned}&\zeta(2)\zeta(1,3)-\zeta_{MT}(2;1)\zeta(3)+\zeta_{MT}(2;1,1,2)\\&\quad+\zeta_{MT}(1,2;1,2)+\frac{1}{2}\zeta_{MT}(1,1,2;2)\\&\quad=\zeta_{MT}(0,2;1,3)+\zeta_{MT}(0,1;2,3)+3\zeta_{MT}(0,1;1,4)\end{aligned}$$

and

$$\begin{aligned}&\zeta(2)\zeta(1,1,2)-\zeta_{MT}(2;1)\zeta(1,2)+\zeta_{MT}(2;2)\zeta(2)\\&\quad-\zeta_{MT}(2;2,2)-\zeta_{MT}(2;1,3)-\zeta_{MT}(1,2;3)\\&\quad=\zeta_{MT}(0,2;1,1,2)+\zeta_{MT}(0,1;2,1,2)\\&\quad+\zeta_{MT}(0,1;1,2,2)+2\zeta_{MT}(0,1;1,1,3),\end{aligned}$$

respectively.

4. Generalized Mordell-Tornheim multiple zeta function

We will generalize Theorem 4 and Theorem 6 in the next section. For this purpose, we need to introduce a new class of the multiple zeta function (4). In this section, we show several propositions on the function (4).

REMARK 8. The function (4) contains the EZ-type MZF, the MT-type MZF and the Miyagawa-type MZF as special cases. For example,

$$\begin{gathered}\zeta_{MT}((s_1,\dots,s_j);(s_{j+1},\dots,s_n))=\zeta(s_1,\dots,s_{j-1},s_j+s_{j+1},s_{j+2},\dots,s_n),\\\zeta_{MT}((s_1),\dots,(s_r);(s_{r+1}))=\zeta_{MT}(s_1,\dots,s_r;s_{r+1}),\end{gathered}$$

and

$$\zeta_{MT}((s_1),\dots,(s_j);(s_{j+1},\dots,s_{r+1}))=\hat{\zeta}_{MT,j,r}(s_1,\dots,s_j;s_{j+1},\dots,s_{r+1}).$$

Therefore, for the case $n_i = 1$ $(1 \le i \le r)$ and $\mathbf{s}_{r+1} = (s_{r+1,1}, \dots, s_{r+1,n_{r+1}})$, we may omit the parentheses. For example, we write $\zeta_{MT}((s_{1,1}), (s_{2,1}, s_{2,2}); (s_{3,1}, s_{3,2}))$ as $\zeta_{MT}(s_{1,1}, (s_{2,1}, s_{2,2}); s_{3,1}, s_{3,2})$. Under this notation, the Miyagawa-type MZF is written as

$$\hat{\zeta}_{MT,r,r+n_{r+1}-1}(s_1, \dots, s_r; s_{r+1,1}, \dots, s_{r+1,n_{r+1}}) = \zeta_{MT}(s_1, \dots, s_r; \mathbf{s}_{r+1}).$$

Proposition 2. *If one of the following conditions is satisfied, then the function* (4) *conveges absolutely.*

(i)

$$\begin{aligned} \Re(s_{i,j}) &\ge 1 && (1 \le i \le r,\ 1 \le j \le n_r), \\ \sum_{i=0}^{k} \Re(s_{r+1,n_{r+1}-i}) &> k+1 && (0 \le k \le n_{r+1}-2), \\ \sum_{i=0}^{n_{r+1}-1} \Re(s_{r+1,n_{r+1}-i}) &> n_{r+1}-1, \end{aligned}$$

(ii)

$$\begin{aligned} \mathbf{s_1} &= (0), \\ \Re(s_{i,j}) &\ge 1 && (2 \le i \le r,\ 1 \le j \le n_r), \\ \sum_{i=0}^{k} \Re(s_{r+1,n_{r+1}-i}) &> k+1 && (0 \le k \le n_{r+1}-1). \end{aligned}$$

Note that these conditions are not a necessary condition for absolute convergence. However, we mainly deal with the case when (ii) is satisfied.

Proof. The series $\sum_{n=1}^{\infty} (N+n)^{-\sigma}$ $(N > 0)$ converges only when $\sigma > 1$, and

$$\sum_{n=1}^{\infty} \frac{1}{(N+n)^{\sigma}} \le \frac{1}{(\sigma-1)N^{\sigma-1}}.$$

Let $\Re(\mathbf{s}_i) = (\Re(s_{i,1}), \dots, \Re(s_{i,n_i}))$ $(1 \le i \le r+1)$ and

$$\mathbf{s}^{(k)} = \left(\Re(s_{r+1,1}), \dots, \Re(s_{r+1,n_{r+1}-k-1}), \sum_{i=0}^{k} \Re(s_{r+1,n_{r+1}-i}) - k \right).$$

(i) We have

$$\begin{aligned}
&\zeta_{MT}(\Re(\mathbf{s}_1),\Re(\mathbf{s}_2),\dots,\Re(\mathbf{s}_r);\Re(\mathbf{s}_{r+1}))\\
&\qquad\le \zeta_{MT}((\{1\}^{n_1}),\dots,(\{1\}^{n_r});\Re(\mathbf{s}_{r+1}))\\
&\qquad\ll \zeta_{MT}((\{1\}^{n_1}),\dots,(\{1\}^{n_r});\mathbf{s}^{(1)})\\
&\qquad\ll \zeta_{MT}((\{1\}^{n_1}),\dots,(\{1\}^{n_r});\mathbf{s}^{(2)})\\
&\qquad\ll \cdots\\
&\qquad\ll \zeta_{MT}\left((\{1\}^{n_1}),\dots,(\{1\}^{n_r});\sum_{i=0}^{n_{r+1}-1}\Re(s_{r+1,n_{r+1}-i})-n_{r+1}+1\right),
\end{aligned}$$

where the implicit constants of $\ll$ depend on $\mathbf{s}_{r+1}=(s_{r+1,1},\dots,s_{r+1,n_{r+1}})$. Let $R=(\sum_{i=0}^{n_{r+1}-1}\Re(s_{r+1,n_{r+1}-i})-n_{r+1}+1)/r$. Then $R>0$ and

$$\frac{1}{(\sum_{v=1}^{r}m_{v,n_v})^{\sum_{i=0}^{n_{r+1}-1}\Re(s_{r+1,n_{r+1}-i})-n_{r+1}+1}}\le\frac{1}{\prod_{v=1}^{r}m_{v,n_v}^{R}}.$$

Hence we have

$$\begin{aligned}
&\zeta_{MT}(\Re(\mathbf{s}_1),\Re(\mathbf{s}_2),\dots,\Re(\mathbf{s}_r);\Re(\mathbf{s}_{r+1}))\\
&\qquad\ll \zeta(\{1\}^{n_1-1},1+R)\zeta(\{1\}^{n_2-1},1+R)\cdots\zeta(\{1\}^{n_r-1},1+R).
\end{aligned}$$

This completes the proof for the case (i).

(ii) In the same way as that of (i), we obtain

$$\begin{aligned}
&\zeta_{MT}(0,\Re(\mathbf{s}_2),\dots,\Re(\mathbf{s}_r);\Re(\mathbf{s}_{r+1}))\\
&\qquad\ll \zeta_{MT}\left(0,(\{1\}^{n_2}),\dots,(\{1\}^{n_r});\sum_{i=0}^{n_{r+1}-1}\Re(s_{r+1,n_{r+1}-i})-n_{r+1}+1\right).
\end{aligned}$$

Let $\varepsilon>0$ such that $\sum_{i=0}^{n_{r+1}-1}\Re(s_{r+1,n_{r+1}-i})-n_{r+1}>\varepsilon$ and let

$$R=\frac{\sum_{i=0}^{n_{r+1}-1}\Re(s_{r+1,n_{r+1}-i})-n_{r+1}-\varepsilon}{r-1}\quad(>0).$$

By using

$$\frac{1}{\left(\sum_{v=1}^{r} m_{v,n_v}\right)^{\sum_{i=0}^{n_{r+1}-1}\Re(s_{r+1,n_{r+1}-i})-n_{r+1}+1}} \le \frac{1}{(m_{1,1})^{1+\varepsilon}\prod_{v=2}^{r} m_{v,n_v}^{R}},$$

we have

$$\begin{aligned}&\zeta_{MT}(0,\Re(\mathbf{s}_2),\dots,\Re(\mathbf{s}_r);\Re(\mathbf{s}_{r+1}))\\ &\qquad \ll \zeta(1+\varepsilon)\zeta(\{1\}^{n_2-1},1+R)\cdots\zeta(\{1\}^{n_r-1},1+R).\end{aligned}$$

This completes the proof. □

REMARK 9. Let

$$A=\begin{pmatrix}
n_1\left\{\underbrace{\begin{matrix}1&&\\ \vdots&\ddots&\\ 1&\cdots&1\end{matrix}}_{n_1}\right. & & & & \\
& n_2\left\{\underbrace{\begin{matrix}1&&\\ \vdots&\ddots&\\ 1&\cdots&1\end{matrix}}_{n_2}\right. & & \text{\Large 0} & \\
& & \ddots & & \\
& \text{\Large 0} & & n_r\left\{\underbrace{\begin{matrix}1&&\\ \vdots&\ddots&\\ 1&\cdots&1\end{matrix}}_{n_r}\right. & \\
\begin{matrix}1&\cdots&\cdots\\ 1&\cdots&\cdots\\ \vdots&&\\ 1&\cdots&\cdots\end{matrix} & \cdots & \cdots & \begin{matrix}\cdots\\ \cdots\\ \\ \cdots\end{matrix} & n_{r+1}\left\{\underbrace{\begin{matrix}1&&&\\ 1&1&&\\ \vdots&&\ddots&\\ 1&\cdots&\cdots&1\end{matrix}}_{n_{r+1}}\right.
\end{pmatrix}.$$

Then we have

$$\zeta_{MT}(\mathbf{s}_1,\mathbf{s}_2,\dots,\mathbf{s}_r;\mathbf{s}_{r+1})=\zeta(\mathbf{s}_1,\mathbf{s}_2,\dots,\mathbf{s}_r,\mathbf{s}_{r+1};A).$$

Therefore, the function (4) is included as a special case of the function (5). From Remark 2, the function (4) can be continued meromorphically to the whole space $\mathbf{C}^{n_1+\cdots+n_{r+1}}$. Moreover, by Lemma 2, especially when the condition (i) of Proposition 2 is satisfied, we have

$$\begin{aligned}
&\zeta_{MT}(\mathbf{s}_1,\dots,\mathbf{s}_r;\mathbf{s}_{r+1})\\
&\quad=\frac{1}{\prod_{i=1}^{r+1}\prod_{j=1}^{n_i}\Gamma(s_{i,j})}\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{r}\prod_{j=1}^{n_i}\frac{t_{i,j}^{s_{i,j}-1}\,dt_{i,j}}{e^{t_{i,j}+\cdots+t_{i,n_i}+t_{r+1,1}+\cdots+t_{r+1,n_{r+1}}}-1}\\
&\qquad\times t_{r+1,1}^{s_{r+1,1}-1}\,dt_{r+1,1}\prod_{u=2}^{n_{r+1}}\frac{t_{r+1,u}^{s_{r+1,u}-1}\,dt_{r+1,u}}{e^{t_{r+1,u}+\cdots+t_{r+1,n_{r+1}}}-1}.
\end{aligned}$$

Let

$$A=\begin{pmatrix}
0 & n_2\left\{\underbrace{\begin{matrix}1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1\end{matrix}}_{n_2}\right. & & {\Large 0} & \\
\vdots & & \ddots & & \\
\vdots & {\Large 0} & & n_r\left\{\underbrace{\begin{matrix}1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1\end{matrix}}_{n_r}\right. & \\
0 & & & & \\
\begin{matrix}1\\ 1\\ \vdots\\ 1\end{matrix} & \begin{matrix}\cdots\\ \cdots\\ \\ \cdots\end{matrix} & \begin{matrix}\cdots\\ \cdots\\ \\ \cdots\end{matrix} & \begin{matrix}\cdots\\ \cdots\\ \\ \cdots\end{matrix} & n_{r+1}\left\{\underbrace{\begin{matrix}1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1\end{matrix}}_{n_{r+1}}\right.
\end{pmatrix}.$$

Then we have

$$\zeta_{MT}(0,\mathbf{s}_2,\dots,\mathbf{s}_r;\mathbf{s}_{r+1})=\zeta(0,\mathbf{s}_2,\dots,\mathbf{s}_r,\mathbf{s}_{r+1};A).$$

Therefore, by Lemma 2, especially when the condition (ii) of Proposition 2 is satisfied, we have

$$\begin{aligned}
(9)\quad&\zeta_{MT}(0,\mathbf{s}_2,\dots,\mathbf{s}_r;\mathbf{s}_{r+1})\\
&\quad=\frac{1}{\prod_{i=2}^{r+1}\prod_{j=1}^{n_i}\Gamma(s_{i,j})}\int_0^\infty\cdots\int_0^\infty\prod_{i=2}^{r}\prod_{j=1}^{n_i}\frac{t_{i,j}^{s_{i,j}-1}\,dt_{i,j}}{e^{t_{i,j}+\cdots+t_{i,n_i}+t_{r+1,1}+\cdots+t_{r+1,n_{r+1}}}-1}\\
&\qquad\times\prod_{u=1}^{n_{r+1}}\frac{t_{r+1,u}^{s_{r+1,u}-1}\,dt_{r+1,u}}{e^{t_{r+1,u}+\cdots+t_{r+1,n_{r+1}}}-1}.
\end{aligned}$$

REMARK 10. For $z\in\mathbf{C}$ with $|z|<1$, let

$$
\begin{aligned}
&\mathrm{Li}_{\mathbf{k}_1,\ldots,\mathbf{k}_r;\mathbf{k}_{r+1}}(z)\\
&\quad= \sum_{\substack{0<m_{1,1}<m_{1,2}<\cdots<m_{1,n_1}\\ \vdots\\ 0<m_{r,1}<m_{r,2}<\cdots<m_{r,n_r}}} \sum_{m_{r+1,1}=1,\ldots,m_{r+1,n_{r+1}-1}=1}^{\infty}\\
&\qquad \times \frac{z^{\sum_{v=1}^{r} m_{v,n_v}+\sum_{w=1}^{n_{r+1}-1} m_{r+1,w}}}{\prod_{i=1}^{r}\prod_{j=1}^{n_i} m_{i,j}^{k_{i,j}} \prod_{u=1}^{n_{r+1}} \left(\sum_{v=1}^{r} m_{v,n_v}+\sum_{w=1}^{u-1} m_{r+1,w}\right)^{k_{r+1,u}}}.
\end{aligned}
$$

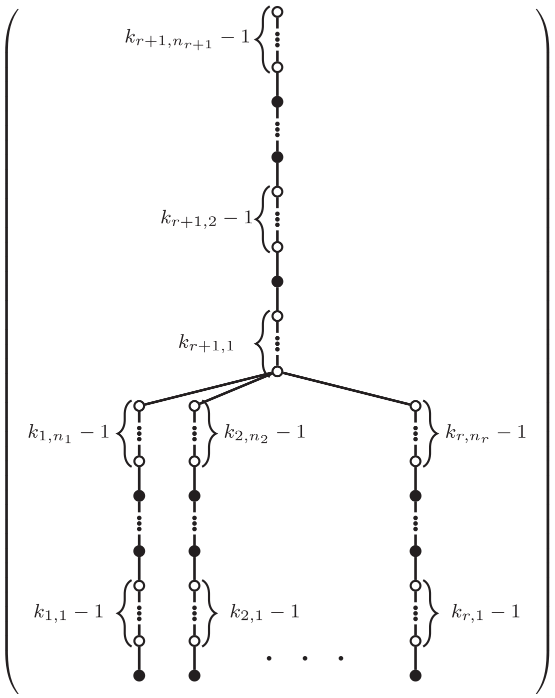
Then we obtain

$$
\begin{aligned}
&\frac{d}{dz}\mathrm{Li}_{\mathbf{k}_1,\ldots,\mathbf{k}_r;\mathbf{k}_{r+1}}(z)\\
&\quad= \begin{cases}
\dfrac{1}{z}\mathrm{Li}_{\mathbf{k}_1,\ldots,\mathbf{k}_r;(\mathbf{k}_{r+1})_-}(z) & (\text{if } k_{r+1,n_{r+1}} > 1),\\[2ex]
\dfrac{1}{1-z}\mathrm{Li}_{\mathbf{k}_1,\ldots,\mathbf{k}_r;k_{r+1,1},\ldots,k_{r+1,n_{r+1}-1}}(z) & (\text{if } k_{r+1,n_{r+1}} = 1,\ n_{r+1} > 1),\\[2ex]
\dfrac{1}{z}\displaystyle\prod_{i=1}^{r}\mathrm{Li}_{\mathbf{k}_i}(z) & (\text{if } k_{r+1,n_{r+1}} = 1,\ n_{r+1} = 1).
\end{cases}
\end{aligned}
$$

Therefore, the special values of the GMT-type MZF are written as follows:

$$
\begin{aligned}
&\zeta_{MT}(\mathbf{k}_1,\ldots,\mathbf{k}_r;\mathbf{k}_{r+1})\\
&\quad= \int_{0<t_1<\cdots<t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}}}<1} \left(\prod_{i=2}^{k_{r+1,n_{r+1}}} \frac{1}{t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}-1}+i}}\right)\\
&\qquad \times \frac{1}{1-t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}-1}+1}} \left(\prod_{i=2}^{k_{r+1,n_{r+1}-1}} \frac{1}{t_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}-2}+i}}\right)\cdots\\
&\qquad \times \frac{1}{1-t_{k_{r+1,1}+k_{r+1,2}+1}} \left(\prod_{i=2}^{k_{r+1,2}} \frac{1}{t_{k_{r+1,1}+i}}\right)\\
&\qquad \times \frac{1}{1-t_{k_{r+1,1}+1}} \left(\prod_{i=1}^{k_{r+1,1}} \frac{1}{t_i}\right) \prod_{i=1}^{r} \mathrm{Li}_{\mathbf{k}_i}(t_1)\, dt_1 \cdots dt_{k_{r+1,1}+\cdots+k_{r+1,n_{r+1}}}
\end{aligned}
$$

$= I$



.

Therefore, GMT-type MZVs can be expressed as a sum of a finite number of EZ-type MZVs by [9, Corollary 2.4].

Proposition 3. *If the condition* (ii) *of Proposition* 2 *is satisfied and all entries of* $\mathbf{s}_i$ $(2 \le i \le r)$ *are positive integers* (*we put* $\mathbf{s}_i = \mathbf{k}_i (= (k_{i,1}, \ldots, k_{i,n_i})) \in \mathbf{N}^{n_i}$ $(2 \le i \le r)$), *then* $\zeta_{MT}(0, \mathbf{k}_2, \ldots, \mathbf{k}_r; \mathbf{s}_{r+1})$ *is expressed as a* $\mathbf{Q}$*-linear combination of EZ-type MZFs.*

Proof. We have

$$(10)\qquad \zeta_{MT}(0, \mathbf{k}_2, \ldots, \mathbf{k}_r; \mathbf{s}_{r+1})$$

$$= \sum_{\substack{0<m_{1,1} \\ 0<m_{2,1}<m_{2,2}<\cdots<m_{2,n_2} \\ \vdots \\ 0<m_{r,1}<m_{r,2}<\cdots<m_{r,n_r}}} \sum_{m_{r+1,1}=1,\ldots,m_{r+1,n_{r+1}-1}=1}^{\infty} \frac{1}{\prod_{i=2}^{r} \prod_{j=1}^{n_i} m_{i,j}^{k_{i,j}} \prod_{u=1}^{n_{r+1}} \left(\sum_{v=1}^{r} m_{v,n_v} + \sum_{w=1}^{u-1} m_{r+1,w}\right)^{s_{r+1,u}}}$$

$$= \sum_{\substack{0<m_{1,1}\\ 0<m_{2,1}<m_{2,2}<\cdots<m_{2,n_2}\\ \vdots\\ 0<m_{r,1}<m_{r,2}<\cdots<m_{r,n_r}}} \sum_{m_{r+1,1}=1,\dots,m_{r+1,n_{r+1}-1}=1}^{\infty} \frac{1}{\prod_{i=2}^{r}\prod_{j=1}^{n_i} m_{i,j}^{k_{i,j}}}$$

$$\times \left(\prod_{u=1}^{n_{r+1}} \frac{1}{\Gamma(s_{r+1,u})}\int_0^{\infty} t_u^{s_{r+1,u}-1} e^{-(\sum_{v=1}^{r} m_{v,n_v}+\sum_{w=1}^{u-1} m_{r+1,w})t_u}\, dt_u\right)$$

$$= \frac{1}{\prod_{u=1}^{n_{r+1}}\Gamma(s_{r+1,u})} \sum_{\substack{0<m_{2,1}<m_{2,2}<\cdots<m_{2,n_2}\\ \vdots\\ 0<m_{r,1}<m_{r,2}<\cdots<m_{r,n_r}}} \frac{1}{\prod_{i=2}^{r}\prod_{j=1}^{n_i} m_{i,j}^{k_{i,j}}}$$

$$\times \int_0^{\infty}\cdots\int_0^{\infty} t_1^{s_{r+1,1}-1}\cdots t_{n_{r+1}}^{s_{r+1,n_{r+1}}-1} e^{-(\sum_{v=2}^{r} m_{v,n_v})(t_1+\cdots+t_{n_{r+1}})}$$

$$\times \frac{1}{e^{t_1+\cdots+t_{n_{r+1}}}-1}\,\frac{1}{e^{t_2+\cdots+t_{n_{r+1}}}-1}\cdots\frac{1}{e^{t_{n_{r+1}}}-1}\, dt_1\cdots dt_{n_{r+1}}$$

$$= \frac{1}{\prod_{u=1}^{n_{r+1}}\Gamma(s_{r+1,u})}\int_0^{\infty}\cdots\int_0^{\infty} t_1^{s_{r+1,1}-1}\cdots t_{n_{r+1}}^{s_{r+1,n_{r+1}}-1}$$

$$\times \frac{\prod_{i=2}^{r}\mathrm{Li}_{\mathbf{k}_i}(e^{-(t_1+\cdots+t_{n_{r+1}})})}{e^{t_1+\cdots+t_{n_{r+1}}}-1}\,\frac{1}{e^{t_2+\cdots+t_{n_{r+1}}}-1}\cdots\frac{1}{e^{t_{n_{r+1}}}-1}\, dt_1\cdots dt_{n_{r+1}}.$$

By using the shuffle product formula for $\prod_{i=2}^{r}\mathrm{Li}_{\mathbf{k}_i}(e^{-(t_1+\cdots+t_{n_{r+1}})})$, we find that (10) is expressed as a $\mathbf{Q}$-linear combination of the form

$$\text{(11)}\qquad \frac{1}{\prod_{u=1}^{n_{r+1}}\Gamma(s_{r+1,u})}\int_0^{\infty}\cdots\int_0^{\infty} t_1^{s_{r+1,1}-1}\cdots t_{n_{r+1}}^{s_{r+1,n_{r+1}}-1}$$

$$\times \frac{\mathrm{Li}_{\mathbf{k}}(e^{-(t_1+\cdots+t_{n_{r+1}})})}{e^{t_1+\cdots+t_{n_{r+1}}}-1}\,\frac{1}{e^{t_2+\cdots+t_{n_{r+1}}}-1}\cdots\frac{1}{e^{t_{n_{r+1}}}-1}\, dt_1\cdots dt_{n_{r+1}}.$$

By applying (10) with $r=2$ to the function (11), we see that the function (11) equals to $\zeta_{MT}(0,\mathbf{k};\mathbf{s}_{r+1})$. Moreover, by the definition of the function (4), we find that

$$\zeta_{MT}(0,\mathbf{k};\mathbf{s}_{r+1}) = \zeta(\mathbf{k},\mathbf{s}_{r+1}).$$

Therefore, (10) is expressed as a $\mathbf{Q}$-linear combination of EZ-type MZFs. □

5. Generalizing Theorem 4 and Theorem 6

In this section, using GMT-type MZFs, we generalize Theorem 4 and Theorem 6.

5.1. Ito zeta function. It is difficult to generalize Theorem 4 and Theorem 6 as their original forms. First, in order to make the idea easier to understand, we rewrite Theorem 4.

PROPOSITION 4. *For* $r \in \mathbf{N}$, $s \in \mathbf{C}$, *we have*

$$\xi_{MT}(\{2\}^r; s)$$
$$= \sum_{a_1+a_2+a_3=r} \frac{r!}{a_1!a_3!} \zeta(2)^{a_1} (-1)^{a_2+a_3} \binom{s+a_2-1}{a_2} \zeta_{MT}(0, \{1\}^{a_2}, \{2\}^{a_3}; a_2+s),$$

where the sum is over all $a_1, a_2, a_3 \in \mathbf{Z}_{\geq 0}$ *satisfying* $a_1 + a_2 + a_3 = r$.

PROOF. By using the first formula of Corollary 1, we have

$$\mathrm{Li}_2(1-e^{-t}) = \zeta(2) - \int_0^\infty \frac{t}{e^{t+u}-1}\,du - \int_0^\infty \frac{u}{e^{t+u}-1}\,du.$$

Therefore, for $\Re(s) > 1$, we have

$$\Gamma(s)\xi_{MT}(\{2\}^r; s)$$
$$= \int_0^\infty \frac{t^{s-1}}{e^t-1} (\mathrm{Li}_2(1-e^{-t}))^r\,dt$$
$$= \int_0^\infty \frac{t^{s-1}}{e^t-1} \left(\zeta(2) - \int_0^\infty \frac{t}{e^{t+u}-1}\,du - \int_0^\infty \frac{u}{e^{t+u}-1}\,du\right)^r dt$$
$$= \sum_{a_1+a_2+a_3=r} \frac{r!}{a_1!a_2!a_3!}$$
$$\times \int_0^\infty \frac{t^{s-1}}{e^t-1} \zeta(2)^{a_1} \left(-\int_0^\infty \frac{t}{e^{t+u}-1}\,du\right)^{a_2} \left(-\int_0^\infty \frac{u}{e^{t+u}-1}\,du\right)^{a_3} dt$$
$$= \sum_{a_1+a_2+a_3=r} \frac{r!}{a_1!a_2!a_3!} \zeta(2)^{a_1} (-1)^{a_2+a_3}$$

$$\times \int_0^\infty \frac{t^{s+a_2-1}}{e^t-1}\left(\int_0^\infty \frac{1}{e^{t+u}-1}\,du\right)^{a_2}\left(\int_0^\infty \frac{u}{e^{t+u}-1}\,du\right)^{a_3} dt$$

$$= \sum_{a_1+a_2+a_3=r} \frac{r!}{a_1!a_2!a_3!}\zeta(2)^{a_1}(-1)^{a_2+a_3}\Gamma(a_2+s)\zeta_{MT}(0,\{1\}^{a_2},\{2\}^{a_3};a_2+s).$$

By the analytic continuation of the Ito zeta function ([3, Theorem 2]) and the MT-type MZF, we obtain the stated theorem. □

The key of this proof is to use the formula

$$\mathrm{Li}_2(1-e^{-t}) = \zeta(2) - \int_0^\infty \frac{t}{e^{t+u}-1}\,du - \int_0^\infty \frac{u}{e^{t+u}-1}\,du$$

directly. We generalize this formula to any k.

Lemma 4. *For $k \in \mathbf{Z}_{\geq 2}$ and $t > 0$, we have*

$$\mathrm{Li}_k(1-e^{-t}) = \sum_{j=0}^{2k-2} f(t;j,k),$$

where

$$f(t;j,k)$$
$$= \begin{cases} (-1)^j\zeta(k-j)\int_0^\infty\cdots\int_0^\infty \prod_{i=1}^{j}\frac{du_i}{e^{t+u_1+\cdots+u_i}-1} & (\mathit{if}\ \ j<k-1), \\ (-1)^{k-1}\int_0^\infty\cdots\int_0^\infty t\prod_{i=1}^{k-1}\frac{du_i}{e^{t+u_1+\cdots+u_i}-1} & (\mathit{if}\ \ j=k-1), \\ (-1)^{k-1}\int_0^\infty\cdots\int_0^\infty u_{j-k+1}\prod_{i=1}^{k-1}\frac{du_i}{e^{t+u_1+\cdots+u_i}-1} & (\mathit{if}\ \ j>k-1), \end{cases}$$

and we understand if $j = 0$ then

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{e^{t+u_1}-1}\cdots\frac{1}{e^{t+u_1+\cdots+u_j}-1}\,du_1\cdots du_j = 1.$$

Proof. We use induction. If $k = 2$, it is true since

$$\mathrm{Li}_2(1-e^{-t}) = \zeta(2) - \int_0^\infty \frac{t}{e^{t+u}-1}\,du - \int_0^\infty \frac{u}{e^{t+u}-1}\,du.$$

Assume that the formula holds for k, and prove it for $k+1$.

$$\mathrm{Li}_{k+1}(1-e^{-t})$$

$$= \zeta(k+1) - \int_0^\infty \frac{\mathrm{Li}_k(1-e^{-(t+u_1)})}{e^{t+u_1}-1}\,du_1$$

$$= \zeta(k+1) - \int_0^\infty \frac{1}{e^{t+u_1}-1}\left(\sum_{j=0}^{2k-2} f(t+u_1; j, k)\right) du_1$$

$$= \zeta(k+1) + \sum_{j=0}^{k-2} f(t; j+1, k+1) + f(t; k, k+1) + f(t; k+1, k+1)$$

$$+ \sum_{j=k}^{2k-2} f(t; j+2, k+1)$$

$$= \sum_{j=0}^{2k} f(t; j, k+1).$$

This completes the proof. □

LEMMA 5. *With the assumption of Lemma* 4, *for* $k_i \geq 2$ $(1 \leq i \leq r)$ *and* $t > 0$, *we have*

$$\prod_{i=1}^r \mathrm{Li}_{k_i}(1-e^{-t}) = \sum_{\substack{0 \leq j_1 \leq 2k_1-2 \\ \vdots \\ 0 \leq j_r \leq 2k_r-2}} \prod_{i=1}^r f(t; j_i, k_i).$$

PROOF. By Lemma 4, we have

$$\prod_{i=1}^r \mathrm{Li}_{k_i}(1-e^{-t}) = \prod_{i=1}^r \sum_{j=0}^{2k_i-2} f(t; j, k_i) = \sum_{\substack{0 \leq j_1 \leq 2k_1-2 \\ \vdots \\ 0 \leq j_r \leq 2k_r-2}} \prod_{i=1}^r f(t; j_i, k_i). \qquad \square$$

THEOREM 8. *For* $l \in \mathbf{Z}_{\geq 0}$, $s \in \mathbf{C}$ *and* $k_i \geq 2$ $(1 \leq i \leq r)$, *we have*

$$\xi_{MT}(\{1\}^l, k_1, \ldots, k_r; s)$$

$$= \sum_{\substack{0 \leq j_1 \leq 2k_1-2 \\ \vdots \\ 0 \leq j_r \leq 2k_r-2}} a_r(\mathbf{j}, \mathbf{k}) \frac{\Gamma(l + b_r(\mathbf{j}, \mathbf{k}) + s)}{\Gamma(s)}$$

$$\times \zeta_{MT}(0, \mathbf{k}(j_1, k_1), \ldots, \mathbf{k}(j_r, k_r); l + b_r(\mathbf{j}, \mathbf{k}) + s),$$

where

$$a_r(\mathbf{j},\mathbf{k}) = \prod_{i=1}^{r} a(j_i,k_i), \quad a(j_i,k_i) = \begin{cases} (-1)^{j_i}\zeta(k_i - j_i) & (j_i < k_i - 1), \\ (-1)^{k_i-1} & (j_i \geq k_i - 1), \end{cases}$$

$$b_r(\mathbf{j},\mathbf{k}) = |\{i \in \{1,\dots,r\} \mid j_i = k_i - 1\}|,$$

and

$$\mathbf{k}(j_i,k_i) = \begin{cases} (\{1\}^{j_i}) & (j_i \leq k_i - 1), \\ (\underbrace{\{1\}^{j_i-k_i}, 2, \{1\}^{2k_i-2-j_i}}_{k_i-1}) & (j_i > k_i - 1). \end{cases}$$

Proof. By Lemma 5 and the formula (9), for $\Re(s) > 1$, we have

$$\begin{aligned} &\Gamma(s)\xi_{MT}(\{1\}^l, k_1,\dots,k_r; s) \\ &= \int_0^\infty \frac{t^{s+l-1}}{e^t - 1}\prod_{i=1}^{r}\mathrm{Li}_{k_i}(1 - e^{-t})\, dt \\ &= \sum_{\substack{0 \leq j_1 \leq 2k_1 - 2 \\ \vdots \\ 0 \leq j_r \leq 2k_r - 2}} \int_0^\infty \frac{t^{s+l-1}}{e^t - 1}\prod_{i=1}^{r} f(t; j_i, k_i)\, dt \\ &= \sum_{\substack{0 \leq j_1 \leq 2k_1 - 2 \\ \vdots \\ 0 \leq j_r \leq 2k_r - 2}} a_r(\mathbf{j},\mathbf{k})\Gamma(l + b_r(\mathbf{j},\mathbf{k}) + s) \\ &\qquad \times \zeta_{MT}(0, \mathbf{k}(j_1,k_1),\dots,\mathbf{k}(j_r,k_r); l + b_r(\mathbf{j},\mathbf{k}) + s). \end{aligned}$$

By the analytic continuation, we obtain the stated theorem. □

By putting $s = m + 1$ in Theorem 8 and using Theorem 3, we can obtain the following relations among GMT-type MZVs with $n_{r+1} = 1$.

Theorem 9. *For* $m \in \mathbf{N}$ *and* $k_i \geq 2$ $(1 \leq i \leq r)$, *we have*

$$\begin{aligned} &\zeta_{MT}(k_1,\dots,k_r,\{1\}^m; 1) \\ &= \sum_{\substack{0 \leq j_1 \leq 2k_1 - 2 \\ \vdots \\ 0 \leq j_r \leq 2k_r - 2}} (b_r(\mathbf{j},\mathbf{k}) + m)!\, a_r(\mathbf{j},\mathbf{k})\zeta_{MT}(0, \mathbf{k}(j_1,k_1),\dots,\mathbf{k}(j_r,k_r); b_r(\mathbf{j},\mathbf{k}) + m + 1). \end{aligned}$$

EXAMPLE 2. In the case $k_1 = 2, \dots, k_r = 2$, Theorem 9 gives relations among MT-type MZVs. For example, if $k_1 = 2$, $r = 1$ then we obtain

$$\begin{aligned}&\zeta_{MT}(2, \{1\}^m; 1)\\&\quad = m!\zeta(2)\zeta_{MT}(0; m+1) - (m+1)!\zeta_{MT}(0, 1; m+2) - m!\zeta_{MT}(0, 2; m+1).\end{aligned}$$

If there exists i satisfying $k_i \geq 3$ then Theorem 9 gives relations among GMT-type MZVs with $n_{r+1} = 1$, not MT-type MZVs. For example, if $k_1 = 3$, $r = 1$ then we obtain

$$\begin{aligned}&\zeta_{MT}(3, \{1\}^m; 1)\\&\quad = m!\zeta(3)\zeta_{MT}(0; m+1) - m!\zeta(2)\zeta_{MT}(0, 1; m+1)\\&\qquad + (m+1)!\zeta_{MT}(0, (1, 1); m+2) + m!\zeta_{MT}(0, (2, 1); m+1)\\&\qquad + m!\zeta_{MT}(0, (1, 2); m+1).\end{aligned}$$

5.2. An analog of the Arakawa-Kaneko zeta function of Miyagawa-type. We generalize Theorem 6 by using Lemma 5.

THEOREM 10. *With the assumption of Theorem* 8, *for* $l \in \mathbf{Z}_{\geq 0}$, $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{Z}_{\geq 2}^n$, $k_{n+1} \in \mathbf{Z}_{\geq 0}$ *and* $s \in \mathbf{C}$, *we have*

$$\begin{aligned}&(-1)^{k_{n+1}}\xi_{MT}((\{1\}^l, \mathbf{k}; k_{n+1}); s)\\&\quad = \sum_{\substack{0 \leq j_1 \leq 2k_1 - 2\\ \vdots \\ 0 \leq j_n \leq 2k_n - 2}} \sum_{c_1 + \cdots + c_{k_{n+1}+1} = l + b_n(\mathbf{j}, \mathbf{k})} (l + b_n(\mathbf{j}, \mathbf{k}))! a_n(\mathbf{j}, \mathbf{k}) \binom{s + c_{k_{n+1}+1} - 1}{c_{k_{n+1}+1}}\\&\qquad \times \zeta_{MT}(0, \mathbf{k}(j_1, k_1), \dots, \mathbf{k}(j_n, k_n); c_1 + 1, \dots, c_{k_{n+1}} + 1, c_{k_{n+1}+1} + s)\\&\qquad - \sum_{i=1}^{k_{n+1}} (-1)^{i-1}\zeta_{MT}(\{1\}^l, k_1, \dots, k_n; i)\zeta(\{1\}^{k_{n+1}-i}, s).\end{aligned}$$

PROOF. Let $q = k_{n+1} + 1$. For $\Re(s) > 1$, let

$$J = \int_0^\infty \cdots \int_0^\infty t_q^{s-1}(t_1 + \cdots + t_q)^l \left(\prod_{i=1}^n \mathrm{Li}_{k_i}(1 - e^{-(t_1 + \cdots + t_q)})\right) \prod_{i=1}^q \frac{dt_i}{e^{t_i + \cdots + t_q} - 1}.$$

We calculate J in two different ways. By using Corollary 1, we have

$$\begin{aligned}
J &= \int_0^\infty \cdots \int_0^\infty t_q^{s-1}(\zeta_{MT}(\{1\}^l, k_1, \dots, k_n; 1) - \mathrm{Li}_{\{1\}^l, k_1, \dots, k_n; 1}(1 - e^{-(t_2+\cdots+t_q)})) \\
&\quad \times \frac{1}{e^{t_2+\cdots+t_q}-1} \cdots \frac{1}{e^{t_q}-1}\, dt_2 \cdots dt_q \\
&= \Gamma(s)\zeta_{MT}(\{1\}^l, k_1, \dots, k_n; 1)\zeta(\{1\}^{q-2}, s) \\
&\quad - \int_0^\infty \cdots \int_0^\infty t_q^{s-1}\, \mathrm{Li}_{\{1\}^l, k_1, \dots, k_n; 1}(1 - e^{-(t_2+\cdots+t_q)}) \\
&\quad \times \frac{1}{e^{t_2+\cdots+t_q}-1} \cdots \frac{1}{e^{t_q}-1}\, dt_2 \cdots dt_q \\
&= \cdots \\
&= \Gamma(s) \sum_{i=1}^{q-1} (-1)^{i-1} \zeta_{MT}(\{1\}^l, k_1, \dots, k_n; i)\zeta(\{1\}^{q-1-i}, s) \\
&\quad + (-1)^{q-1}\Gamma(s)\xi_{MT}((\{1\}^l, \mathbf{k}; q-1); s).
\end{aligned}$$

On the other hand, by using Lemma 5, we have

$$\begin{aligned}
J &= \sum_{\substack{0 \le j_1 \le 2k_1 - 2 \\ \vdots \\ 0 \le j_n \le 2k_n - 2}} \int_0^\infty \cdots \int_0^\infty t_q^{s-1}(t_1 + \cdots + t_q)^l \\
&\quad \times \left(\prod_{i=1}^n f(t_1 + \cdots + t_q; j_i, k_i) \right) \prod_{i=1}^q \frac{dt_i}{e^{t_i+\cdots+t_q}-1} \\
&= \sum_{\substack{0 \le j_1 \le 2k_1 - 2 \\ \vdots \\ 0 \le j_n \le 2k_n - 2}} \int_0^\infty \cdots \int_0^\infty t_q^{s-1}(t_1 + \cdots + t_q)^{l+b_n(\mathbf{j},\mathbf{k})} \\
&\quad \times \left(\prod_{i=1}^n \frac{f(t_1 + \cdots + t_q; j_i, k_i)}{(t_1 + \cdots + t_q)^{b_n(\mathbf{j},\mathbf{k})}} \right) \prod_{i=1}^q \frac{dt_i}{e^{t_i+\cdots+t_q}-1} \\
&= \sum_{\substack{0 \le j_1 \le 2k_1 - 2 \\ \vdots \\ 0 \le j_n \le 2k_n - 2}} \sum_{c_1+\cdots+c_q = l+b_n(\mathbf{j},\mathbf{k})} \frac{(l+b_n(\mathbf{j},\mathbf{k}))!}{c_1! \cdots c_q!} \int_0^\infty \cdots \int_0^\infty t_q^{s-1} t_1^{c_1} \cdots t_q^{c_q}
\end{aligned}$$

$$\times \left(\prod_{i=1}^{n} \frac{f(t_1 + \cdots + t_q; j_i, k_i)}{(t_1 + \cdots + t_q)^{b_n(\mathbf{j},\mathbf{k})}} \right) \prod_{i=1}^{q} \frac{dt_i}{e^{t_i + \cdots + t_q} - 1}$$

$$= \sum_{\substack{0 \le j_1 \le 2k_1 - 2 \\ \vdots \\ 0 \le j_n \le 2k_n - 2}} \sum_{c_1 + \cdots + c_q = l + b_n(\mathbf{j},\mathbf{k})} \frac{(l + b_n(\mathbf{j},\mathbf{k}))!}{c_1! \cdots c_q!} a_n(\mathbf{j},\mathbf{k}) \left(\prod_{i=1}^{q-1} \Gamma(c_i + 1) \right)$$

$$\times \Gamma(c_q + s) \zeta_{MT}(0, \mathbf{k}(j_1, k_1), \ldots, \mathbf{k}(j_n, k_n); c_1 + 1, \ldots, c_{q-1} + 1, c_q + s)$$

$$= \sum_{\substack{0 \le j_1 \le 2k_1 - 2 \\ \vdots \\ 0 \le j_n \le 2k_n - 2}} \sum_{c_1 + \cdots + c_q = l + b_n(\mathbf{j},\mathbf{k})} \frac{(l + b_n(\mathbf{j},\mathbf{k}))!}{c_q!} a_n(\mathbf{j},\mathbf{k}) \Gamma(c_q + s)$$

$$\times \zeta_{MT}(0, \mathbf{k}(j_1, k_1), \ldots, \mathbf{k}(j_n, k_n); c_1 + 1, \ldots, c_{q-1} + 1, c_q + s).$$

By the analytic continuation, we obtain the stated theorem. □

By putting $s = m + 1$ in Theorem 10 and using Theorem 5, we can obtain the following relations among GMT-type MZVs.

THEOREM 11. *For* $m \in \mathbf{N}, k_{n+1}, l \in \mathbf{Z}_{\ge 0}$ *and* $k_i \ge 2 \ (1 \le i \le n)$, *we have*

$$(-1)^{k_{n+1}} \sum_{a_1 + \cdots + a_{k_{n+1}+1} = m} \frac{1}{a_{k_{n+1}+1}!}$$

$$\times \zeta_{MT}(\{1\}^{a_{k_{n+1}+1}+l}, k_1, \ldots, k_n; {}_{-}((a_1 + 1, \ldots, a_{k_{n+1}} + 1, 2)^*))$$

$$= \sum_{\substack{0 \le j_1 \le 2k_1 - 2 \\ \vdots \\ 0 \le j_n \le 2k_n - 2}} \sum_{c_1 + \cdots + c_{k_{n+1}+1} = l + b_n(\mathbf{j},\mathbf{k})} (l + b_n(\mathbf{j},\mathbf{k}))! a_n(\mathbf{j},\mathbf{k}) \binom{m + c_{k_{n+1}+1}}{c_{k_{n+1}+1}}$$

$$\times \zeta_{MT}(0, \mathbf{k}(j_1, k_1), \ldots, \mathbf{k}(j_n, k_n); c_1 + 1, \ldots, c_{k_{n+1}} + 1, c_{k_{n+1}+1} + m + 1)$$

$$- \sum_{i=1}^{k_{n+1}} (-1)^{i-1} \zeta_{MT}(\{1\}^l, k_1, \ldots, k_n; i) \zeta(\{1\}^{k_{n+1}-i}, m + 1).$$

EXAMPLE 3. In the case $k_{n+1} = 0$, Theorem 11 coincides with Theorem 9. If $k_1 = 2, \ldots, k_n = 2$ then Theorem 11 gives relations among Miyagawa-type MZVs.

For example, if $m = 1$, $l = 0$, $k_1 = 2$, $k_2 = 1$, $n = 1$ then we obtain

$$\begin{aligned} &-\zeta_{MT}(2;1,2) - \zeta_{MT}(1,2;2) \\ &\quad = \zeta(2)\zeta_{MT}(0;1,2) - \zeta_{MT}(0,1;2,2) - 2\zeta_{MT}(0,1;1,3) \\ &\qquad - \zeta_{MT}(0,2;1,2) - \zeta_{MT}(2;1)\zeta(2). \end{aligned}$$

If there exists i satisfying $k_i \geq 3$ then Theorem 11 gives relations among GMT-type MZVs, not Miyagawa-type MZVs or GMT-type MZVs with $n_{r+1} = 1$. For example, if $m = 1$, $l = 0$, $k_1 = 3$, $k_2 = 1$, $n = 1$ then we obtain

$$\begin{aligned} &-\zeta_{MT}(3;1,2) - \zeta_{MT}(1,3;2) \\ &\quad = \zeta(3)\zeta_{MT}(0;1,2) - \zeta(2)\zeta_{MT}(0,1;1,2) + \zeta_{MT}(0,(1,1);2,2) \\ &\qquad + 2\zeta_{MT}(0,(1,1);1,3) + \zeta_{MT}(0,(2,1);1,2) + \zeta_{MT}(0,(1,2);1,2) \\ &\qquad - \zeta_{MT}(3;1)\zeta(2). \end{aligned}$$

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