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COMPUTATION METHODS OF LOGARITHMIC VECTOR FIELDS ASSOCIATED TO SEMI-WEIGHTED HOMOGENEOUS ISOLATED HYPERSURFACE SINGULARITIES

By

Katsusuke Nabeshima and Shinichi Tajima

Abstract. Methods for computing logarithmic vector fields along a semi-weighted homogeneous hypersurface with an isolated singularity are considered in the context of symbolic computation. The main idea of our approach is based on the concept of polar variety and of algebraic local cohomology. New algorithms are introduced for computing a set of generators of the modules of logarithmic vector fields. The keys of the resulting algorithms are a notion of parametric syzygy system and that of parametric local cohomology system.

1. Introduction

The concept of logarithmic vector fields along a hypersurface, introduced by K. Saito [38], is of considerable importance in complex analysis and singularity theory. Logarithmic vector fields have been extensively studied and utilized by several authors in diverse fields and in many different problems such as the theory of Saito free divisors [2, 5, 11, 12], logarithmic comparison problems [8, 9], singular holomorphic vector fields [3, 16, 39, 41], I-versal deformation theory [13, 14, 37]. H. Terao [49] and J. W. Bruce [5] studied the modules of the logarithmic vector fields along the bifurcation set of a semiuniversal deformation of an isolated hypersurface singularity and decided its structure. These authors also gave a method of explicit computation for its free base.

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In singularity theory, A. Aleksandrov [1] and J. Wahl [50] independently gave, among other results, a closed formula of the generators of logarithmic vector fields along quasi-homogeneous complete intersection singularities. Later, H. Hauser and G. Müller [19, 20] investigated Gröbner correspondences and showed in particular that two germs of hypersurfaces with an isolated singular point are biholomorphicaly isomorphic if and only if the corresponding Lie algebras of logarithmic vector fields are isomorphic.

For non-quasi homogeneous cases, no closed formula and no algorithmic method for computing logarithmic vector fields are known. Structure of logarithmic vector fields has not been studied systematically even for the case of semi-weighted homogeneous hypersurface isolated singularities. Many problems that involve logarithmic vector fields still remain unsolved.

In this paper, we consider logarithmic vector fields along semi-weighted homogeneous hypersurface isolated singularities. Based on results given in [42], we propose an effective method for computing a set of generators of the module of logarithmic vector fields. The keys of our approach are the concept of a polar variety and a set of local cohomology classes associated to the polar variety. We generaize the proposed method to parametric cases for studying deformation of hypersurface singularities. An innovation of this paper is a notion of parametric syzygy system. The resulting algorithms can compute in particular the parameter dependency of the structure of the module of logarithmic vector fields associated to μ-constant deformations of weighted homogeneous hypersurface isolated singularities.

To be more precise, let f be a semi-weighted homogeneous polynomial in $K[x_1, \dots, x_n]$, w.r.t. a weighted vector $\mathbf{w} \in \mathbf{N}^n$, where K is the field of rational numbers or complex numbers. We assume that the polynomial f defines an isolated singularity at the origin and the sequence $\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}\right)$ is a regular sequence ([24, 25]).

In section 3, we describe an algorithm for computing a basis of local cohomology classes associated to the polar variety, namely local cohomology classes associated to the ideal generated by $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}$ in the local ring. As we have given in [31] only an outline of the algorithm, we illustrate here in section 3 a complete algorithm. We also show the effectivity of the proposed algorithm and how Poincaré polynomials work well, together with results of the benchmark tests.

In section 4, first we describe relations between logarithmic vector fields and local cohomology classes and we see that these local cohomology classes can be used to reveal the structure of logarithmic vector fields. Second, we propose an algorithm for computing standard basis of the annihilator ideal of local cohomology classes mentioned above. The resulting algorithms will be utilized in the next section.

In section 5, we provide two different computational methods of logarithmic vector fields (with parameters). The first method utilizes Lazard's homogenization technique [23]. The second method utilizes a Gröbner basis computation of a syzygy module. We describe these algorithms with many details and examples. We also present empirical data and comparison of the two computational methods.

This paper extends our conference paper [31] by many details, algorithms, computation experiments and examples. The first method, described in section 5, has been introduced in [31]. The second method is newly obtained in the present paper.

All algorithms in this paper have been implemented in the computer algebra system Risa/Asir [35]. All tests presented in this paper, have been performed on a machine [OS: Windows 7 (64bit), CPU: Intel(R) Core i-7-5930K CPU @ 3.50 GHz 3.50 GHz, RAM: 64 GB] and the computer algebra system Risa/Asir version 20150126 [35].

2. Preliminaries

Throughout this paper, we use the notation x as the abbreviation of n variables $x_1, \dots, x_n$. The set of natural numbers $\mathbf{N}$ includes zero. K is the field of rational numbers $\mathbf{Q}$ or the field of complex numbers $\mathbf{C}$.

Let $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbf{N}^n$ be a weight vector with positive entries (i.e., $w_i > 0$ for all i) for a given coordinate system $x = (x_1, x_2, \dots, x_n)$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n)$. Set $|\alpha|_{\mathbf{w}} = \sum_{i=1}^{n} w_i \alpha_i$ for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{N}^n$. The weighted degree of a term $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ is defined by $\deg_{\mathbf{w}}(x^\alpha) = |\alpha|_{\mathbf{w}}$. Let $\deg_{\mathbf{w}}(f)$ denote the weighted degree of f, defined to be $\deg_{\mathbf{w}}(f) = \max\{|\alpha|_{\mathbf{w}} \mid x^\alpha \text{ is a term of } f\}$. Let $\mathrm{ord}_{\mathbf{w}}(f) = \min\{|\alpha|_{\mathbf{w}} \mid x^\alpha \text{ is a term of } f\}$. ($\mathrm{ord}_{\mathbf{w}}(0) = -1$).

Definition 2.1 ([4]). (i) A nonzero polynomial f in $K[x]$ is weighted homogeneous of type $(d; \mathbf{w})$ if all terms of f have the same weighted degree d with respect to $\mathbf{w}$, i.e., $f = \sum_{|\alpha|_{\mathbf{w}} = d} c_\alpha x^\alpha$ where $c_\alpha \in K$.

(ii) A polynomial f is called semi-weighted homogeneous (or semi-quasihomogeneous) of type $(d; \mathbf{w})$ if f is of the form $f = f_0 + g$ where f_0 is a weighted homogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at the origin, $f = f_0$ or $\mathrm{ord}_{\mathbf{w}}(f - f_0) > d$.

DEFINITION 2.2 (weighted term orders). For two multi-indices $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_n)$ in $\mathbf{N}^n$, we write $\xi^{\lambda'} \prec \xi^{\lambda}$ or $\lambda' \prec \lambda$ if $|\lambda'|_{\mathbf{w}} < |\lambda|_{\mathbf{w}}$, or if $|\lambda'|_{\mathbf{w}} = |\lambda|_{\mathbf{w}}$ and there exists $j \in \mathbf{N}$ so that $\lambda'_i = \lambda_i$ for $i < j$ and $\lambda'_j < \lambda_j$.

DEFINITION 2.3 (inverse orders). Let $\prec$ be a local or global term order. Then, the **inverse order** $\prec^{-1}$ of $\prec$ is defined by $x^\alpha \prec x^\beta \Leftrightarrow x^\beta \prec^{-1} x^\alpha$ where $\alpha, \beta \in \mathbf{N}^n$.

Note that if $\prec$ is a global term order (1 is the minimal term), then $\prec^{-1}$ is the local term order (1 is the maximal term). Conversely, if $\prec$ is a local term order, then $\prec^{-1}$ is the global term order.

DEFINITION 2.4 (minimal bases). A basis $\{x^{\gamma_1}, \dots, x^{\gamma_l}\}$ for a monomial ideal I is said to be **minimal** if no x^{γ_i} in the basis divides other x^{γ_j} for $i \neq j$, where $\gamma_1, \dots, \gamma_l \in \mathbf{N}^n$.

3. Algorithms for computing algebraic local cohomology classes

In this section we describe algorithms for computing algebraic local cohomology classes associated to a polar variety, and give results of the benchmark tests.

3.1. Algebraic local cohomology

Here we briefly review algebraic local cohomology classes, and give notation and definitions that will be used in this paper. The details are in [17, 18, 34, 43, 44, 45].

Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin O in $\mathbf{C}^n$, where X is an open neighborhood of the origin O and f is a holomorphic defining function. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions, $\mathcal{O}_{X,O}$ the stalk at the origin of the sheaf $\mathcal{O}_X$. Let $\mathcal{H}^n_{\{O\}}(\mathcal{O}_X)$ be the local cohomology supported at O. Consider the pair $(X, X - O)$ and its relative Čech covering. Then, any section of $\mathcal{H}^n_{\{O\}}(\mathcal{O}_X)$ can be represented as an element of relative Čech cohomology. All local cohomology classes we handle in this paper are actually algebraic local cohomology classes that belong to the set defined by

$$H^n_{[O]}(K[x]) := \lim_{k \to \infty} \operatorname{Ext}^n_{K[x]}(K[x]/\langle x_1, x_2, \dots, x_n \rangle^k, K[x]),$$

where $\langle x_1, \dots, x_n \rangle$ is the maximal ideal generated by $x_1, \dots, x_n$. We identify $H^n_{[O]}(K[x])$ with $K[\xi_1, \dots, \xi_n]$. An algebraic local cohomology class $\sum c_\lambda \begin{bmatrix} 1 \\ x^{\lambda+1} \end{bmatrix}$ is represented as a polynomial in n variables $\sum c_\lambda \xi^\lambda$ where $x^{\lambda+1} = x_1^{\lambda_1+1} x_2^{\lambda_2+1} \cdots x_n^{\lambda_n+1}$, $c_\lambda \in K$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{N}^n$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n)$. The multiplication by x^α is defined as

$$x^\alpha * \xi^\lambda = \begin{cases} \xi^{\lambda-\alpha}, & \lambda_i \geq \alpha_i,\ i = 1, \dots, n, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{N}^n$, and $\lambda - \alpha = (\lambda_1 - \alpha_1, \dots, \lambda_n - \alpha_n)$.

Let fix a global term order $\prec$ on $K[\xi]$. For a given algebraic local cohomology class of the form

$$\psi = c_\lambda \xi^\lambda + \sum_{\xi^{\lambda'} \prec \xi^\lambda} c_{\lambda'} \xi^{\lambda'}, \quad c_\lambda \neq 0,$$

we call ξ^λ the **head term**, c_λ the **head coefficient**, $c_\lambda \xi^\lambda$ the **head monomial** and $\xi^{\lambda'}$ the **lower terms**. Let $\mathrm{ht}(\psi)$, $\mathrm{hc}(\psi)$ and $\mathrm{hm}(\psi)$ denote the head term, the head coefficient and the head monomial respectively. Furthermore, let $\mathrm{Term}(\psi) := \{\xi^\kappa \mid \psi = \sum_{\kappa \in \mathbf{N}^n} c_\kappa \xi^\kappa, c_\kappa \neq 0, c_\kappa \in K\}$, the set of terms of ψ, $\mathrm{Coef}(\psi) := \{c_\kappa \mid \psi = \sum_{\kappa \in \mathbf{N}^n} c_\kappa \xi^\kappa, c_\kappa \neq 0, c_\kappa \in K\}$, the set of coefficients of ψ and let $\mathrm{LL}(\psi) := \{\xi^\kappa \in \mathrm{Term}(\psi) \mid \xi^\kappa \neq \mathrm{ht}(\psi)\}$, the set of lower terms of ψ.

Let Ψ be a finite subset of $H^n_{[O]}(K[x])$. Set $\mathrm{ht}(\Psi) := \{\mathrm{ht}(\psi) \mid \psi \in \Psi\}$, $\mathrm{Term}(\Psi) := \bigcup_{\psi \in \Psi} \mathrm{Term}(\psi)$, $\mathrm{Coef}(\Psi) := \bigcup_{\psi \in \Psi} \mathrm{Coef}(\psi)$ and $\mathrm{LL}(\Psi) := \bigcup_{\psi \in \Psi} \mathrm{LL}(\psi)$. Moreover, let $\mathrm{ML}(\Psi)$ denote the set of monomial elements of Ψ, $\mathrm{SL}(\Psi)$ the set of linear combination elements of Ψ. For instance, let $\Psi = \{2\xi_1^2\xi_2 - 3\xi_1^2 + \xi_2, \xi_1\xi_2^2 + \xi_1, \xi_1^3\xi_2^2, \xi_1\xi_2\}$ in $\mathbf{C}[\xi_1, \xi_2]$, then $\mathrm{ML}(\Psi) = \{\xi_1^3\xi_2^2, \xi_1\xi_2\}$ and $\mathrm{SL} = \{2\xi_1^2\xi_2 - 3\xi_1^2 + \xi_2, \xi_1\xi_2^2 + \xi_1\}$.

Let ξ^λ be a term and let Φ be a set of terms in $K[\xi]$ where $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{N}^n$. We call $\xi^\lambda \cdot \xi_i$ a **neighbor** of ξ^λ for each $i = 1, \dots, n$. We define the neighbor of Φ as $\mathbf{Neighbor}(\Phi) := \{\varphi \cdot \xi_i \mid \varphi \in \Phi, i = 1, \dots, n\}$.

Note that for a polynomial and a set of polynomials in $K[x]$, we use the same notation as above, too.

Definition 3.1 (changing variables). Let G be a set of polynomials in $K[x]$ and $g \in G$. A map $\mathscr{CV}$ is defined as changing variables x_i into ξ_i, for all $i \in \{1, \dots, n\}$. The inverse map $\mathscr{CV}^{-1}$ is defined as changing variables ξ_i into x_i.

That is, $\mathscr{CV}(g)$ is in $K[\xi]$. The set $\mathscr{CV}(G)$ is also defined as $\mathscr{CV}(G) = \{\mathscr{CV}(g) \mid g \in G\}$.

For instance, $f = -2x_1^3x_2 + 2/5x_1^2 + 3 \in \mathbf{Q}[x_1, x_2]$ and $\psi = 3/2\xi_1^2 - 2\xi_2 + 2\xi_3 \in \mathbf{Q}[\xi_1, \xi_2, \xi_3]$. Then, $\mathscr{CV}(f) = -2\xi_1^3\xi_2 + 2/5\xi_1^2 + 3$ and $\mathscr{CV}^{-1}(\psi) = 3/2x_1^2 - 2x_2 + 2x_3$.

3.2. Algorithms for computing algebraic local cohomology classes

Here we illustrate an algorithm for computing a basis of the vector space $H_{\Gamma(f)}$ associated to a polar variety $\Gamma(f)$ of a hypersurface S.

Let $f = f_0 + g$ be a semi-weighted homogeneous polynomial of type $(d; \mathbf{w})$ in $K[x]$, where f_0 is a weighted homogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at the origin, and $\mathbf{w}$ is a weight vector. Let $\Gamma(f)$ be a polar variety [24, 46, 47, 48] of the hypersurface S defined to be

$$\Gamma(f) = \left\{ x \in X \,\middle|\, \frac{\partial f}{\partial x_2}(x) = \frac{\partial f}{\partial x_3}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0 \right\}.$$

Set

$$\begin{aligned} H_{\Gamma(f)} = \Bigg\{ \psi \in H^n_{[O]}(K[x]) \mid f * \psi &= \left(\frac{\partial f}{\partial x_2}\right) * \psi \\ &= \left(\frac{\partial f}{\partial x_3}\right) * \psi = \cdots = \left(\frac{\partial f}{\partial x_n}\right) * \psi = 0 \Bigg\}. \end{aligned}$$

Here, the system of coordinates $(x_1, x_2, \ldots, x_n)$ is assumed to be *generic* in a sense that $H_{\Gamma(f)}$ is a finite dimensional subspace of $H^n_{[O]}(K[x])$. That is, $\left\{ f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\}$ has an isolated common root at the origin.

Remark 1. Let $I = \left\langle f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$ and $\mathfrak{m} = \langle x_1, x_2, \ldots, x_n \rangle$ in $K[x]$. Since $\mathbf{V}(I : \mathfrak{m}^\infty) = \overline{\mathbf{V}(I) \backslash \{O\}}$, if there exists p in the ideal quotient $I : \mathfrak{m}^\infty$ sucht that $p(O) \neq 0$, then $f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$ has an isolated common root at the origin. Hence, by computing a Gröbner basis of $I : \mathfrak{m}^\infty$ in $K[x]$, one can know whether $f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$ has the isolated common root at the origin or not.

The aim of this subsection is to give an efficient algorithm for computing a basis of the vector space $H_{\Gamma(f)}$. First we present an algorithm for computing a basis of $H_{\Gamma(f_0)}$. Second, we design an algorithm for computing a basis of $H_{\Gamma(f)}$ by using the basis of $H_{\Gamma(f_0)}$. The essential point of the proposed algorithm is a use of Poincaré polynomials.

Now we recall the notion of Poincaré polynomial for the ideal $\left\langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$ [7].

Definition 3.2. Let $f = f_0 + g$ be a semi-weighted homogeneous polynomial of type $(d; \mathbf{w})$. Then, the **Poincaré polynomial** of the ideal $\left\langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$ is defined by

$$P_{\Gamma(f)}(s) = \frac{(s^d - 1)(s^{d-w_2} - 1)(s^{d-w_3} - 1) \cdots (s^{d-w_n} - 1)}{(s^{w_1} - 1)(s^{w_2} - 1)(s^{w_3} - 1) \cdots (s^{w_n} - 1)}.$$

Let $P_{\Gamma(f)}(s) = \sum_{i=1}^{p} m_i s^{d_i}$ be the Poincaré polynomial of the ideal $\left\langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$. We introduce the multiset $D_{P_{\Gamma(f)}}$ of weighted degrees as

$$D_{P_{\Gamma(f)}} = \bigcup_{i=1}^{p} \{\underbrace{d_i, d_i, \dots, d_i}_{m_i \text{ elements}}\}.$$

Notice that $D_{P_{\Gamma(f_0)}} = D_{P_{\Gamma(f)}}$.

The following two results are essentially same as our previous results presented in [30, 34].

Proposition 3.3. *Using the same notation as above, there exists a basis Ψ_0 of $H_{\Gamma(f_0)}$ that satisfies the following conditions*

(i) Ψ_0 *consists of weighted homogeneous polynomials.*

(ii) $D_{P_{\Gamma(f_0)}} = \{\deg_{\mathbf{w}}(\psi) \mid \psi \in \Psi_0\}$.

As f_0 is a weighted homogeneous polynomial of type $(d; \mathbf{w})$, the multiset of weighted degrees of elements of a basis of $H_{\Gamma(f_0)}$ equal to the multiset $D_{P_{\Gamma(f_0)}}$.

The next two lemmas [30, 45] are needed to construct the algorithm.

Lemma 3.4. *Let T be the minimal basis of $\left\langle \mathrm{Term}\left(\left\{ f, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\}\right) \right\rangle$ in $K[x]$ and let M be the set of standard monomials of $\langle T \rangle$. Then, for all $\xi^\lambda \in \mathscr{CV}(M)$,*

$$f * \xi^\lambda = \left(\frac{\partial f}{\partial x_2}\right) * \xi^\lambda = \cdots = \left(\frac{\partial f}{\partial x_n}\right) * \xi^\lambda = 0.$$

Let $\mathrm{MB}(H_{\Gamma(f)})$ denote the set $\mathscr{CV}(M)$.

All monic monomial elements of a basis of the vector space $H_{\Gamma(f_0)}$ can be also obtained from the minimal basis of $\left\langle \mathrm{Term}\left(\left\{f_0, \frac{\partial f_0}{\partial x_2}, \dots, \frac{\partial f_0}{\partial x_n}\right\}\right)\right\rangle$.

Let Λ_{H_0} denote the set of exponents of head terms in $H_{\Gamma(f_0)}$ and let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{N}^n$. Let $\Lambda_{H_0}^{(\lambda)} = \{\lambda' \in \Lambda_{H_0} \mid \lambda' \prec \lambda\}$.

The following lemma tells us a condition of head terms of $H_{\Gamma(f_0)}$.

LEMMA 3.5. *If $\lambda \in \Lambda_{H_0}$, then, for each $j = 1, 2, \dots, n$, $(\lambda_1, \dots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \dots, \lambda_n)$ is in $\Lambda_{H_0}^{(\lambda)}$ provided $\lambda_j \geq 1$.*

The property above, denoted by (C), will be used in Algorithm 1 as a condition to select candidates of head terms. Proposition 3.3 together with Lemma 3.4 and Lemma 3.5, allows us to design an algorithm to compute a basis of $H_{\Gamma(f_0)}$.

As the set D is finite, the termination is obvious. The correctness follows from Proposition 3.3 together with Lemma 3.4 and Lemma 3.5.

We illustrate Algorithm 1 with the following example.

EXAMPLE 1. A polynomial $f_0 = x_1^4 x_2 + x_2^4 \in \mathbf{C}[x_1, x_2]$ (W_{13} singularity) is a weighted homogeneous polynomial of type $(16; (3, 4))$ and defines an isolated singularity at the origin of $\mathbf{C}^2$. As the sequence $\left(f_0, \frac{\partial f_0}{\partial x_2}\right)$ is a regular sequence, we are able to apply Algorithm 1 for computing a basis of $H_{\Gamma(f_0)}$. The variables ξ_1, ξ_2 correspond x_1, x_2. Let $\prec$ be the weighted term order s.t. $\xi_2 \prec \xi_1$.

As the Poincaré polynomial of the ideal $\left\langle f_0, \frac{\partial f_0}{\partial x_2}\right\rangle$ is

$$
\begin{aligned}
P_{\Gamma(f_0)}(s) &= \frac{(s^{16}-1)(s^{16-4}-1)}{(s-3)(s^2-4)} \\
&= s^{21} + s^{18} + s^{17} + s^{15} + s^{14} + s^{13} + s^{12} + s^{11} \\
&\quad + s^{10} + s^9 + s^8 + s^7 + s^6 + s^4 + s^3 + 1,
\end{aligned}
$$

we obtain $D_{\Gamma(F_0)} = \{0, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 21\}$. Next we compute the set M_0 of monic monomial elements of $H_{\Gamma(f_0)}$. By Lemma 3.4,

$$M_0 = \{1, \xi_1, \xi_1^2, \xi_1^3, \xi_2, \xi_1\xi_2, \xi_1^2\xi_2, \xi_1^3\xi_2, \xi_2^2, \xi_1\xi_2^2, \xi_1^2\xi_2^2, \xi_1^3\xi_2^2\}.$$

$D = D_{\Gamma(f_0)} \backslash \deg_{\mathbf{w}}(M_0) = \{12, 15, 18, 21\}$. (See Figure 1 and Figure 2). Set $\Psi_0 := M_0$.

The minimum number in D is 12. D is renewed as $D \backslash \{12\} = \{15, 18, 21\}$. Select terms whose weighted degree is 12. Then, from Figure 2, $L = \{\xi_1^4, \xi_2^3\}$. Since $\xi_2^3 \prec \xi_1^4$ and ξ_2^3 satisfies the condition (C), set $\psi = \xi_1^4 + c_{(0,3)}\xi_2^3$ where $c_{(0,3)}$

Algorithm 1. Coho_Weighted

Specification: Coho_Weighted($f_0, \prec$)
Computing a basis of the vector space $H_{\Gamma(f_0)}$.
Input: f_0 : a weighted homogeneous polynomial of type $(d; \mathbf{w})$ with an isolated singularity at the origin. $\prec$: a weighted term order.
Output: Ψ_0 : a basis of the vector space $H_{\Gamma(f_0)}$.

```
BEGIN
D_{P_{Γ(f0)}} ← Compute D_{P_{Γ(f0)}} from the Poincaré polynomial of type (d; w);
M ← Compute all monic monomial elements of a basis of H_{Γ(f0)} according to Lemma 3.4;
D ← D_{P_{Γ(f0)}} \ deg_w(M); Ψ0 ← M;
while D ≠ ∅ do
k ← Select the minimum number from D; D ← D\{k};
L ← {ξ^λ | deg_w(ξ^λ) = k, ξ^λ ∉ ht(Ψ0)};
L' ← Select the 1st and 2nd smallest elements from L w.r.t. ≺;
L ← L\L';
Flag ← 0;
  while Flag ≠ 1 do
   ξ^λ ← Select the greatest element from L' w.r.t. ≺;
    if λ satisfies the condition (C) then
     ψ ← ξ^λ + Σ_{λ'∈L'\{ξ^λ}, ξ^λ' ≺ ξ^λ} c_λ' ξ^λ' (where c_λ' is an undetermined coefficient)
     F ← { f * ψ, (∂f/∂x2) * ψ, ..., (∂f/∂xn) * ψ };
     E ← {b = 0 | b ∈ Coef(F)};
     A ← Solve the system E of linear equations;
      if E has a solution then
       ψ' ← Substitute A into ψ;
       Ψ0 ← Ψ0 ∪ {ψ'};
       Flag ← 1;
      end-if
    end -if
   ξ^κ ← Select the smallest element in L;
   L ← L\{ξ^κ};
   L' ← L' ∪ {ξ^κ}
  end-while
end-while
return Ψ0;
END
```

is an undetermined coefficient. From $f_0 * \psi = 0$, $\left(\frac{\partial f_0}{\partial x_2}\right) * \psi = 1 + 4c_{(0,3)} = 0$, we have $c_{(0,3)} = -1/4$. Hence, $\xi_1^4 - 1/4\xi_2^3$ is a member of the basis. Ψ_0 is renewed as $\Psi_0 \cup \{\xi_1^4 - 1/4\xi_2^3\}$.

The minimum number in D is 15. D is renewed as $D\backslash\{15\} = \{18, 21\}$. Select terms whose weighted degree is 15. Then, from Figure 2, $L = \{\xi_1^5, \xi_1\xi_2^3\}$. Since $\xi_1\xi_2^3 \prec \xi_1^5$ and $\xi_1\xi_2^3$ satisfies the condition (C), set $\psi = \xi_1^5 + c_{(1,3)}\xi_1\xi_2^3$ where $c_{(1,3)}$ is an undetermined coefficient. From $f_0 * \psi = 0$, $\left(\frac{\partial f_0}{\partial x_2}\right) * \psi = (1 + 4c_{(1,3)})\xi_1 = 0$, we have $c_{(1,3)} = -1/4$. Thus, $\xi_1^5 - 1/4\xi_1\xi_2^3$ is a member of the basis. Ψ_0 is renewed as $\Psi_0 \cup \{\xi_1^5 - 1/4\xi_1\xi_2^3\}$.

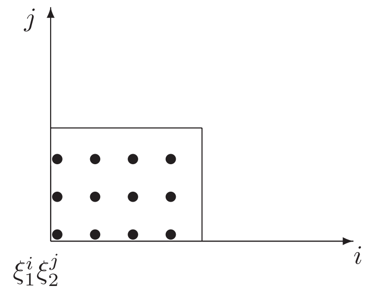

Figure 1. Monic monomial elements.

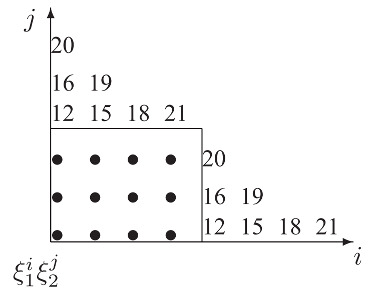

Figure 2. Weighted degrees.

The minimum number in D is 18. D is renewed as $D \backslash \{18\} = \{21\}$. Select terms whose weighted degree is 18. Then, from Figure 2, $L = \{\xi_1^6, \xi_1^2 \xi_2^3\}$. Since $\xi_1^2 \xi_2^3 \prec \xi_1^6$ and $\xi_1^2 \xi_2^3$ satisfies the condition (C), set $\psi = \xi_1^6 + c_{(2,3)} \xi_1^2 \xi_2^3$ where $c_{(2,3)}$ is an undetermined coefficient. From $f_0 * \psi = 0$, $\left(\frac{\partial f_0}{\partial x_2}\right) * \psi = (1 + 4c_{(2,3)}) \xi_1^2 = 0$, we have $c_{(2,3)} = -1/4$, and $\xi_1^6 - 1/4 \xi_1^2 \xi_2^3$ is a member of the basis. Ψ_0 is renewed as $\Psi_0 \cup \{\xi_1^6 - 1/4 \xi_1^2 \xi_2^3\}$.

The minimum number in D is now 21. D is renewed as $D \backslash \{21\} = \varnothing$. We Select terms whose weighted degree is 21. Then, from Figure 2, $L = \{\xi_1^7, \xi_1^3 \xi_2^3\}$. Since $\xi_1^3 \xi_2^3 \prec \xi_1^7$ and $\xi_1^3 \xi_2^3$ satisfies the condition (C), set $\psi = \xi_1^7 + c_{(3,3)} \xi_1^3 \xi_2^3$. From $f_0 * \psi = 0$, $\left(\frac{\partial f_0}{\partial x_2}\right) * \psi = (1 + 4c_{(3,3)}) \xi_1^3 = 0$, we have $c_{(3,3)} = -1/4$. Thus, $\xi_1^7 - 1/4 \xi_1^3 \xi_2^3$ is a member of the basis. Ψ_0 is renewed as $\Psi_0 \cup \{\xi_1^7 - 1/4 \xi_1^3 \xi_2^3\}$.

Therefore,

$$\Psi_0 = M_0 \cup \{\xi_1^4 - 1/4 \xi_2^3, \xi_1^5 - 1/4 \xi_1 \xi_2^3, \xi_1^6 - 1/4 \xi_1^2 \xi_2^3, \xi_1^7 - 1/4 \xi_1^3 \xi_2^3\}$$

is a basis of $H_{\Gamma(f_0)}$.

The following theorem that follows immediately from Proposition 3.2 of [34], shows the relations between a basis of $H_{\Gamma(f_0)}$ and that of $H_{\Gamma(f)}$.

THEOREM 3.6. *Let $\Psi_0 = \{\rho_1, \dots, \rho_{r_0}\}$ be a basis of the vector space $H_{\Gamma(f_0)}$ that satisfies properties given in Proposition* 3.3. *Then, there exists a basis $\Psi = \{\psi_1, \dots, \psi_{r_0}\}$ of the vector space $H_{\Gamma(f)}$ s.t.*

(i) $\psi_i = \rho_i + v_i,\ i = 1, \dots, r_0,$

(ii) $\deg_{\mathbf{w}}(\rho_i) > \deg_{\mathbf{w}}(v_i).$

The theorem says that, in semi-weighted case, the weighted degree of the basis of $H_{\Gamma(f)}$ is completely determined by the Poincaré polynomial $P_{\Gamma(f)}(s)$ associated to the ideal $\left\langle f, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$.

Theorem 3.6 together with Lemma 3.4 allows us to design an efficient algorithm to compute a basis of $H_{\Gamma(f)}$.

Algorithm 2. **Coho_SemiW**

Specification: Coho_SemiW($f, \prec$)
Computing a basis of the vector space $H_{\Gamma(f)}$.
Input: $f = f_0 + g$: a semi-weighted homogeneous polynomial of type $(d; \mathbf{w})$ where f_0 is a weighted-homogeneous polynomial of type $(d; \mathbf{w})$; $\prec$: a weighted term order.
Output: Ψ : a basis of the vector space $H_{\Gamma(f)}$.
BEGIN
$\Psi_0 \leftarrow$ **Coho_Weighted**$(f_0, \prec)$;
$M \leftarrow$ Compute all monic monomial elements of a basis of $H_{\Gamma(f)}$ according to Lemma 3.4;
$T \leftarrow \Psi_0 \backslash M$; $\Psi \leftarrow M$;
while $T \neq \emptyset$ **do**
 $\rho \leftarrow$ Select an element whose head term is the smallest in $\mathrm{ht}(T)$ w.r.t. $\prec$, from T;
 $T \leftarrow T \backslash \{\rho\}$;
 if $\left(\forall i \in \{2, 3, \dots, n\}, \left(\frac{\partial f}{\partial x_i}\right) * \rho = 0\right) \wedge (f * \rho = 0)$ **then**
  $\Psi \leftarrow \Psi \cup \{\rho\}$;
 else
  $L \leftarrow \{\xi^\lambda \mid \deg_{\mathbf{w}}(\xi^\lambda) < \deg_{\mathbf{w}}(\rho), \xi^\lambda \notin \mathrm{ht}(\Psi)\}$; $(\diamondsuit)$
  $\psi \leftarrow \rho + \sum_{\xi^\lambda \in L} c_\lambda \xi^\lambda$;
  $F \leftarrow \left\{ f * \psi, \left(\frac{\partial f}{\partial x_2}\right) * \psi, \dots, \left(\frac{\partial f}{\partial x_n}\right) * \psi \right\}$;
  $E \leftarrow \{b = 0 \mid b \in \mathrm{Coef}(F)\}$;
  $A \leftarrow$ Solve the system E of linear equations;
  $\psi' \leftarrow$ Substitute A into ψ;
  $\Psi \leftarrow \Psi \cup \{\psi'\}$;
 end-if
end-while
return Ψ;
END

REMARK 2. In [33] the conditions of lower monomials are introduced. It is possible to improve Algorithm 2 by utilizing the conditions at ($\diamondsuit$). In fact, our implementations contain these optimizations.

As the algorithm **Coho_Weighted** terminates, Algorithm 2 also terminates. The correctness is also guaranteed by the algorithm **Coho_SemiW** and Theorem 3.6.

We illustrate Algorithm 2 with the following example.

EXAMPLE 2. A polynomial $f = f_0 + x_1^6 \in \mathbf{C}[x_1, x_2]$ (W_{13} singularity) is a semi-weighted homogeneous polynomial of type $(16; (3, 4))$ where $f_0 = x_1^4 x_2 + x_2^4$. From Example 1, $\Psi_0 = M_0 \cup \{\xi_1^4 - 1/4\xi_2^3, \xi_1^5 - 1/4\xi_1\xi_2^3, \xi_1^6 - 1/4\xi_1^2\xi_2^3, \xi_1^7 - 1/4\xi_1^3\xi_2^3\}$. We compute the set M of monic monomial elements of $H_{\Gamma(f)}$. By Lemma 3.4, $M = M_0$. Set $T = \Psi_0 \backslash M = \{\xi_1^4 - 1/4\xi_2^3, \xi_1^5 - 1/4\xi_1\xi_2^3, \xi_1^6 - 1/4\xi_1^2\xi_2^3, \xi_1^7 - 1/4\xi_1^3\xi_2^3\}$ and $\Psi = M$.

Take the element whose head term is the smallest, w.r.t. $\prec$, in $\mathrm{ht}(T)$, that is $\xi_1^4 - 1/4\xi_2^3$. Set $\rho = \xi_1^4 - 1/4\xi_2^3$ and T is renewed as $T \backslash \{\xi_1^4 - 1/4\xi_2^3\}$. Since ρ satisfies $f * \rho = \left(\frac{\partial f}{\partial x_2}\right) * \rho = 0$, $\xi_1^4 - 1/4\xi_2^3$ is a member of the basis. Ψ is renewed as $\Psi \cup \{\xi_1^4 - 1/4\xi_2^3\}$.

Take the element whose head term is the smallest, w.r.t. $\prec$, in $\mathrm{ht}(T)$, that is $\xi_1^5 - 1/4\xi_1\xi_2^3$. Set $\rho = \xi_1^5 - 1/4\xi_1\xi_2^3$ and T is renewed as $T \backslash \{\xi_1^5 - 1/4\xi_1\xi_2^3\}$. Since ρ satisfies $f * \rho = \left(\frac{\partial f}{\partial x_2}\right) * \rho = 0$, $\xi_1^5 - 1/4\xi_1\xi_2^3$ is a member of the basis. Ψ is renewed as $\Psi \cup \{\xi_1^5 - 1/4\xi_1\xi_2^3\}$.

Take the element whose head term is the smallest, w.r.t. $\prec$, in $\mathrm{ht}(T)$, that is $\xi_1^6 - 1/4\xi_1^2\xi_2^3$. Set $\rho = \xi_1^6 - 1/4\xi_1^2\xi_2^3$ and T is renewed as $T \backslash \{\xi_1^6 - 1/4\xi_1^2\xi_2^3\}$. Then, as $f * \rho = 1 \neq 0$ and $\left(\frac{\partial f}{\partial x_2}\right) * \rho = 0$, we have to decide additional lower terms of ρ. $L = \{\xi^\lambda \mid \deg_{\mathbf{w}}(\xi^\lambda) < 18, \xi^\lambda \notin \mathrm{ht}(\Psi)\} = \{\xi_2^3, \xi_1\xi_2^3, \xi_1^4\xi_2, \xi_2^4\}$. Set $\psi = \rho + c_{(0,3)}\xi_2^3 + c_{(1,3)}\xi_1\xi_2^3 + c_{(4,1)}\xi_1^4\xi_2 + c_{(0,4)}\xi_2^4$ and solve $\left[f * \psi = c_{(4,1)} + c_{(0,1)} + 1 = 0, \left(\frac{\partial f}{\partial x_2}\right) * \psi = (c_{(4,1)} + 4c_{(0,4)})\xi_2 + 4c_{(0,3)} + 4c_{(1,3)}\xi_1 = 0\right]$. Then, we obtain $c_{(0,3)} = c_{(1,3)} = 0$, $c_{(4,1)} = -4/3$, $c_{(0,4)} = 1/3$. Hence, $\xi_1^6 - 1/4\xi_1^2\xi_2^3 - 4/3\xi_1^4\xi_2 + 1/3\xi_2^4$ is a member of the basis. Ψ is renewed as $\Psi \cup \{\xi_1^6 - 1/4\xi_1^2\xi_2^3 - 4/3\xi_1^4\xi_2 + 1/3\xi_2^4\}$.

Set $\rho = \xi_1^7 - 1/4\xi_1^3\xi_2^3$. Then, as $f * \rho = \xi_1 \neq 0$ and $\left(\frac{\partial f}{\partial x_2}\right) * \rho = 0$, we have to decide additional lower terms of ρ. $L = \{\xi^\lambda \mid \deg_{\mathbf{w}}(\xi^\lambda) < 21, \xi^\lambda \notin \mathrm{ht}(\Psi)\} = \{\xi_2^3, \xi_1\xi_2^3, \xi_1^4\xi_2, \xi_2^4, \xi_1\xi_2^3, \xi_2^5, \xi_1^5\xi_2, \xi_1\xi_2^4, \xi_1\xi_2^2\}$. Set $\psi = \rho + c_{(0,3)}\xi_2^3 + c_{(1,3)}\xi_1\xi_2^3 + c_{(4,1)}\xi_1^4\xi_2 + c_{(0,4)}\xi_2^4 + c_{(1,3)}\xi_1\xi_2^3 + c_{(0,5)}\xi_2^5 + c_{(5,1)}\xi_1^5\xi_2 + c_{(1,4)}\xi_1\xi_2^4 + c_{(4,2)}\xi_1^4\xi_2^2$ and solve $\left[f * \psi = 0, \left(\frac{\partial f}{\partial x_2}\right) * \psi = 0\right]$. Then, we obtain $\xi_1^7 - 1/4\xi_1^3\xi_2^3 + 1/3\xi_1\xi_2^4 -$

$4/3\xi_1^5\xi_2$ as a member of the basis. Ψ is renewed as $\Psi \cup \{\xi_1^7 - 1/4\xi_1^3\xi_2^3 + 1/3\xi_1\xi_2^4 - 4/3\xi_1^5\xi_2\}$.

Therefore, $\Psi = M \cup \{\xi_1^4 - 1/4\xi_2^3, \xi_1^5 - 1/4\xi_1\xi_2^3, \xi_1^6 - 1/4\xi_1^2\xi_2^3 - 4/3\xi_1^4\xi_2 + 1/3\xi_2^4, \xi_1^7 - 1/4\xi_1^3\xi_2^3 + 1/3\xi_1\xi_2^4 - 4/3\xi_1^5\xi_2\}$ is a basis of $H_{\Gamma(f)}$.

In our previous paper [45], an algorithm has been introduced for computing a basis of the vector space H_F of local cohomology classes in $H^n_{[O]}(K[x])$ annihilated by the zero-dimensional ideal $\langle f_1, \dots, f_s\rangle$ generated by $F = \{f_1, \dots, f_s\} \subset K[x]$. The algorithm mentioned above can also compute a basis of $H_{\Gamma(F)}$ by giving $\left\{f, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right\}$ as an input data.

The algorithm **Coho_SemiW** has been implemented in the computer algebra system Risa/Asir. Here we give results of the benchmark tests. Table 1 shows a comparison of the implementation of **Coho_SemiW** with our previous Risa/Asir implementation [45] (Prev. alg.) in computation time (CPU time). x_1, x_2, x_3 are variables. The time is given in second. (The term order is the total degree lexicographic term order s.t. $\xi_3 \prec \xi_2 \prec \xi_1$.) $\mu(f)$ is the Milnor number of f at the origin. $\tau(f)$ is the Tjurina number of f at the origin. Note that in Prob. 5, $(x_1^4 + x_2^9)^4 + 3x_1^{16}$ is a weighted homogeneous polynomial.

As is evident from Table 1, the algorithm **Coho_SemiW** results in better performance compare to our previous algorithm. In semi-weighted cases, as a Poincaré polynomial tells us candidates of head terms and a number of elements of a basis of $H_{\Gamma(f)}$, the computation cost of selecting candidates of head terms and lower terms, becomes smaller than that of our previous algorithm.

Prob.	Semi-weighted homogeneous polynomial f	$\mu(f)$	$\tau(f)$	Prev. alg.	**Coho_SemiW**
1	$(x_1^4 + x_2^6 + x_1^2x_2^3)^2 + x_1^2x_2^9 + 2x_1^9$	77	67	0.8424	0.0312
2	$(x_1^5 + x_2^7)^2 + 3x_2^{14} + x_1^{10}x_2^5 + 3x_1x_2^{14}$	117	99	1.42	0.234
3	$(x_1^3 + x_2^{13})^2 + x_1^6 - 5x_1^3x_2^{20}$	125	115	2.278	0.2496
4	$(x_1^4 + x_2^6 + x_1^2x_2^3)^3 + x_1^8x_2^6 + 3x_1^{11}x_2^2$	163	137	29.8	1.716
5	$(x_1^4 + x_2^9)^4 + 3x_1^{16}$	525	525	393.8	0.8112
6	$(x_1^4 + x_2^9)^4 + 3x_1^{16} + 4x_1^{15}x_2^3$	525	439	1132	36.84
7	$(x_1^3x_2 + x_2^7 + x_1^2x_2^3)^4 + x_1^{14} + 3x_1^{13}x_2^3$	351	293	816.1	46.32
8	$(x_1^3 + x_1x_3^3 + x_2^4)^2 + x_2^8 + x_3^9 + x_1x_2^7$	280	221	3760	75.08

Table 1. Comparison of the algorithm [45] and **Coho_SemiW**.

3.3. Parametric local cohomology systems

We turn to the parametric cases. Let $f = f_0 + g$ be a semi-weighted homogeneous polynomial of type $(d; \mathbf{w})$ with **parameters** $t = (t_1, \dots, t_m) \in \bar{K}^m$, where f_0 is the weighted homogeneous part and $\bar{K}$ is an algebraic closure of K. We assume that for generic values of the parameters t, f_0 has an isolated singularity at the origin.

In order to treat the parametric cases, we require the following notation and definitions. Let $t = (t_1, t_2, \dots, t_m)$ denote parameters in $\bar{K}^m$. For $q_1, \dots, q_r \in K[t]$, $\mathbf{V}(q_1, \dots, q_r) \subseteq \bar{K}^m$ denotes the affine variety of $q_1, \dots, q_r$, i.e., $\mathbf{V}(q_1, \dots, q_r) := \{\bar{a} \in \bar{K}^m \mid q_1(\bar{a}) = \cdots = q_r(\bar{a}) = 0\}$ and $\mathbf{V}(0) := \bar{K}^m$. We call an algebraically constructible set of the form $\mathbf{V}(q_1, \dots, q_r) \backslash \mathbf{V}(q'_1, \dots, q'_{s'}) \subseteq \bar{K}^m$ with $g_1, \dots, q_r$, $q'_1, \dots, q'_{s'} \in K[t]$, a **stratum**. (Notation $\mathbf{A}, \mathbf{A}', \mathbf{A}'', \mathbf{A}_1, \dots, \mathbf{A}_l, \mathbf{B}_1, \dots, \mathbf{B}_k$ are frequently used to represent strata.)

We define the localization of $K[t]$ w.r.t. the stratum $\mathbf{A} \subseteq \bar{K}^m$ as follows: $K[t]_{\mathbf{A}} = \{c/b \mid c, b \in K[t], b(t) \neq 0 \text{ for } t \in \mathbf{A}\}$. Then for every $\bar{a} \in \mathbf{A}$, the specialization homomorphism $\sigma_{\bar{a}} : K[t]_{\mathbf{A}}[x] \to \bar{K}[x]$ ($\sigma_{\bar{a}} : K[t]_{\mathbf{A}}[\xi] \to \bar{K}[\xi]$ or $\sigma_{\bar{a}} : (K[t]_{\mathbf{A}}[x])^s \to (\bar{K}[x])^s$, $s \in \mathbf{N}_{>0}$) is defined as the map that substitutes $\bar{a}$ into m variables t. When we say that $\sigma_{\bar{a}}(h)$ makes sense for $h \in K(t)[x]$, it has to be understood that $h \in K[t]_{\mathbf{A}}[x]$ for some $\mathbf{A}$ with $\bar{a} \in \mathbf{A}$ and for $F \subset K[t]_{\mathbf{A}}[x]$, $\sigma_{\bar{a}}(F) = \{\sigma_{\bar{a}}(h) \mid h \in F\}$.

In order to treat parametric polynomial systems, we require comprehensive Gröbner systems.

DEFINITION 3.7 (CGS). Let fix a term order. Let F be a subset of $(K[t])[x]$, $\mathbf{A}_1, \dots, \mathbf{A}_\ell$ strata in $\bar{K}^m$ and $G_1, \dots, G_\ell$ subsets of $(K[t])[x]$. A finite set $\mathscr{G} = \{(\mathbf{A}_1, G_1), \dots, (\mathbf{A}_\ell, G_\ell)\}$ of pairs is called a **comprehensive Gröbner system (CGS)** on $\mathbf{A}_1 \cup \cdots \cup \mathbf{A}_\ell$ for $\langle F \rangle$ if $\sigma_a(G_i)$, $a \in \mathbf{A}_i$, is a Gröbner basis of the ideal $\langle \sigma_a(F) \rangle$ in $\bar{K}[x]$ for each $i = 1, \dots, \ell$. We simply say $\mathscr{G}$ is a comprehensive Gröbner system for $\langle F \rangle$ if $\mathbf{A}_1 \cup \cdots \cup \mathbf{A}_\ell = \bar{K}^m$.

There exist several implementations [21, 27, 29] for computing comprehensive Gröbner systems.

As f has parameters, the structure of the vector spaces $H_{\Gamma(f)}$ may change with the values of parameters t. In order to deal with this issue, we introduce now a notion of parametric local cohomology system of $H_{\Gamma(f)}$.

DEFINITION 3.8. Let $\mathbf{A}_i$, $\mathbf{B}_j$ be strata in $\bar{K}^m$ and S_i a subset of $(K[t]_{\mathbf{A}_i})[\xi]$ where $1 \leq i \leq \ell$ and $1 \leq j \leq k$. Set $\mathscr{S} = \{(\mathbf{A}_1, S_1), \dots, (\mathbf{A}_\ell, S_\ell)\}$ and $\mathscr{D} = \{\mathbf{B}_1, \dots,$

$\mathbf{B}_k\}$. Then, a pair $(\mathscr{S}, \mathscr{D})$ is called a **parametric local cohomology system (PLCS)** of $H_{\Gamma(f)}$ on $\mathbf{A}_1 \cup \cdots \cup \mathbf{A}_\ell \cup \mathbf{B}_1 \cup \cdots \cup \mathbf{B}_k$, if for all $i \in \{1, \ldots, l\}$ and $\bar{a} \in \mathbf{A}_i$, $\sigma_{\bar{a}}(S_i)$ is a basis of the vector space $H_{\Gamma(\sigma_{\bar{a}}(f))}$, and for all $j \in \{1, \ldots, k\}$ and $\bar{b} \in \mathbf{B}_j$, $\left\{x \in X \mid \sigma_{\bar{b}}(f)(x) = \sigma_{\bar{b}}\left(\frac{\partial f}{\partial x_2}\right)(x) = \cdots = \sigma_{\bar{b}}\left(\frac{\partial f}{\partial x_n}\right)(x) = 0\right\}$ is not zero-dimensional for any sufficiently small neighborhood X of O, where $H_{\Gamma(\sigma_{\bar{a}}(f))} = \left\{\psi \in H^n_{[O]}(K[x]) \mid \sigma_{\bar{a}}(f) * \psi = \sigma_{\bar{a}}\left(\frac{\partial f}{\partial x_2}\right) * \psi = \cdots = \sigma_{\bar{a}}\left(\frac{\partial f}{\partial x_n}\right) * \psi = 0\right\}$.

In the case where the weighted homogeneous part f_0 contains parameters, there is a possibility that f_0 has non-isolated singularities for some values of the parameters.

Let $J_0 = \left\{\frac{\partial f_0}{\partial x_1}, \ldots, \frac{\partial f_0}{\partial x_n}\right\}$ (or $\Gamma_0 = \left\{f_0, \frac{\partial f_0}{\partial x_2}, \ldots, \frac{\partial f_0}{\partial x_n}\right\}$) and $\mathscr{G} = \{(\mathbf{A}_1, G_1), \ldots, (\mathbf{A}_l, G_\ell)\}$ is a CGS on $\bar{K}^m$ for J_0 (or Γ_0). Since for all $a \in \mathbf{A}_i$, $\sigma_a(G_i)$ is a Gröbner basis, the dimension of J_0 (or Γ_0), on $\mathbf{A}_i$, can be easily computed. Because, as f_0 is weighted homogeneous, $\langle J_0 \rangle$ (or $\langle \Gamma_0 \rangle$) is weighted homogeneous, and thus $\langle J_0 \rangle$ (or Γ_0) is zero dimensional on $\mathbf{A}$ in $K[x]$ if and only if $\langle J_0 \rangle$ is zero dimensional on $\mathbf{A}$ in the ring $\mathscr{O}_{X,O}$ of convergent power series.

As Algorithm 2 consists of only linear algebra computation, by utilizing the Gaussian elimination method with parameter [40], the algorithm can be naturally extended to parametric cases. Here, we give an outline of an algorithm for computing parametric local cohomology systems of $H_{\Gamma(f)}$.

Algorithm 3. Para_SemiW

Specification: Para_SemiW($f, \prec$)
Computing a parametric local cohomology system of $H_{\Gamma(f)}$.
Input: $f = f_0 + g$: a semi-weighted homogeneous polynomial of type $(d; \mathbf{w})$ with parameters where f_0 is a weighted homogeneous polynomial of type $(d; \mathbf{w})$; $\prec$: a weighted term order.
Output: $(\mathscr{S}, \mathscr{D})$: a PLCS of $H_{\Gamma(f)}$.
BEGIN
$\mathscr{D}_1 \leftarrow$ Compute strata on which f_0 has non-isolated singularities;
$\mathscr{D}_2 \leftarrow$ Compute strata on which $\left\langle f_0, \frac{\partial f_0}{\partial x_2}, \ldots, \frac{\partial f_0}{\partial x_n}\right\rangle$ is not of zero dimension;
$\mathscr{S}_0 \leftarrow$ Compute a PLCS of $H_{\Gamma(f_0)}$ on $\bar{K}^m \backslash (\bigcup_{\mathbf{B}_i \in \mathscr{D}_1 \cup \mathscr{D}_2} \mathbf{B}_i)$;
$\mathscr{S} \leftarrow$ Compute a PLCS of $H_{\Gamma(f)}$ from $\mathscr{S}_0$;
END

Note that as we described in Remark 1 of subsection 3.2, $\mathscr{D}_1$, $\mathscr{D}_2$ can be obtained by utilizing comprehensive Gröbner systems.

We illustrate a PLCS of $H_{\Gamma(f)}$ with the following examples. In the examples, variables ξ_1, ξ_2 correspond to variables x_1, x_2.

Example 3. A polynomial $f = x_1^4 + x_2^5 + tx_1x_2^4 \in (\mathbf{C}[t])[x_1, x_2]$ is semi-weighted of type $(20; (5,4))$ where x_1, x_2 are variables and t is a parameter. (The weight vector is $\mathbf{w} = (5,4)$.) Then, a PLCS of $H_{\Gamma(f)} = \left\{\psi \in H^2_{[O]}(K[x]) \mid f * \psi = \left(\frac{\partial f}{\partial x_2}\right) * \psi = 0\right\}$ w.r.t. the weighted term order, is the following:

– if the parameter t belongs to $\mathbf{C}$, then the set

$$\Psi = \{1, \xi_2, \xi_2^2, \xi_2^3, \xi_2, \xi_1\xi_2, \xi_1\xi_2^2, \xi_1^2, \xi_1^2\xi_2, \xi_1^2\xi_2^2, \xi_1^3, \xi_1^3\xi_2, \xi_1^3\xi_2,$$

$$4/25t^2\xi_1^5 - 16/125t^3\xi_2\xi_1^4 + \xi_2^3\xi_1^3 - 4/5t\xi_2^4\xi_1^2$$

$$+ 16/25t^2\xi_2^5\xi_1 - 64/125t^3\xi_2^6, 4/25t^2\xi_1^4 + \xi_2^3\xi_1^2$$

$$- 4/5t\xi_2^4\xi_1 + 16/25t^2\xi_2^5, \xi_2^3\xi_1 - 4/5t\xi_2^4\}$$

is a basis of $H_{\Gamma(f)}$. In this case, the parameter space $\mathbf{C}$ has not been decomposed.

Example 4. A polynomial $f = x_1^3 + x_2^9 + sx_1^2x_2^3 \in (\mathbf{C}[s])[x_1, x_2]$ is weighted homogeneous of type $(9; (3,1))$ where x_1, x_2 are variables and t is a parameter. (A weight vector is $\mathbf{w} = (3,1)$.) Then, a PLCS of $H_{\Gamma(f)} = \left\{\psi \in H^2_{[O]}(K[x]) \mid f * \psi = \left(\frac{\partial f}{\partial x_2}\right) * \psi = 0\right\}$ w.r.t. the weighted term order, is the following:

– if the parameter s belongs to $\mathbf{V}(4s^3 + 27)$, then f has non-isolated singularity,

– if the parameter s belongs to $\mathbf{V}(s)$, then

$$\{1, \xi_2, \xi_2^2, \xi_2^3, \xi_2^4, \xi_2^5, \xi_2^6, \xi_2^7, \xi_1, \xi_1\xi_2, \xi_1\xi_2^2, \xi_1\xi_2^3, \xi_1\xi_2^4, \xi_1\xi_2^5, \xi_1\xi_2^6,$$

$$\xi_1\xi_2^7, \xi_1^2, \xi_1^2\xi_2, \xi_1^2\xi_2^2, \xi_1^2\xi_2^3, \xi_1^2\xi_2^4, \xi_1^2\xi_2^5, \xi_1^2\xi_2^6, \xi_1^2\xi_2^7\}$$

is a basis of $H_{\Gamma(f)}$, and

– if the parameter s belongs to $\mathbf{C}\backslash\mathbf{V}(4s^4 + 27s)$, then

$$\{1, \xi_2, \xi_2^2, \xi_2^3, \xi_2^4, \xi_2^5, \xi_2^6, \xi_2^7, \xi_1, \xi_1\xi_2, \xi_1\xi_2^2, \xi_1\xi_2^3, \xi_1\xi_2^4, \xi_1\xi_2^5, \xi_1\xi_2^6, \xi_1\xi_2^7, \xi_1^2, \xi_1^2\xi_2,$$

$$\xi_2\xi_1^4 - 3/(2s)\xi_2^4\xi_1^3 + 9/(4s^2)\xi_2^7\xi_1^2 + 1/2\xi_2^{10}\xi_1 - 3/(4s)\xi_2^{13},$$

$$\xi_1^4 - 3/(2s)\xi_2^3\xi_1^3 + 9/(4s^2)\xi_2^6\xi_1^2 + 1/2\xi_2^9\xi_1 - 3/(4s)\xi_2^{12},$$

$$\xi_2^2\xi_1^3 - 3/(2s)\xi_2^5\xi_1^2 - s/3\xi_2^8\xi_1 + 1/2\xi_2^{11}, \xi_2\xi_1^3 - 3/(2s)\xi_2^4\xi_1^2 + 1/2\xi_2^{10},$$

$$\xi_1^3 - 3/(2s)\xi_2^3\xi_1^2 + 1/2\xi_2^9, \xi_2^2\xi_1^2 - 1/3s\xi_2^8\}$$

is a basis of $H_{\Gamma(f)}$.

	Semi-weighted homogeneous polynomial	Prev. alg.		**Para_SemiW**	
		No. str.	time	No. str.	time
1	$x_1^4x_2 + x_2^6 + t_1x_1x_2^5 + t_2x_1^2x_2^4 + t_3x_1x_2^{14}$	15	0.2184	1	0.0156
2	$x_1^2x_2 + x_2^4 + x_3^5 + t_1x_2^2x_3^3 + t_2x_2^3x_3^2$	7	0.156	1	0.0936
3	$x_1^3x_3 + x_2^3 + x_2x_3^2 + s_1x_2x_1^3 + t_1x_1x_3^3$	9	0.312	2	0.0312
4	$x_1^4 + x_2^9 + t_1x_1x_2^7 + t_2x_1^2x_2^5$	8	0.3774	1	0.0312
5	$(x_1^4 + x_2^6 + x_1^2x_2^3)^2 + x_1^2x_2^9 + t_1x_1^3x_2^8 + t_2x_1^7x_2^2$	28	19.17	1	0.2652
6	$(x_1^4 + x_2^6 + x_1^2x_2^3)^3 + x_1^8x_2^6 + t_1x_1^{11}x_2^2 + t_2x_1^{11}x_2^3$	—	$> 1h$	1	1.997
7	$x_1^4 + s_1x_1^3x_2^2 + s_2x_1^2x_2^4 + x_2^8$	—	$> 1h$	11	2.886
8	$x_1^4 + s_1x_1^3x_2^2 + s_2x_1^2x_2^4 + x_2^8 + t_1x_2^9 + t_2x_2^{10}$	—	$> 1h$	11	2.98

Table 2. Comparison of the algorithm [33] and **Para_SemiW**.

The algorithm **Para_SemiW** has been implemented in the computer algebra system Risa/Asir. We give results of the benchmark tests. Table 2 shows comparisons between the implementation of **Para_SemiW** and our previous Risa/Asir implementation (Prev. alg.) of the algorithm[1] [33] in numbers of strata (No. str.) and computation time (CPU time). x_1, x_2, x_3 are variables and s_1, s_2, t_1, t_2, t_3 are parameters. The time is given in second. $> 1h$ means it takes more than 1 hour.

As is evident from Table 2, the algorithm **Para_SemiW** results in better performance in contrast to our previous algorithm [33]. For semi-weighted homogeneous polynomials, the algorithm is quite effective and gives a suitable decomposition of the parameter space depending on the structure of the parametric local cohomology classes. As results, the algorithm **Para_SemiW** gives small numbers of strata.

4. Logarithmic vector fields and local cohomology

Here, we show the relations between logarithmic vector fields and local cohomology classes. Second, we review a method to compute a standard basis of the annihilator ideal of a certain subspace of $H_{\Gamma(f)}$, which will be exploited to construct an algorithm for computing logarithmic vector fields.

[1] In our previous paper [33], an algorithm for computing a PLCS of H_F has been introduced where $F = \{f_1, f_2, \dots, f_s\}$. By the algorithm, we are able to compute a PLCS of $H_{\Gamma(f)}$ if we input $F = \left\{f, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right\}$. The term order is the total degree lexicographic term order s.t. $\xi_n \prec \cdots \prec \xi_2 \prec \xi_1$.

Recall that $S = \{x \in X \mid f(x) = 0\}$ is a hypersurface with an isolated singularity at the origin O in X.

4.1. Logarithmic vector fields

DEFINITION 4.1 ([38]). A holomorphic vector field

$$v = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n},$$

$a_i(x) \in \mathcal{O}_X$, $i = 1, \dots, n$, is **logarithmic** along S if $v(f)$ belongs to the ideal $\langle f \rangle$ generated by f in $\mathcal{O}_X$.

Let $\mathscr{D}\mathrm{er}_X(-\log S)$ denote the sheaf of logarithmic vector fields along S and $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$ the stalk at O of $\mathscr{D}\mathrm{er}_X(-\log S)$.

Let $\pi_\Gamma : H_{\Gamma(f)} \to H_{\Gamma(f)}$ be the map defined by $\pi_\Gamma(\psi) = \left(\frac{\partial f}{\partial x_1}\right) * \psi$ and let $H_{\Phi(f)}$ denote the image of the map π_Γ:

$$H_{\Phi(f)} = \left\{ \left(\frac{\partial f}{\partial x_1}\right) * \psi \,\middle|\, \psi \in H_{\Gamma(f)} \right\}.$$

Let $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Gamma(f)})$ denote the annihilator ideal in $\mathcal{O}_{X,O}$ of $H_{\Gamma(f)}$:

$$\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Gamma(f)}) = \{a(x) \in \mathcal{O}_{X,O} \mid a(x) * \psi = 0,\ \forall \psi \in H_{\Gamma(f)}\}.$$

LEMMA 4.2. *$Ann_{\mathcal{O}_{X,O}}(H_{\Gamma(f)}) = \left\langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$.*

PROOF. As $H_{\Gamma(f)} = \left\{ \psi \in H^n_{[O]}(K[x]) \mid f * \psi = \left(\frac{\partial f}{\partial x_2}\right) * \psi = \cdots = \left(\frac{\partial f}{\partial x_n}\right) * \psi = 0 \right\}$, the Grothendieck local duality theorem on residue [17] implies that $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Gamma(f)}) = \left\langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$. □

The following theorem is of basic importance.

THEOREM 4.3 ([42]). *Let $a(x) \in \mathcal{O}_{X,O}$. Then, the following conditions are equivalent.*

(i) *$a(x) \in Ann_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$.*

(ii) *There exists a logarithmic vector field v along S ($v \in \mathscr{D}\mathrm{er}_{X,O}(-\log S)$) such that*

$$v = a(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$$

where $a_2(x), \dots, a_n(x) \in \mathcal{O}_{X,O}$.

PROOF. It is sufficient to show that the annihilator ideal, in the local ring $\mathcal{O}_{X,O}$, of $H_{\Phi(f)}$ is the ideal quotient $\left\langle f, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle : \left\langle \frac{\partial f}{\partial x_1} \right\rangle$. Let $a(x) \in \mathcal{O}_{X,O}$. Then, $a(x)$ is in the annihilator ideal $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ if and only if

$$a(x) * \left(\left(\frac{\partial f}{\partial x_1} \right) * \psi \right) = \left(a(x) \left(\frac{\partial f}{\partial x_1} \right) \right) * \psi = 0, \quad \forall \psi \in H_{\Gamma(f)}.$$

Since $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Gamma(f)}) = \left\langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$, the condition above is equivalent to the following.

$$a(x) \in \left\langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle : \left\langle \frac{\partial f}{\partial x_1} \right\rangle.$$

Namely $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)}) = \left\langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle : \left\langle \frac{\partial f}{\partial x_1} \right\rangle$, which completes the proof. □

A logarithmic vector field v generated over $\mathcal{O}_{X,O}$ by

$$f\left(\frac{\partial}{\partial x_1}\right), \dots, f\left(\frac{\partial}{\partial x_n}\right) \quad \text{and} \quad \left(\frac{\partial f}{\partial x_j}\right)\left(\frac{\partial}{\partial x_i}\right) - \left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial}{\partial x_j}\right),$$

$(1 \le i < j \le n)$, is called trivial.

LEMMA 4.4. *Let $v' = a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$ be a germ of holomorphic vector field. If v' is a logarithmic vector field along S, then v' is trivial.*

PROOF. Since $\left(f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right)$ is a regular sequence, this lemma follows immediately from the definition of regular sequences. □

This yields the following.

PROPOSITION 4.5. *Let $v = a(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$ be a logarithmic vector field along S. Then, the following conditions are equivalent.*

(i) *v is trivial.*

(ii) $a(x) \in \left\langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$.

In the next subsection, we consider an algorithm for computing a standard basis of the ideal $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ which will be utilized to reveal the structure of logarithmic vector fields along S.

4.2. Local cohomology and standard bases

Here, we present an algorithm for computing standard bases as an application of the algorithms Algorithm 2 and 3.

Let us consider first, for simplicity, the case where f has no parameters.

DEFINITION 4.6 ([30]). Let $\prec$ be a global term order on $K[\xi]$, Ψ a finite subset of $H^n_{[O]}(K[x])$ and ψ an element of $\mathrm{SL}(\Psi)$ with $\mathrm{ht}(\psi) = \xi^\gamma$ where $\gamma \in \mathbf{N}^n$. Let $c_{(\gamma,\kappa)} \in K$ denote the coefficient of the lower term ξ^κ of ψ, i.e., $\psi = \xi^\gamma + \sum_{\xi^\kappa \prec \xi^\gamma} c_{(\gamma,\kappa)}\xi^\kappa$.

Let Ξ be a set of terms in $K[\xi]$, then, for $\xi^\lambda \in \Xi$, the transfer SB_Ψ is defined by the following:

$$\begin{cases} \mathrm{SB}_\Psi(\xi^\lambda) = x^\lambda - \sum_{\xi^\kappa \in \mathrm{ht}(\mathrm{SL}(\Psi))} c_{(\kappa,\lambda)} x^\kappa & \text{in } K[x], \text{ if } \xi^\lambda \in \mathrm{LL}(\Psi), \\ \mathrm{SB}_\Psi(\xi^\lambda) = x^\lambda & \text{in } K[x], \text{ if } \xi^\lambda \notin \mathrm{LL}(\Psi). \end{cases}$$

The set $\mathrm{SB}_\Psi(\Xi)$ is also defined by $\mathrm{SB}_\Psi(\Xi) = \{\mathrm{SB}_\Psi(\xi^\lambda) \mid \xi^\lambda \in \Xi\}$.

The next theorem describes how to compute a standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ from a basis of the vector space $H_{\Phi(f)}$.

THEOREM 4.7 ([30, 45]). *Let Φ be a basis of the vector space $H_{\Phi(f)}$ such that, for all $\varphi \in \Phi$, $\mathrm{hc}(\varphi) = 1$ and $\mathrm{ht}(\varphi) \notin \mathrm{LL}(\Phi)$. Let $\prec$ be a global term order in $K[\xi]$ and Ξ be the minimal basis of $\langle \mathbf{Neighbor}(\mathrm{ht}(\Phi)) \backslash \mathrm{ht}(\Phi)\rangle$ in $K[\xi]$. Then, $\mathrm{SB}_\Phi(\Xi)$ is a reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. the local term order $\prec^{-1}$ in the ring $\mathcal{O}_{X,O}$ the ring of power series.*

EXAMPLE 5. Let us consider Example 2. $f = x_1^4x_2 + x_2^4 + x_1^6 \in \mathbf{C}[x_1, x_2]$ (W_{13} singularity) is a semi-weighted homogeneous polynomial of type $(16; (3,4))$. The set Ψ is a basis of the vector space $H_{\Gamma(f)}$ and the set

$$\pi_\Gamma(\Psi) = \left\{ \left(\frac{\partial f}{\partial x_1}\right) * \psi \mid \psi \in \Psi \right\} = \{4, 4\xi_2, 2/3\xi_1, 6, -8\xi_2 - \xi_2^2 + 2/3\xi_1^2\}.$$

The basis Φ of the vector space $H_{\Phi(f)}$ that satisfies, for all $\psi \in \pi_\Gamma(\Psi)$, $\mathrm{hc}(\psi) = 1$ and $\mathrm{ht}(\psi) \notin \mathrm{LL}(\pi_\Gamma(\Psi))$ w.r.t. the lexicographical term order $\prec$ s.t. $\xi_2 \prec \xi_1$, is $\{1, \xi_1, \xi_2, \xi_1^2 - 3/2\xi_2^2\}$. The minimal basis Θ of $\langle \mathbf{Neighbor}(\mathrm{ht}(\pi_\Gamma(\Psi))) \backslash \mathrm{ht}(\pi_\Gamma(\Psi))\rangle$ is $\{\xi_1^3, \xi_1\xi_2, \xi_2^2\}$.

Therefore, $\mathrm{SB}_\Psi(\Theta) = \{x_1^3, x_1x_2, x_2^2 + 3/2x_1^2\}$ is the reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. $\prec^{-1}$ in the local ring.

We turn to the parametric cases. In order to treat standard bases with parameters, we introduce now a notion of parametric standard basis.

Definition 4.8. Let F be a subset of $(K[t])[x]$, $\mathbf{A}_1, \dots, \mathbf{A}_\ell$ strata in $\overline{K}^m$, $S_1, \dots, S_\ell$ subsets of $K(t)\{x\}$ and $\prec$ a local term order. A finite set $\mathscr{S} = \{(\mathbf{A}_1, S_1), \dots, (\mathbf{A}_\ell, S_\ell)\}$ of pairs is called a **parametric standard basis** on $\mathbf{A}_1 \cup \dots \cup \mathbf{A}_\ell$ of $\langle F \rangle$ w.r.t. $\prec$ if $S_i \subset (K[t]_{\mathbf{A}_i})[x]$ and $\sigma_{\bar{a}}(S_i)$ is a standard basis of the ideal $\langle \sigma_{\bar{a}}(F) \rangle$ in $\overline{K}\{x\}$ w.r.t. $\prec$ for each $i = 1, \dots, \ell$ and $\bar{a} \in \mathbf{A}_i$ where $K(t)$ is the field of rational functions and $\overline{K}\{x\}$ is the ring of power series.

As the method for computing standard bases from a basis of $H_{\Phi(f)}$ consists of only linear algebra computation, the method can be generalized to parametric cases, like Algorithm 3. This algorithm is the same as our previous algorithm [30], essentially. Notably the algorithm also performs simultaneously a decomposition of a given stratum into finer strata according to the structure of resulting vector spaces. We sketch the resulting method for computing a parametric standard basis of $\mathrm{Ann}_{\mathscr{O}_{X,o}}(H_{\Phi(f)})$ in Algorithm 4.

Algorithm 4. PSB

Specification: PSB($f, \prec$)
Computing a parametric standard basis of $\mathrm{Ann}_{\mathscr{O}_{X,o}}(H_{\Phi(\sigma_a(f))})$ w.r.t. $\prec$.
Input: f : a semi-weighted homogeneous polynomial of type $(d; \mathbf{w})$ with parameters t.
$\prec$: a local term order.
Output: $(\mathscr{P}, \mathscr{D})$:
$\mathscr{P} = \{(\mathbf{A}_1, P_1), (\mathbf{A}_2, P_2), \dots, (\mathbf{A}_l, P_\ell)\}$ is a parametric standard basis on $\mathbf{A}_1 \cup \mathbf{A}_2 \cup \dots \cup \mathbf{A}_\ell$, of the ideal $\mathrm{Ann}_{\mathscr{O}_{X,o}}(H_{\Phi(f)})$ w.r.t. $\prec$. For all $\bar{a} \in \mathbf{A}_i$, $\sigma_{\bar{a}}(P_i)$ is the **reduced standard basis** of $\mathrm{Ann}_{\mathscr{O}_{X,o}}(H_{\Phi(\sigma_a(f))})$ w.r.t. $\prec$, $1 \le i \le \ell$.
$\mathscr{D} = \{\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_k\}$ is a set of strata s.t. the weighted homogeneous part of f does not define an isolated singularity at the origin on $\mathbf{B}_i$ for $1 \le i \le k$.
BEGIN
$(\mathscr{S}, \mathscr{D}) \leftarrow$ **Para_SemiW**$(f, \prec_{\mathbf{w}})$ where $\prec_{\mathbf{w}}$ a weighted term order.
$\mathscr{S}' \leftarrow \{(\mathbf{A}, \pi_\Gamma(\Psi)) \mid (\mathbf{A}, \Psi) \in \mathscr{S}\}$; $\mathscr{P} \leftarrow \emptyset$;
while $\mathscr{S}' \neq \emptyset$ **do**
Select $(\mathbf{A}', \Phi')$ from $\mathscr{S}'$; $\mathscr{S}' \leftarrow \mathscr{S}' \backslash \{(\mathbf{A}', \Phi')\}$;
$v \leftarrow {}^t(\xi^{\alpha_1}, \dots, \xi^{\alpha_u})$ where $\mathrm{Term}(\Phi') = \{\xi^{\alpha_1}, \dots, \xi^{\alpha_u}\}$ and $\xi^{\alpha_u} \prec^{-1} \dots \prec^{-1} \xi^{\alpha_1}$;
$\mathscr{H} \leftarrow$ Compute a maximal linearly independent subset of Φ' whose coefficient matrix is the row reduced echelon matrix w.r.t. v on $\mathbf{A}$;
  while $\mathscr{H} \neq \emptyset$ **do**
  Select $(\mathbf{A}'', \Phi)$ from $\mathscr{H}$; $\mathscr{H} \leftarrow \mathscr{H} \backslash \{(\mathbf{A}'', \Phi)\}$;
  $(\mathbf{A}'', P) \leftarrow$ Compute the reduced standard basis P of $\mathrm{Ann}_{\mathscr{O}_{X,o}}(H_{\Phi(f)})$ on $\mathbf{A}''$ from Φ;
  $\mathscr{P} \leftarrow \mathscr{P} \cup \{(\mathbf{A}'', P)\}$;
  end-while
end-while
return$(\mathscr{P}, \mathscr{D})$;
END

The correctness and termination of Algorithm 4 follow from Algorithm 3 and Theorem 4.7. We have implemented the algorithm for computing parametric standard bases of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$, in the computer algebra system Risa/Asir.

We illustrate parametric standard bases of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ with Example 6.

EXAMPLE 6. Let us consider Example 3, again. Ψ is the basis of the vector space $H_{\Gamma(f)}$ and $\pi_{\Gamma}(\Psi) = \left\{\left(\frac{\partial f}{\partial x_1}\right) * \psi \mid \psi \in \Psi\right\}$ is $\{-4/5t^2, -4/25t^2\xi_1 + 16/25t^3\xi_2, -4/25t^2\xi_1^2 + 16/125t^3\xi_1\xi_2 + 4\xi_2^3 - 64/125t^4\xi_2^2, 4, 4\xi_2, 4\xi_2^2\}$. Hence, we obtain a PLCS of $H_{\Phi(f)}$ from the set $\pi_{\Gamma}(\Psi)$. The maximal linearly independent subset of $H_{\Phi(f)}$ whose coefficient matrix is a row reduced echelon matrix w.r.t. the total degree lexicographic term order $\prec$ s.t. $\xi_2 \prec \xi_1$, is the following;

- if the parameter t belongs to $\mathbf{V}(t)$, then $\Phi = \{\xi_2^3, \xi_2^2, \xi_2, 1\}$ is a basis of $H_{\Phi(f)}$, and
- if the parameter t belongs to $\mathbf{C}\backslash\mathbf{V}(t)$, then $\Phi = \{\xi_2^3 - 1/25t^2\xi_1^2 + 4/125t^3\xi_1\xi_2, \xi_2^2, \xi_2, \xi_1, 1\}$ is a basis of $H_{\Phi(f)}$.

By Algorithm 4, the reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. $\prec^{-1}$ is easily obtained from a PLCS of $H_{\Phi(f)}$, as follows;

- if the parameter t belongs to $\mathbf{V}(t)$, then $\{x_1, x_2^4\}$ is the reduced standard basis, and
- if the parameter t belongs to $\mathbf{C}\backslash\mathbf{V}(t)$, then $\{x_1^2 + 1/25t^2x_2^3, x_1x_2 - 4/125t^3x_2^3, x_2^4\}$ is the reduced standard basis.

5. Main results

Here we introduce our main results that are algorithms for computing logarithmic vector fields along S. We present two computation methods. The main difference is; the first method involves syzygy computation in a "*local*" ring, and the second method performs syzygy computation in a "*global*" ring. We will compare the first and the second methods in numbers of strata and computation time.

5.1. Method 1

In order to explain the main idea of the method, let us consider first, for simplicity, the case where f has no parameters. Assume that the reduced standard basis $\{q_1, q_2, \ldots, q_r\}$ of the annihilating ideal $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. a local term order $\prec$ and a standard basis M_j of the module of syzygies w.r.t. the generators $q_j\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}, f$ in $\mathcal{O}_{X,O}$ for each $j = 1, 2, \ldots, r$, are given. Note that, the module order is POT ("top down" order, see [10]) with $\prec$. Then, we have the following theorem.

THEOREM 5.1. *Under the setup above, there exists a vector* $(c_{j_1}, c_{j_2}, \dots, c_{j_n}, c_{j_{n+1}}) \in M_j$ *such that* c_{j_1} *contains a term of degree* 0, *i.e., a non-zero constant term is in* c_{j_1}. *The holomorphic vector field*

$$v_j = q_j \frac{\partial}{\partial x_1} + (c_{j_2}/c_{j_1}) \frac{\partial}{\partial x_2} + \cdots + (c_{j_n}/c_{j_1}) \frac{\partial}{\partial x_n}$$

is logarithmic along S, *for each* $j \in \{1, \dots, r\}$.

PROOF. As the coefficients of $\frac{\partial}{\partial x_1}$ are generated by the reduced standard basis $\{q_1, \dots, q_r\}$ w.r.t. $\prec$ in $\mathcal{O}_{X,O}$ by Theorem 4.3, there exists a $(c_{j_1}, c_{j_2}, \dots, c_{j_n}, c_{j_{n+1}}) \in M_j$ that satisfies the property because M_j is a standard basis w.r.t. POT with $\prec$. Since $(c_{j_1}, c_{j_2}, \dots, c_{j_n}, c_{j_{n+1}})$ is a syzygy,

$$c_{j_1} q_j \frac{\partial f}{\partial x_1} + c_{j_2} \frac{\partial f}{\partial x_2} + \cdots + c_{j_n} \frac{\partial f}{\partial x_n} = -c_{j_{n+1}} f.$$

Hence, $v_j(f) \in \langle f \rangle$ holds. □

COROLLARY 5.2. *Using the same notation as in Theorem* 5.1, *let* M *be a standard basis of the module of syzygies w.r.t. the generators* $\frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f$ *in* $\mathcal{O}_{X,O}$. *Set* $T = \left\{ c_2 \frac{\partial}{\partial x_2} + \cdots + c_n \frac{\partial}{\partial x_n} \,\middle|\, (c_2, \dots, c_n, c_{n+1}) \in M \right\}$ *w.r.t. a POT module order with* $\prec$. *Then,* $\{v_1, v_2, \dots, v_r\} \cup T$ *is a set of generators of* $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$.

PROOF. By Proposition 4.5, $v_1, v_2, \dots, v_r$ and elements of T generate $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$ over $\mathcal{O}_{X,O}$. □

REMARK. For an arbitrary defining polynomial of a hypersurface, a set of generators of the logarithmic vector fields with polynomial coefficients can be directly computed as a syzygy module over the polynomial ring, which also generates the logarithmic vector fields with analytic coefficients because of the flatness of the power series ring over the polynomial ring. However, it is difficult in general to extract local analytic properties of the module $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$ from the generators obtained by the syzygy computation in the polynomial rings. Note also that if we construct logarithmic vector fields directly by computing standard bases of the module of syzygies w.r.t. the generators $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f$ in $\mathcal{O}_{X,O}$, then the output of the computation are, in general, not suitable to know the local analytic properties of logarithmic vector fields. See for instance, Example 7. In contrast, the proposed method that utilize the standard basis $\{q_1, q_2, \dots, q_r\}$ gives a nice set of generators of $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$ for analyzing *complex analytic properties*, near the singular point in question, of logarithmic vector fields.

In the non-parametric case, it is possible to compute a standard basis of a module of syzygies w.r.t. a given local term order in $\mathcal{O}_{X,O}$. In fact, the computer algebra system SINGULAR [15] has a command of computing them.

Now we turn to the parametric case. Before describing the algorithm, we introduce a notion of parametric syzygy systems.

DEFINITION 5.3 (PSS). Let fix a term order. Let $f_1, \dots, f_s$ be a subset of $(K[t])[x]$, $\mathbf{A}_1, \dots, \mathbf{A}_\ell$ strata in $\bar{K}^m$ and $G_1, \dots, G_\ell$ subsets of $(K[t])[x]$. A finite set $\mathcal{G} = \{(\mathbf{A}_1, G_1), \dots, (\mathbf{A}_\ell, G_\ell)\}$ of pairs is called a **parametric syzygy system (PSS)** on $\mathbf{A}_1 \cup \dots \cup \mathbf{A}_\ell$ of $(f_1, \dots, f_s)$ if $\sigma_a(G_i)$, $a \in \mathbf{A}_i$, is a standard basis (or Gröbner basis) of the module of syzygies w.r.t. the generators $\sigma_a(f_1), \sigma_a(f_2), \dots, \sigma_a(f_s)$ in $\bar{K}[[x]]$ (or $\bar{K}[x]$) for each $i = 1, \dots, \ell$. We simply say $\mathcal{G}$ is a parametric syzygy system of $(f_1, \dots, f_s)$ if $\mathbf{A}_1 \cup \dots \cup \mathbf{A}_\ell = \bar{K}^m$.

We write for clarity a parametric syzygy system in a local ring as $\mathrm{PSS}_{\mathrm{sb}}$ (for standard bases) and parametric syzygy system in a global ring as $\mathrm{PSS}_{\mathrm{gb}}$ (for Gröbner bases).

It is easy to see that Theorem 5.1 can be generalized to the parametric case by $\mathrm{PSS}_{\mathrm{sb}}$. The outline of the algorithm for computing logarithmic vector fields is therefore the following.

Step 1. Compute a parametric standard basis of the annihilator ideal $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ by Algorithm 1.
Step 2. Compute a $\mathrm{PSS}_{\mathrm{sb}}$ of $\left(q_j \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f\right)$ where q_j is an element of the standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$.
Step 3. Select an element $(c_1, c_2, \dots, c_n, c_{n+1})$ from a $\mathrm{PSS}_{\mathrm{sb}}$ in $\mathcal{O}_X$, whose first component has a non-zero constant term.
Step 4. Set $v_j = q_j \frac{\partial}{\partial x_1} + (c_2/c_1) \frac{\partial}{\partial x_2} + \dots + (c_n/c_1) \frac{\partial}{\partial x_n}$.

In step 2, it is necessary to compute a $\mathrm{PSS}_{\mathrm{sb}}$ of $\left(q_j \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f\right)$ in the rings of power series. However, to the best of our knowledge, there is currently no implementation of such syzygy computation. Thus, we provide a new alternative efficient algorithm for computing the $\mathrm{PSS}_{\mathrm{sb}}$ in the rings of power series.

In [28], an efficient algorithm for computing $\mathrm{PSS}_{\mathrm{gb}}$ in a "*polynomial ring*", has been introduced. One can generalize the algorithm to a local ring by using Lazard's homogenization technique [23]. The algorithm of parametric syzygies is described in Appendix A.

Note that as we apply Lazard's homogenization technique, we obtain a standard basis of the module of syzygies w.r.t. a local "*total degree*" term order $\prec$. Thus, we compute, beforehand, a parametric standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. the same term order $\prec$.

The complete algorithm for computing logarithmic vector fields along S with parameters, is Algorithm 5.

The correctness clearly follows from Algorithm 4 (PSB) and Theorem 5.1. As we use the Lazard's homogenization technique, it follows from [28] and Algorithm 4 that the algorithm for computing a $\mathrm{PSS}_{\mathrm{sb}}$, at $(*)$, terminates. Since the set $\mathscr{P}$ and $\mathscr{M}$ have only finite number of pairs, the algorithm terminates. Note that the part of $(\triangle)$ will be used in Algorithm 8, too.

We illustrate the algorithm with the following examples.

Example 7. Let us consider Example 5. From Example 5, the reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. $\prec^{-1}$ is $\{x_1^3, x_1x_2, x_2^2+3/2x_1^2\}$. Then, a syzygy basis of $\left(x_1^3\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f\right)$ is

$$\begin{aligned}
\{&(9+64x_2, -6x_1^4+96x_1^2x_2^2+12x_1^2x_2-8x_2^3, -384x_1^2x_1-48x_1^2+32x_2^2),\\
&(-24x_1^4-16x_1^2x_2, 4x_1^2+3x_2, 6x_1^4x_2^4+4x_1^2x_2^2, 0),\\
&(-24x_1^2x_2-3x_1^2-2x_2^2, 2x_1^6-36x_1^4x_2^2-4x_1^4x_2, 144x_1^4x_2+16x_1^4),\\
&(-3x_1^2+16x_2^2, 2x_1^6-4x_1^4x_2+24x_1^2x_2^3, 16x_1^4-96x_1^2x_2^2),\\
&(x_1^4+4x_2^3, -6x_1^8-4x_1^6x_2, 0)\}.
\end{aligned}$$

We take $(9+64x_2, -6x_1^4+96x_1^2x_2^2+12x_1^2x_2-8x_2^3, -384x_1^2x_1-48x_1^2+32x_2^2)$ (because the first component has a non-zero constant term) and set

$$v_1 = x_1^3\frac{\partial}{\partial x_1} + (-6x_1^4+96x_1^2x_2^2+12x_1^2x_2-8x_2^3)/(9+64x_2)\cdot\frac{\partial}{\partial x_2}.$$

As $1/(9+64x_2) = 1/4\sum_{i=0}^{\infty}(-64/9)^i x_2^i$, v_1 is a holomorphic vector field

$$x_1^3\frac{\partial f}{\partial x_1} + (-6x_1^4+96x_1^2x_2^2+12x_1^2x_2-8x_2^3)\sum_{i=0}^{\infty}(-64/9)^i x_2^i\frac{\partial}{\partial x_2}.$$

Algorithm 5. **Method 1**

Specification: Method1($f, \prec$)
Computing bases of $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$.
Input: f : a semi-weighted homogeneous polynomial of type $(d; \mathbf{w})$ with parameters t.
$\prec$: a local term order.
Output: $(\mathscr{V}, \mathscr{D})$:
$\mathscr{V} = \{(\mathbf{A}_1, V_1), \dots, (\mathbf{A}_\ell, V_\ell)\}$, V_i is a set of logarithmic vector fields along S on $\mathbf{A}_i$ for each $i \in \{1, \dots, \ell\}$.
$\mathscr{D} = \{\mathbf{B}_1, \dots, \mathbf{B}_k\}$ is a set of strata s.t. the weighted homogeneous part of f does not define an isolated singularity at the origin on $\mathbf{B}_i$ for $1 \leq i \leq k$.
BEGIN

$\mathscr{L} \leftarrow \emptyset$; $(\mathscr{P}, \mathscr{D}) \leftarrow$ **PSB**$(f, \prec)$;
$\mathscr{T} \leftarrow$ Compute a $\mathrm{PSS}_{\mathrm{sb}}$ of $\left(\frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f\right)$ in $\mathscr{O}_X$;
while $\mathscr{T} \neq \emptyset$ **do**
  Select $\{(\mathbf{A}_0, M)\}$ from $\mathscr{T}$; $\mathscr{T} \leftarrow \mathscr{T} \backslash \{(\mathbf{A}_0, M)\}$; $L_0 \leftarrow \emptyset$;
    while $M \neq \emptyset$ **do**
      Select $(c_2, \dots, c_n, c_{n+1})$ from M; $M \leftarrow M \backslash \{(c_2, \dots, c_n, c_{n+1})\}$; $(\triangle)$
      $L_0 \leftarrow \left\{c_2 \frac{\partial}{\partial x_2} + \cdots + c_n \frac{\partial}{\partial x_n}\right\}$;
    end-while
end-while
$\mathscr{L} \leftarrow \mathscr{L} \cap \{(\mathbf{A}_0, L_0)\}$;

while $\mathscr{P} \neq \emptyset$ **do**
  Select $(\mathbf{A}, \{q_1, \dots, q_r\})$ from $\mathscr{P}$; $\mathscr{P} \leftarrow \mathscr{P} \backslash \{(\mathbf{A}, \{q_1, \dots, q_r\})\}$;
  `/*`$\{q_1, \dots, q_r\}$ `is the reduced standard basis*/`
   for each $j \in \{1, \dots, r\}$ **do**
     $\mathscr{M} \leftarrow$ Compute a $\mathrm{PSS}_{\mathrm{sb}}$ of $\left(q_j \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f\right)$ on $\mathbf{A}$; $(*)$
     $\mathscr{S} \leftarrow \emptyset$;
     while $\mathscr{M} \neq \emptyset$ **do**
       Select $(\mathbf{A}', M)$ from $\mathscr{M}$; $\mathscr{M} \leftarrow \mathscr{M} \backslash \{(\mathbf{A}', M)\}$;
       $(c_1, \dots, c_{n+1}) \leftarrow$ Select an element from M whose 1st component is a nonzero constant;
         $v \leftarrow q_j \frac{\partial}{\partial x_1} + (c_2/c_1) \frac{\partial}{\partial x_2} + \cdots + (c_n/c_1) \frac{\partial}{\partial x_n}$;
       while $\mathscr{V} \neq \emptyset$ **do**
         Select $(\mathbf{A}'', V)$ from $\mathscr{V}$; $\mathscr{V} \leftarrow \mathscr{V} \backslash \{(\mathbf{A}'', V)\}$;
         if $\mathbf{A}' \cap \mathbf{A}'' \neq \emptyset$ **then**
           $\mathscr{S} \leftarrow \mathscr{S} \cup \{(\mathbf{A}' \cap \mathbf{A}'', V \cup \{v\})\}$;
         end-if
       end-while
       end-while
       $\mathscr{V} \leftarrow \mathscr{S}$;
   end-for
end-while
return $(\mathscr{V}, \mathscr{D})$;
END

Likewise, we take the following vector from a syzygy basis of $\left(x_1x_2\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f\right)$:

$$(9 + 64x_2, 8x_1^4 + 2x_1^2x_2 + 96x_2^3 + 12x_2^2, -8x_1^2 - 384x_2^2 - 48x_2).$$

Moreover, we take the following vector from a syzygy basis of $\left(x_2^2 + 3/2x_1^2\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f\right)$:

$$(27 - 292x_1^2 - 27x_2, 4x_1^5 - 629x_1^3x_2 - 27x_1^3 - 438x_1x_2^3 + 54x_1x_2,$$
$$2624x_1^3 + 1752x_1x_2^2 + 216x_1).$$

Hence, we have the following as non-trivial logarithmic vector fields

$$v_2 = x_1x_2 \cdot \frac{\partial}{\partial x_1} + (8x_1^4 + 2x_1^2x_2 + 96x_2^3 + 12x_2^2)/(9 + 64x_2) \cdot \frac{\partial}{\partial x_2},$$
$$v_3 = x_2^2 + 3/2x_1^2 \cdot \frac{\partial}{\partial x_1} + (4x_1^5 - 629x_1^3x_2 - 27x_1^3 - 438x_1x_2^3 + 54x_1x_2)/$$
$$(27 - 292x_1^2 - 27x_2) \cdot \frac{\partial}{\partial x_2}.$$

Thus, v_1, v_2, v_3 and trivial vector fields generate $\mathscr{D}er_{X,O}(-\log S)$.

Note that the expansion of a polynomial $(9 + 64x_2)x_1^3$ is $9x_1^3 + 64x_1^3x_2$. If the expansion of a polynomial is given, then we cannot obtain the really important factor x_1^3. If we compute logarithmic vector fields with expanded polynomials in coefficients (for example the command "syz" of Singular [15]), then as, in general, a coefficient polynomial cannot be factored into polynomials, we cannot get really important information as outputs and we need further computation to find the essential factor. In contrast, our algorithm tells us the essential information on coefficients $a_1(x)$'s, at the isolated singularity, by computing a standard basis of an annihilating ideal $\mathrm{Ann}_{O_{X,O}}(H_{\Phi(f)})$. This is a significant feature of the proposed algorithm.

The next example handles a parametric case.

Example 8. Let us consider Example 3 and Example 6, again. Now, we know a parametric standard basis of the annihilator ideal $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. $\prec^{-1}$ where $\prec$ is the local total degree lexicographic term order s.t. $x_2 \prec x_1$.

– If the parameter t belongs to $\mathbf{V}(t)$, then $\{x_1, x_2^4\}$ is the reduced standard basis. Compute a $\mathrm{PSS}_{\mathrm{sb}}$ of $\left(x_1\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f\right)$ on $\mathbf{V}(t)$. Then, $\{(-5, 4x_2, 20),$

$(5x_2^4, -4x_1^4, 0)\}$ is the $\mathrm{PSS_{sb}}$. Select $(-5, 4x_2, 20)$ and set

$$v_1 = x_1 \frac{\partial}{\partial x_1} - 4/5x_2 \frac{\partial}{\partial x_2}$$

which is an Euler logarithmic vector field. Next, we compute a $\mathrm{PSS_{sb}}$ of $\left(x_2^4 \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f\right)$ on $\mathbf{V}(t)$. Then, $\{(0, -x_1^4 - x_2^5, 5x_2^4), (-5, 4x_1^3, 0)\}$ is the $\mathrm{PSS_{sb}}$. Select $(-5, 4x_1^3, 0)$ and set

$$v_2 = x_2^4 \frac{\partial}{\partial x_1} - 4/5x_1^3 \frac{\partial}{\partial x_2}$$

which is a trivial logarithmic vector field. Thus, v_1 and trivial vector fields generate $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$.

– If the parameter t belongs to $\mathbf{C}\backslash\mathbf{V}(t)$, then $\{x_1^2 + 1/25t^2x_2^3, x_1x_2 - 4/125t^3x_2^3, x_2^4\}$ is the reduced standard basis. By the same way, we can obtain the following three non-trivial logarithmic vector fields u_1, u_2, u_3;

$$u_1 = (x_1^2 + 1/25t^2x_2^3)\frac{\partial}{\partial x_1} + 1/25((64t^6x_1^3 - (16t^5x_2 + 625t)x_1^2$$
$$- (1180t^4x_2^2 - 12500x_2)x_1 + 64t^7x_2^4 - 125t^3x_2^3)/(625 - 64t^4x_2))\frac{\partial}{\partial x_2},$$

$$u_2 = (x_1x_2 - 4/125t^3x_2^3)\frac{\partial}{\partial x_1} + 1/125((-256t^7x_1^3 + (64t^6x_2 + 2500t^2)x_1^2$$
$$- (80t^5x_2^2 + 3125tx_2)x_1 - 256t^8x_2^4 - 5900t^4x_2^3$$
$$+ 62500x_2^2)/(625 - 64t^4x_2))\frac{\partial}{\partial x_2},$$

$$u_3 = x_2^4\frac{\partial}{\partial x_1} + (((64t^4x_2 - 500)x_1^3 - 16t^3x_2^2x_1^2 + 20t^2x_2^3x_1$$
$$+ 64t^5x_2^5 - 525tx_2^4)/(625 - 64t^4x_2))\frac{\partial}{\partial x_2}.$$

5.2. Method 2

Here we introduce another new algorithm for computing logarithmic vector fields along S. The key ideal of the new algorithm is the next lemma.

LEMMA 5.4. *Let $f_1, f_2, \ldots, f_\ell$ be polynomial in $K[x]$ s.t. $\{x \in X \mid f_1(x) = f_2(x) = \cdots = f_\ell(x) = 0\} = \{O\}$ where X be a neighborhood of the origin O of $\mathbf{C}^n$. Let $\mathscr{I}_O$ be an ideal generated by $f_1, f_2, \ldots, f_\ell$ in $\mathcal{O}_{X,O}$ (local ring) and I be an ideal* *generated by* $f_1, f_2, \dots, f_\ell$ *in* $K[x]$ *(global ring). Let* h *be a polynomial in* $K[x]$, *s.t.* $h \in \mathscr{I}_O$. *Then, there exists a polynomial* $g \in I : \langle h \rangle$ *s.t.* $g \notin \mathfrak{m}$, *where* $I : \langle h \rangle = \{g \in K[x] \mid gh \in I\}$ *is the ideal quotient and* $\mathfrak{m} = \langle x_1, x_2, \dots, x_n \rangle$ *is the maximal ideal in* $\mathcal{O}_{X,O}$.

Proof. As I has a minimal primary decomposition and $\{x \in X \mid f_1(x) = \cdots = f_s(x) = 0\} = \{O\}$, I can be written as $I = I_0 \cap I_1 \cap I_2 \cap \cdots \cap I_r$ where $I_0, I_1, \dots, I_r$ are primary ideals and $\mathbf{V}(I_0) = \{O\}$, $O \notin \mathbf{V}(I_i)$ for each $i \in \{1, \dots, r\}$. Notice that $\mathscr{I}_O = \mathcal{O}_{X,O} \otimes I_0$ where $\otimes$ is a tensor product. Recall that $\mathbf{V}(I : \langle I_0 \rangle) = \bigcup_{1 \le i \le r} \mathbf{V}(I_i)$. Since, $h \in I_0 = \mathscr{I}_0 \cap K[x_1, \dots, x_n]$, we have $\mathbf{V}(I : \langle h \rangle) \subseteq \bigcup_{1 \le i \le r} \mathbf{V}(I_i)$, which immediately implies that there exists a polynomial $g \in K[x]$ s.t. $gh \in I$ and $g(O) \neq 0$. $\square$

Let $\{q_1, \dots, q_r\}$ be the reduced standard basis of the annihilating ideal of $H_{\Phi(f)}$ w.r.t. a local term order $\prec$. Then, by the proof of Theorem 4.3, for each $j \in \{1, \dots, r\}$,

$$q_j \frac{\partial f}{\partial x_1} \in \left\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}, f \right\rangle.$$

Therefore, there exists $g_j \in \left\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}, f \right\rangle : \left\langle q_j \frac{\partial f}{\partial x_1} \right\rangle$ with $g_j(O) \neq 0$. Since, $g_j\left(q_j \frac{\partial f}{\partial x_1}\right) \in \left\langle \frac{\partial f}{\partial_2}, \frac{\partial f}{\partial_3}, \dots, \frac{\partial f}{\partial x_n}, f \right\rangle$, $g_j\left(q_j \frac{\partial f}{\partial x_1}\right)$ can be written as

$$g_j\left(q_j \frac{\partial f}{\partial x_1}\right) = p_2 \frac{\partial f}{\partial x_2} + \cdots + p_n \frac{\partial f}{\partial x_n} + p_{n+1} f$$

where $p_2, \dots, p_n, p_{n+1} \in K[x]$. The condition $g_j(O) \neq 0$ implies that $q_j = p_j/g_j$ and p_j/g_j, are well-defined as elements of $\mathcal{O}_{X,O}$ for $j = 2, \dots, n$. Hence, if we have polynomial $g_j, p_2, \dots, p_n, p_{n+1}$, then $q_j \frac{\partial f}{\partial x_1}$ can be written as follows

$$q_j \frac{\partial f}{\partial x_1} = (p_2/g_j) \cdot \frac{\partial f}{\partial x_2} + \cdots + (p_n/g_j) \cdot \frac{\partial f}{\partial x_n} + (p_{n+1}/g_j) \cdot f.$$

This implies

$$q_j \frac{\partial f}{\partial x_1} - (p_2/g_j) \cdot \frac{\partial f}{\partial x_2} - \cdots - (p_n/g_j) \cdot \frac{\partial f}{\partial x_n} \in \langle f \rangle$$

in $\mathcal{O}_{X,O}$, namely,

$$v_j = q_j \frac{\partial}{\partial x_1} + (-p_2/g_j) \frac{\partial}{\partial x_2} + \cdots + (-p_n/g_j) \frac{\partial}{\partial x_n}$$

is a logarithmic vector field along S.

The denominator g_j can be obtained by using an algorithm for computing ideal quotients, and polynomials $p_2, \dots, p_n, p_{n+1}$ can also be obtained in a polynomial ring $K[x]$ by utilizing an algorithm for computing syzygies.

Algorithm 6. OneElement

Specification: OneElement(f, q)
Computing a logarithmic vector field along S.
Input: f : a semi-weighted homogeneous polynomial of type $(d; \mathbf{w})$.
$q \in K[x]$: $q \in \mathrm{SB}$ where SB is a reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$.
Output: $v = q\frac{\partial}{\partial x_1} + d_2\frac{\partial}{\partial x_2} + \cdots + d_n\frac{\partial}{\partial x_n}$: v is logarithmic along S.
BEGIN
1: $G \leftarrow$ Compute a Gröbner basis of $\left\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}, f \right\rangle : \left\langle q\frac{\partial f}{\partial x_1} \right\rangle$ in $K[x]$;
2: $g \leftarrow$ Select a polynomial g from G s.t. $g(O) \neq 0$;
3: Syz $\leftarrow$ Compute a Gröbner basis of a module of syzygies w.r.t. the generators $\left(gq\frac{\partial f}{\partial x_1}\right), \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$, f, w.r.t. a POT module order in $K[x]^{n+1}$;
4: $(c_1, \dots, c_n, c_{n+1}) \leftarrow$ Select $(c_1, \dots, c_n, c_{n+1})$ from Syz s.t. c_1 contains a nonzero constant;
5: For each $i \in \{2, \dots, n\}$,
$d_i \leftarrow c_i/(c_1 g)$;
return $q\frac{\partial}{\partial x_1} + d_2\frac{\partial}{\partial x_2} + \cdots + d_n\frac{\partial}{\partial x_n}$;
END

Theorem 5.5. *Algorithm* 4 *outputs a logarithmic vector field along S and terminates.*

Proof. We prove that there exists a vector $(c_1, \dots, c_n, c_{n+1})$ in Syz s.t. c_1 is a nonzero constant. By Lemma 5.4, there exists $g \in G$ s.t. $g(O) \neq 0$. Since $g\left(q\frac{\partial f}{\partial x_1}\right) \in \left\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}, f \right\rangle$, there exist $p_2, \dots, p_n, p_{n+1} \in K[x]$ s.t. $g\left(q\frac{\partial f}{\partial x_1}\right) = p_2\frac{\partial f}{\partial x_2} + \cdots + p_n\frac{\partial f}{\partial x_n} + p_{n+1}f$. Let $(u_2, \dots, u_n, u_{n+1})$ be a syzygy of $\left(\frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f\right)$, i.e., $u_2\frac{\partial f}{\partial x_2} + \cdots + u_n\frac{\partial f}{\partial x_n} + u_{n+1}f = 0$. Then,

$$\left(g\left(q\frac{\partial f}{\partial x_1}\right) - \left(p_2\frac{\partial f}{\partial x_2} + \cdots + p_n\frac{\partial f}{\partial x_n} + p_{n+1}f\right)\right)$$

$$+ \left(u_2\frac{\partial f}{\partial x_2} + \cdots + u_n\frac{\partial f}{\partial x_n} + u_{n+1}f\right) = 0, \quad \text{i.e.,}$$

$$\left(gq\frac{\partial f}{\partial x_1}\right) + (u_2 - p_2)\frac{\partial f}{\partial x_2} + \cdots + (u_n - p_n)\frac{\partial f}{\partial x_n} + (u_{n+1} - p_{n+1})f = 0.$$

Hence, $(1, u_2 - p_2, \dots, u_n - p_n, u_{n+1} - p_{n+1})$ is a syzygy of $\left(gq\frac{\partial f}{\partial x_1}\right), \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f$. As Syz is a Gröbner basis of the syzygy module w.r.t. a POT order in $K[x]^{n+1}$ and $(1, u_2 - p_2, \dots, u_n - p_n, u_{n+1} - p_{n+1}) \in \langle \mathrm{Syz} \rangle$, there exists an element $(c_1, \dots,$

$c_n, c_{n+1}) \in \mathrm{Syz}$ such that the first component is a nonzero constant. Therefore, obviously,

$$q\frac{\partial f}{\partial x_1} + (c_2/(c_1 g))\frac{\partial f}{\partial x_2} + \cdots + (c_n/(c_1 g))\frac{\partial f}{\partial x_n} \in \langle f \rangle.$$

As an algorithm for computing Gröbner bases terminates in $K[x]$, Algorithm 4 also terminates. □

Corollary 5.6. *Using the same notation as in Theorem* 5.5 *and Corollary* 5.2, *let* $V = \{v \mid \textit{for each } i \in \{1, \dots, r\}, v = \mathbf{OneElement}(f, q_i)\}$. *Then,* $V \cup T$ *is a set of generators of* $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$.

Proof. By Proposition 4.5 and Theorem 5.5, obviously, elements of $V \cup T$ generate $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$ over $\mathscr{O}_{X,O}$. □

Let us consider parametric cases. It is possible to extend Algorithm 6 to parametric cases, naturally, by utilizing CGS and $\mathrm{PSS_{gb}}$, as follows.

Algorithm 7. ParaOneElement

Specification: ParaOneElement($f, q, \mathbf{A}$)
Computing a logarithmic vector field along S with parameters on $\mathbf{A}$.
Input: f : a semi-weighted homogeneous polynomial of type $(d; \mathbf{w})$ with parameters.
$(q, \mathbf{A}) : q \in \mathrm{SB}$ and $(\mathbf{A}, \mathrm{SB}) \in \mathscr{P}$ where $\mathscr{P}$ is an output of Algorithm 4.
Output: $\mathscr{V} = \{(\mathbf{A}_1, \{v_1\}), \dots, (\mathbf{A}_\ell, \{v_\ell\})\}$: $v_j = q\frac{\partial}{\partial x_1} + d_{j2}\frac{\partial}{\partial x_2} + \cdots + d_{jn}\frac{\partial}{\partial x_n}$ is a logarithmic along S on $\mathbf{A}_j$, for each $j \in \{1, \dots, \ell\}$.
BEGIN
$\mathscr{V} \leftarrow \emptyset$; $\mathscr{G} \leftarrow$ Compute a CGS of $\left\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}, f \right\rangle : \left\langle q\frac{\partial f}{\partial x_1} \right\rangle$ on $\mathbf{A}$; $(\diamond)$
while $\mathscr{G} \neq \emptyset$ **do**
 $(\mathbf{A}', G) \leftarrow$ Select $(\mathbf{A}', G)$ from $\mathscr{G}$; $\mathscr{G} \leftarrow \mathscr{G} \backslash \{(\mathbf{A}', G)\}$;
 $g \leftarrow$ Select a polynomial g from G s.t. $g(O) \neq 0$;
 $\mathscr{Y} \leftarrow$ Compute a $\mathrm{PSS_{gb}}$ of $\left(\left(gq\frac{\partial f}{\partial x_1}\right), \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f\right)$ on $\mathbf{A}'$;
  while $\mathscr{Y} \neq \emptyset$ **do**
   $(\mathbf{A}'', \mathrm{Syz}) \leftarrow$ Select $(\mathbf{A}'', \mathrm{Sy})$ from $\mathscr{Y}$; $\mathscr{Y} \leftarrow \mathscr{Y} \backslash \{(\mathbf{A}'', \mathrm{Syz})\}$;
   $(c_1, \dots, c_n, c_{n+1}) \leftarrow$ Select $(c_1, \dots, c_n, c_{n+1})$ from Syz s.t. c_1 is a nonzero constant;
    for each $i \in \{2, \dots, n\}$ **do**
     $d_i \leftarrow c_i/(c_1 g)$;
    end-for
   $\mathscr{V} \leftarrow \mathscr{V} \cup \left\{\left(\mathbf{A}'', \left\{q\frac{\partial}{\partial x_1} + d_2\frac{\partial}{\partial x_2} + \cdots + d_n\frac{\partial}{\partial x_n}\right\}\right)\right\}$;
  end-while
end-while
return $\mathscr{V}$;
END

REMARK. At $(\Diamond)$, an algorithm for computing CGS of ideal quotients is required. This algorithm is described in [22]. It is possible to compute CGS of the ideal quotients.

The consideration above yields the following new algorithm for computing logarithmic vector fields along S with parameters.

Algorithm 8. Method 2

Specification: Method2($f, \prec$)
Computing bases of $\mathscr{D}\mathrm{er}_{X,O}(-\log S)$.
Input: f : a semi-weighted homogeneous polynomial of type $(d;\mathbf{w})$ with parameters t.
$\prec$: a local term order.
Output: $(\mathscr{V}, \mathscr{D})$:
$\mathscr{V} = \{(\mathbf{A}_1, V_1), \dots, (\mathbf{A}_l, V_l)\}$, V_i is a set of logarithmic vector fields along S on $\mathbf{A}_i$ for each $i \in \{1, \dots, l\}$.
$\mathscr{D} = \{\mathbf{B}_1, \dots, \mathbf{B}_k\}$ is a set of strata s.t. the weighted homogeneous part of f does not define an isolated singularity at the origin on $\mathbf{B}_i$ for $1 \le i \le k$.
BEGIN

$\boxed{(\triangle) \text{ of Method 1}}$

while $\mathscr{P} \neq \emptyset$ **do**
  Select $(\mathbf{A}, \{q_1, \dots, q_r\})$ from $\mathscr{P}$; $\mathscr{P} \leftarrow \mathscr{P} \backslash \{(\mathbf{A}, \{q_1, \dots, q_r\})\}$; `/*standard basis*/`
  for each j from 1 to r **do**
    $\mathscr{V} \leftarrow$ **ParaOneElement**$(f, q_i, \mathbf{A})$; $\mathscr{S} \leftarrow \emptyset$;
    while $\mathscr{V} \neq \emptyset$ **do**
      Select $(\mathbf{A}', V)$ from $\mathscr{V}$; $\mathscr{V} \leftarrow \mathscr{M} \backslash \{(\mathbf{A}', V)\}$;
      while $\mathscr{L} \neq \emptyset$ **do**
        Select $(\mathbf{A}'', L)$ from $\mathscr{L}$; $\mathscr{L} \leftarrow \mathscr{L} \backslash \{(\mathbf{A}'', L)\}$;
        if $\mathbf{A}' \cap \mathbf{A}'' \neq \emptyset$ **then** $\mathscr{S} \leftarrow \mathscr{S} \cup \{(\mathbf{A}' \cap \mathbf{A}'', V \cup L)\}$; **end-if**
      end-while
    end-while
    $\mathscr{L} \leftarrow \mathscr{S}$;
  end-for
end-while
return $(\mathscr{V}, \mathscr{D})$;
END

Let us remark that the first part of **Method 2** is the same as $(\triangle)$ of **Method 1**. The correctness follows from Theorem 5.5. Since the set $\mathscr{P}$ and $\mathscr{M}$ have only finite number of pairs, the algorithm terminates.

We illustrate the algorithm with the following examples.

EXAMPLE 9. A polynomial $f = f_0 + x_1^6$ (W_{13} singularity) is a semi-weighted homogeneous polynomial of type $(16;(3,4))$ in $\mathbf{C}[x_1, x_2]$ where $f_0 = x_1^4 x_2 + x_2^4$

is a weighted homogeneous polynomial. From Example 5, $\mathrm{SB} = \{x_1^3, x_1x_2^2, x_2^2 + 3/2x_1^2\}$ is the reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t the local weighted degree lexicographic term order $\prec$ with the coordinate (x_1, x_2).

For each element of SB, we apply the algorithm **OneElement** for computing logarithmic vector fields.

1. Take x_1^3 from SB. Then, $\{27 - 256x_1^2, 9 + 64x_2\}$ is a Gröbner basis of the ideal quotient $\left\langle \frac{\partial f}{\partial x_2}, f \right\rangle : \left\langle (x_1^3) \frac{\partial f}{\partial x_1} \right\rangle$ w.r.t. the total degree lexicographic term order $\prec_{td}$ (global term order) with the coordinate (x_1, x_2). Set $g = 9 + 64x_2$ and compute a Gröbner basis of a module of syzygies w.r.t. the generators $g(x_1^3)\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f$. Then, the Gröbner basis (w.r.t. $\prec_{td}$) is
$$\{(-1, 6x_1^4 - 96x_1^2x_2^2 - 12x_1^2x_2 + 8x_2^3, 384x_1^2x_1 + 48x_1^2 - 32x_2^2),$$
$$(0, x_1^6 + x_1^4x_2 + x_2^4, -x_1^4 - 4x_2^3)\}.$$
From the first element, we get
$$v_1 = -x_1^3\frac{\partial}{\partial x_1} + (6x_1^4 - 96x_1^2x_2^2 - 12x_1^2x_2 + 8x_2^3)/(9 + 64x_2)\frac{\partial}{\partial x_2}$$
as a logarithmic vector field along S because the first component of the first element is a constant.
2. Take $x_1x_2^2$ from SB. Then, $\{27 - 256x_1^2, 9 + 64x_2\}$ is a Gröbner basis of $\left\langle \frac{\partial f}{\partial x_2}, f \right\rangle : \left\langle x_1x_2^2 \frac{\partial f}{\partial x_1} \right\rangle$ w.r.t. $\prec_{td}$. Set $g = 9 + 64x_2$ and compute a Gröbner basis of a module of syzygies w.r.t. the generators $g(x_1x_2^2)\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f$. Then, the Gröbner basis (w.r.t. $\prec_{td}$) is
$$\{(-1, -8x_1^4 - 2x_1^2x_2 - 96x_2^3 - 12x_2^2, 8x_1^2 + 384x_2^2 + 48x_2),$$
$$(0, x_1^6 + x_1^4x_2 + x_2^4, -x_1^4 - 4x_2^3)\}$$
Hence, we get
$$v_2 = -x_1x_2^2\frac{\partial}{\partial x_1} + (-8x_1^4 - 2x_1^2x_2 - 96x_2^3 - 12x_2^2)/(9 + 64x_2)\frac{\partial}{\partial x_2}$$
as a logarithmic vector field along S.
3. Take $x_2^2 + 3/2x_1^2$ from SB. Then, $\{27 - 256x_1^2, 9 + 64x_2\}$ is a Gröbner basis of $\left\langle \frac{\partial f}{\partial x_2}, f \right\rangle : \left\langle (x_2^2 + 3/2x_1^2) \frac{\partial f}{\partial x_1} \right\rangle$ w.r.t. $\prec_{td}$. Set $g = 9 + 64x_2$ and compute a Gröbner basis of a module of syzygies w.r.t. the generators $g(x_3)\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f$. Then, the Gröbner basis (w.r.t. $\prec_{td}$)

$$\{(1, -96x_1^5 - 88x_1^3x_2 - 9x_1^3 + 146x_1x_2^2 + 18x_1x_2, 96x_1^3 - 584x_1x_2 - 72x_1),$$
$$(0, -x_1^6 - x_1^4x_2 - x_2^4, x_1^4 + 4x_2^3)\}.$$

Hence, we get

$$v_3 = (x_2 + 3/2x_1^2)\frac{\partial}{\partial x_1} + (-96x_1^5 - 88x_1^3x_2 - 9x_1^3 + 146x_1x_2^2 + 18x_1x_2)/(9 + 64x_2)\frac{\partial}{\partial x_2}$$

as a logarithmic vector field along S.
Therefore, v_1, v_2, v_3 and trivial vector fields generate $\mathscr{D}er_{X,O}(-\log S)$.

The next example handles a parametric case.

EXAMPLE 10. A polynomial $f = f_0 + t_1x_2^4x_3 + t_2x_3^3 \in \mathbf{C}[x_1, x_2, x_3]$ (S_{17} singularity) is a semi-weighted homogeneous polynomial of type $(24; (7, 4, 10))$ where $f_0 = x_1^2x_3 + x_2x_3^2 + x_2^6$ and t_1, t_2 are parameters.

By applying the method [32] for computing Tjurina stratification, the list of Tjurina numbers of f is obtained as follows.

- If $(t_1, t_2) \in \mathbf{V}(t_1, t_2)$, then the Tjurina number $\tau(f)$ of f at the origin, is 17.
- If $(t_1, t_2) \in \mathbf{V}(t_1)\backslash\mathbf{V}(t_1, t_2)$, then the Tjurina number $\tau(f)$ of f at the origin, is 16.
- If $(t_1, t_2) \in \mathbf{C}^2\backslash\mathbf{V}(t_1)$, then the Tjurina number $\tau(f)$ of f at the origin, is 15.

Let us consider logarithmic vector fields along S with the parameters. Algorithm 4 outputs

$$\mathscr{PS} = \{(\mathbf{V}(t_1, t_2), \{x_2^6, x_2x_3, 6x_2^5 + x_3^2, x_1\}), (\mathbf{V}(t_1)\backslash\mathbf{V}(t_1, t_2),$$
$$\{x_2^6, x_1^2 + 18/7t_2x_2^5, x_1x_2, x_1x_3, -72/7t_2x_2^5 + x_1x_3, 6x_2^5 + x_3^2\}),$$
$$(\mathbf{C}^2\backslash\mathbf{V}(t_1), \{x_2^6, x_1x_2^2, x_1^2 + 18/7t_2x_2^5 - 1/7t_1x_2^4, x_1x_3,$$
$$-72/7t_2x_2^5 + 4/7t_1x_2^4 + x_2x_3, 6x_2^5 + x_3^2\})\}$$

as a parametric standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(\sigma_a(f))})$.

Notice that the decomposition of the parameter space $\mathbf{C}^2$ is the same as the Tjurina stratification.

1. Take $(\mathbf{V}(t_1, t_2), \{x_2^6, x_2x_3, 6x_2^5 + x_3^2, x_1\})$ from $\mathscr{PS}$. Then, on $\mathbf{V}(t_1, t_2)$, the algorithm outputs the following non-trivial logarithmic vector fields

along S:

$$\mathscr{V}_{11} = \left\{7x_1\frac{\partial}{\partial x_1} + 10x_3\frac{\partial}{\partial x_3} + 4x_2\frac{\partial}{\partial x_2}, (6x_2^5 + x_3^2)\frac{\partial}{\partial x_1} - 2x_1x_3\frac{\partial}{\partial x_2},\right.$$
$$-7x_2x_3\frac{\partial}{\partial x_1} + 12x_1x_3\frac{\partial}{\partial x_3} + 2x_1x_2\frac{\partial}{\partial x_2},$$
$$\left.-7x_2^6\frac{\partial}{\partial x_1} - 2x_1x_3^2\frac{\partial}{\partial x_3} + 2x_1x_2x_3\frac{\partial}{\partial x_2}\right\}$$

and the following trivial logarithmic vector fields along S:

$$\mathscr{V}_{12} = \left\{(-6x_1^2x_3 - 5x_2x_3^2)\frac{\partial}{\partial x_3} + (-x_1^2x_2 - 2x_2^2x_3)\frac{\partial}{\partial x_2},\right.$$
$$\left.(6x_2^5 + x_3^2)\frac{\partial}{\partial x_3} + (-x_1^2 - 2x_2x_3)\frac{\partial}{\partial x_2}, (x_1^2x_3 + x_2^6 + x_2x_3^2)\frac{\partial}{\partial x_2}\right\}.$$

Hence, if $(t_1, t_2) \in \mathbf{V}(t_1, t_2)$, then $\mathscr{V}_{11} \cup \mathscr{V}_{12}$ generates $\mathscr{D}er_{X,O}(-\log S)$.

2. Take $(\mathbf{V}(t_1)\backslash\mathbf{V}(t_1, t_2), \{x_2^6, x_1^2 + 18/7t_2x_2^5, x_1x_2, x_1x_3, -72/7t_2x_2^5 + x_1x_3, 6x_2^5 + x_3^2\})$, from $\mathscr{PS}$. Then, on $\mathbf{V}(t_1)\backslash\mathbf{V}(t_1, t_2)$, the algorithm outputs the following non-trivial logarithmic vector fields along S:

$$\mathscr{V}_{21} = \left\{(6x_2^5 + x_3^2)\frac{\partial}{\partial x_1} - 2x_1x_3\frac{\partial}{\partial x_2}, (-7)(-72/7t_2x_2^5 + x_2x_3)\frac{\partial}{\partial x_1}\right.$$
$$+ 12x_1x_3\frac{\partial}{\partial x_3} + (2x_1x_2 - 24t_2x_1x_3)\frac{\partial}{\partial x_2},$$
$$(1492992b^5x_3^3 - 16807)(x_1x_3)\frac{\partial}{\partial x_1} + (-74088t_2x_2^4x_3 + 127008t_2^2x_2^3x_3^2$$
$$- 217728t_2^3x_2^2x_3 + 373248t_2^4x_2x_3^4 + 1492992t_2^5x_3^5 - 24010x_3^2)\frac{\partial}{\partial x_3}$$
$$+ (-12348t_2x_2^5 + 21168t_2^2x_2^4x_3 - 36288t_2^3x_2^3x_3^2 + 62208t_2^4x_2^2x_3^3$$
$$+ 746496t_2^5x_2x_3^4 - 9604x_2x_3 + 14406t_2x_3^2)\frac{\partial}{\partial x_2},$$
$$(1492992t_2^5x_3^3 - 16807)x_1x_2\frac{\partial}{\partial x_1} + (127008t_2^2x_2^4x_3 - 217728t_2^3x_2^3x_2^2$$
$$+ 373248t_2^4x_2^2x_3^3 + 1492992t_2^5x_2x_3^4 - 24010x_2x_3 + 12348t_2x_3^2)\frac{\partial}{\partial x_3}$$

$$+ (21168t_2^2x_2^5 - 36288t_3^3x_2^4 + 62208t_2^4x_2^3x_3^2 + 746496t_2^5x_2^2x_3^3$$

$$- 9604x_2^2 + 2058t_2x_2x_3 - 24696t_2^2x_3^2)\frac{\partial}{\partial x_2}, -7(x_1^2 + 18/7t_2x_2^5)\frac{\partial}{\partial x_1}$$

$$- 10x_1x_3\frac{\partial}{\partial x_3} + (-4x_1x_2 + 6t_2x_1x_3)\frac{\partial}{\partial x_2},$$

$$(-1492992t_2^5x_3^3 + 16807)x_2^6\frac{\partial}{\partial x_1} + (49392t_2x_2^4x_3 - 84672t_2^2x_2^3x_3^2$$

$$+ 145152t_2^3x_2^2x_3^3 - 248832t_2^4x_2x_3^4 + 4802x_3^2)x_1\frac{\partial}{\partial x_3}$$

$$+ (8232t_2x_2^5 - 14112t_2^2x_2^4x_1 + 24192t_2^3x_2^3x_3^2 - 41472t_2^4x_2^2x_3^3$$

$$+ (497664t_2^5x_3^4 - 4802x_3)x_2 - 9604t_2x_3^2)x_1\frac{\partial}{\partial x_2}\Big\},$$

and the trivial logarithmic vector fields along S are $\mathscr{V}_{12}$.

Hence, if $(t_1, t_2) \in \mathbf{V}(t_1)\backslash\mathbf{V}(t_1, t_2)$, then $\mathscr{V}_{21} \cup \mathscr{V}_{12}$ generates $\mathscr{D}er_{X,O}(-\log S)$.

3. Take the last segment from $\mathscr{PS}$. In this case, the algorithm decomposes $\mathbf{C}^2\backslash\mathbf{V}(t_1)$ into 4 strata

$$\mathbf{V}(t_2)\backslash\mathbf{V}(t_1,t_2), \mathbf{V}(4t_1^3 + 27t_2)\backslash\mathbf{V}(t_1,t_2), \mathbf{V}(64t_1^6 + 912t_1^3t_2 + 3969t_2^2)\backslash\mathbf{V}(t_1,t_2),$$

$$\mathbf{C}^2\backslash\mathbf{V}(256t_1^{10}t_2 + 5376t_1^7t_2 + 40500t_1^4t_2^3 + 107163t_1t_2^4).$$

This decomposition happens when we compute a Gröbner basis of an ideal quotient $\left\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, f \right\rangle : \left\langle q\frac{\partial f}{\partial x_1}\right\rangle$ in Algorithm 6, namely, for each stratum, the structure of the ideal quotient is different from others.

Due to the saving of pages, we omit the output of logarithmic vector fields along S, because the output is quite huge.

The algorithm **Method 2** (with total degree lexicographic term order s.t. $x_3 \prec x_2 \prec x_1$) has been implemented in the computer algebra system Risa/Asir. Here we give results of the benchmark tests. Table 3 shows a comparison of the implementation of **Method 1** with **Method 2** in numbers of strata (No. strata) and computation time (CPU time). x_1, x_2, x_3 are variables and s_1, s_2, t_1, t_2 are parameters. The time is given in second. $> 3h$ means it takes more than 3 hours. Note that in Prob. 5, if $s_1 \neq 0 \wedge s_2^2 - 4 \neq 0$, then $x_1^4 + s_1x_1^3x_2^2 + s_2x_1^2x_2^4 + x_2^8$ is a weighted homogeneous polynomial.

Prob.	Semi-weighted homogeneous poly.	Method 1		Method 2	
		No. strata	time	No. strata	time
1	$x_1^3 + x_2x_3^2 + x_2^8 + t_1x_1x_2^6$	2	0.0468	2	0.0312
2	$x_1^3x_2 + x_2^3 + x_2x_3^2 + x_1^3x_3 + t_1x_1x_3^3$	12	0.4050	2	0.078
3	$x_1^3x_3 + x_2^3 + x_2x_3^2 + s_1x_2x_1^3 + t_1x_1x_3^3$	14	0.4056	5	0.3276
4	$x_1^2x_3 + x_2^3 + s_1x_2x_3^7 + t_1x_3^{11} + x_3^{12}$	9	3.292	3	0.4836
5	$x_1^4 + s_1x_1^3x_2^2 + s_2x_1^2x_2^4 + x_2^8$	24	1.451	15	1.045
6	$x_1^3 + x_2x_3^2 + x_2^{11} + t_1x_1x_2^8 + t_2x_1x_2^9$	12	26.08	5	3.931
7	$x_1^3x_2 + x_2^{15} + t_1x_1x_2^{11} + t_2x_1x_2^{12}$	18	43.6	6	8.19
8	$x_1^3x_2 + x_1^2x_2^4 + x_2^{10} + t_1x_2^{11} + t_2x_2^{12}$	—	$> 3h$	9	510.5

Table 3. Comparison of Method 1 and Method 2.

In all tests of Table 3, Method 2 results in better performance compared to Method 1. The essential point of Method 2 is computing Gröbner bases of ideal quotients, instead of standard bases. In general, a size of output of PSS_{gb} in Algorithm 7 is smaller than that of PSS_{sb} in Algorithm A (Lazard's homogenization technique). Thus, the numbers of strata of Method 2 is smaller than that of Method 1.

We can use various term order in Method 2 unlike Method 1.

In this paper, we have introduced two algorithms for computing logarithmic vector fields along a semi-weighted homogeneous isolated hypersurface singularity.

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Appendix A. Parametric syzygies

Here, we describe how to compute $\mathrm{PSS}_{\mathrm{sb}}$ in a "*local ring*". Our main idea for computing $\mathrm{PSS}_{\mathrm{sb}}$, is to combine the algorithm for computing $\mathrm{PSS}_{\mathrm{sb}}$ in a polynomial ring [28] with Lazard's homogenization technique [23].

DEFINITION A.1. Let $g = \sum_{i=0}^{d} g_i \in K[x]$ be a polynomial of total degree d where g_i is a homogeneous polynomial of degree i. Then, $g^h(x_0, x) = \sum_{i=0}^{d} g_i(x) x_0^{d-i}$ is a homogeneous polynomial of total degree d in $K[x_0, x]$ where x_0 is the extra variable. We call g^h the homogenization of g. Let q be a homogenization of g, i.e., $q = g^h$. The dehomogenization of q is $q^e = q(1, x)$, i.e., $q^e = g^h(1, x) = g(x)$.

We generalize the algorithm [28] to compute $\mathrm{PSS}_{\mathrm{gb}}$ in a local ring by using Lazard's homogenization technique [23]. The following algorithm outputs $\mathrm{PSS}_{\mathrm{sb}}$.

Algorithm A. PSYZ$_{\mathrm{sb}}$

Specification: PSYZ$_{\mathrm{sb}}$(($f_1, \dots, f_s$), A)
Computing a $\mathrm{PSS}_{\mathrm{sb}}$ of $(f_1, \dots, f_s)$ on $\mathbf{A}$.
Input: $f_1, \dots, f_s$: polynomials with parameters t, $\mathbf{A} \subset \bar{K}^m$.
Output: $\{(\mathbf{A}_1, G_1'), \dots, (\mathbf{A}_\ell, G_\ell')\}$: For all $\bar{a} \in \mathbf{A}_i'$, $\sigma_{\bar{a}}(G_i')$ is a standard basis of a syzygy module of $(f_1, \dots, f_s)$ in $\bar{K}\{x\}$ where G_i is a subset of $((K[t])\{x\})^s$, $1 \le i \le \ell$ and $\mathbf{A} = \mathbf{A}_1 \cup \cdots \cup \mathbf{A}_\ell$.
BEGIN
$f_1^h, \dots, f_s^h \leftarrow$ Homogenize $f_1, \dots, f_s$;
$\{(\mathbf{A}_1, G_1), \dots, (\mathbf{A}_\ell, G_\ell)\} \leftarrow$ Compute $\mathrm{PSS}_{\mathrm{gb}}$ of $(f_1^h, \dots, f_s^h)$ on $\mathbf{A}$ w.r.t. a total degree term order s.t. $x_0 \gg x$ in a polynomial ring, by [28];
$\{(\mathbf{A}_1, G_1'), \dots, (\mathbf{A}_\ell, G_\ell')\} \leftarrow$ Dehomogenize G_i for each $1 \le i \le \ell$, i.e., $G_i' = \{q^e \mid q \in G_i\}$;
return$\{(\mathbf{A}_1, G_1'), \dots, (\mathbf{A}_\ell, G_\ell')\}$;
END

The correctness and termination follow from [23] and [28].

References

[1] A. Aleksandrov, Cohomology of a quasihomogeneous complete intersection. Math. USSR Izvestiya **26** (1986), 437–477.
[2] A. Aleksandrov, Nonisolated Saito singularities. Math. USSR Sb. **65** (1990), 561–574.
[3] A. Aleksandrov, The index of vector fields and logarithmic differential forms. Functional Analysis and its Applications **39-4** (2005), 245–255.
[4] V. Arnold, Normal forms of functions in neighbourhoods of degenerate critical points. Russian Math. Survey **29** (1974), 10–50.
[5] J. Bruce, Vector fields on discriminants and bifurcation varieties. Bull. London Math. Soc. **17** (1985), 257–262.
[6] J. W. Bruce and R. M. Roberts, Critical points of functions on an analytic varieties. Topology **27** (1988), 57–90.
[7] W. Bruns and J. Herzog, Cohen-Macaulay rings, revised edition. Cambridge Univ. Press, 1998.
[8] F. J. Calderón-Moreno, L. N.-M. D. Mond and F. J. Castro-Jiménez, Logarithmic cohomology of the complement of a plane curve. Comment. Math. Helv. **77** (2002), 24–38.
[9] F. J. Castro-Jiménez and J. M. Ucha-Enríquez, Gröbner bases and logarithmic D-modules. Journal of Symbolic Computation **41** (2006), 317–335.

[10] D. Cox, J. Little and D. O'Shea, Ideals, Varieties, and Algorithms (3rd ed.). Springer, N.Y., 2007.

[11] J. Damon, On the legacy of free divisors: Discriminants and Morse-type singularities. Amer. J. Math. **120** (1998), 453–492.

[12] J. Damon, On the legacy of free divisors II: Free* divisors and complete intersections, Moscow mathematical J., **3** (2003), 361–395.

[13] J. F. de Bobadilla, Relative Morsification theory. Topology **43** (2004), 925–982.

[14] T. de Jong and D. van Straten, A deformation theory for nonisolated singularities. Abh. Math. Sem. Univ. Hamburg **60** (1990), 177–208.

[15] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, Singular 3-1-6—A computer algebra system for polynomial computations. (2012) http://www.singular.uni-kl.de.

[16] X. Gómez-Mont, J. Seade and A. Verjovsky, The index of a holomorphic flow with an isolated singularity. Mathematische Annalen **291** (1991), 737–751.

[17] A. Grothendieck, Théorèmes de dualité pour les faisceaux algébriques cohérents. Séminaire Bourbaki **149**, 1957.

[18] A. Grothendieck, Local Cohomology, notes by R. Hartshorne, Lecture Notes in Math., **41**, Springer, 1967.

[19] H. Hauser and G. Müller, Affine varieties and Lie algebras of vector fields. Manuscripta Math. **80-2** (1993), 309–337.

[20] H. Hauser and G. Müller, On the Lie algebra $\theta(x)$ of vector fields on a singularity. J. Math. Sci. Univ. Tokyo **1** (1994), 239–250.

[21] D. Kapur, Y. Sun and D. Wang, A new algorithm for computing comprehensive Gröbner systems. In: S. Watt (Ed.), International Symposium on Symbolic and Algebraic Computation (ISSAC), (2010), 29–36.

[22] D. Kapur, D. Lu, M. Monagan, Y. Sun and D. Wang, An Efficient Algorithm for Computing Parametric Multivariate Polynomial GCD. In: C. Arreche (Ed.), International Symposium on Symbolic and Algebraic Computation (ISSAC), (2018), 239–246.

[23] D. Lazard, Gröbner bases, Gaussian elimination, and resolution of systems of algebraic equations. In: G. Goos and J. Hartmanis (Eds.), European Conference on Computer Algebra (EUROCAL), Lecture Notes in Computer Science **162** (1985). Springer, 146–156.

[24] D. T. Lê, Calcul du nombre de cycles évanouissants d'une hypersueface complexe. Ann. Inst. Fourier, Grenoble **23** (1973), 261–270.

[25] D. T. Lê and B. Teissier, Variétés polaires locales et classes de Chern des variétés singulières. Annals of Mathematics **114** (1981), 457–491.

[26] M. Manubens and A. Montes, Improving DISPGB algorithm using the discriminant ideal. Journal of Symbolic Computation **41** (2006), 1245–1263.

[27] A. Montes and M. Wibmer, Gröbner bases for polynomial systems with parameters. Journal of Symbolic Computation **45/12** (2010), 1391–1425.

[28] K. Nabeshima, On the computation of parametric Gröbner bases for modules and syzygies. Japan Journal of Industrial and Applied Mathematics **27** (2010), 217–238.

[29] K. Nabeshima, Stability conditions of monomial bases and comprehensive Gröbner systems. In: V. Gerdt, W. Koepf, E. Mayr and E. Vorozhtsov (Eds.), Computer Algebra in Scientific Computing (CASC), Lecture Notes in Computer Science **7442**, (2012). Springer, 248–259.

[30] K. Nabeshima and S. Tajima, On efficient algorithms for computing parametric local cohomology classes associated with semi-quasihomogeneous singularities and standard bases. In: K. Nabeshima (Ed.), International Symposium on Symbolic and Algebraic Computation (2014). ACM, 351–358.

[31] K. Nabeshima and S. Tajima, Computing logarithmic vector fields associated with parametric semi-quasihomogeneous hypersurface isolated singularities. In: D. Robertz (Ed.), International Symposium on Symbolic and Algebraic Computation (ISSAC) (2015). ACM, 334–348.

[32] K. Nabeshima and S. Tajima, Computing Tjurina stratifications of μ-constant deformations via parametric local cohomology systems, Applicable Algebra in Engineering, Communication and Computing, **27** (2016), 451–467.

[33] K. Nabeshima and S. Tajima, Algebraic local cohomology with parameters and parametric standard bases for zero-dimensional ideals. Journal of Symbolic Computation, **82** (2017), 91–122.

[34] Y. Nakamura and S. Tajima, On weighted-degrees for algebraic local cohomologies associated with semiquasihomogeneous singularities. Advanced Studies in Pure Mathematics **46** (2007), 105–117.

[35] M. Noro and T. Takeshima, Risa/Asir—A computer algebra system. In: Wang, P. (Ed.), International Symposium on Symbolic and Algebraic Computation (ISSAC) (1992). ACM, 387–396.

[36] J. J. Nuño-Ballesteros, B. Oréfice and J. M. Tomazella, The Bruce-Roberts number of a function on a weighted homogeneous hypersurface. The Quarterly J. Math. **64** (2013), 269–280.

[37] R. Pellikaan, Deformations of hypersurfaces with a one-dimensional singular locus. J. Pura and Applied Algebra **67** (1990), 49–71.

[38] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields. J. Fac. Sci. Univ. Tokyo, Sect. IA Math **27** (1980), 265–291.

[39] F. Sancho de Salas, Residues of a Pfaff system relative to an invariant subscheme. Trans. Amer. Math. Soc. **352-9** (2000), 4019–403.

[40] W. Sit, An algorithm for solving parametric linear systems. Journal of Symbolic Computation **13** (1992), 353–394.

[41] T. Suwa, Indices of holomorphic vector fields relative to invariant curves on surface. Proc. Amer. Math. Soc. **123-10** (1995), 2989–2997.

[42] S. Tajima, On polar varieties, logarithmic vector fields and holonomic D-modules. RIMS Kôkyûroku Bessatsu **40** (2013), 41–51.

[43] S. Tajima and Y. Nakamura, Algebraic local cohomology classes attached to quasi-homogeneous isolated hypersurface singularities. Publications of the Research Institute for Mathematical Sciences **41** (2005), 1–10.

[44] S. Tajima and Y. Nakamura, Algebraic local cohomology classes attached to unimodal singularities. Publications of the Research Institute for Mathematical Sciences **48** (2012), 21–43.

[45] S. Tajima, Y. Nakamura and K. Nabeshima, Standard bases and algebraic local cohomology for zero dimensional ideals. Advanced Studies in Pure Mathematics **56** (2009), 341–361.

[46] B. Teissier, Cycles évanescents, sections planes et conditions de Whitney, Singularités à Cargèse. Astérisque **7-8** (1973), 285–362.

[47] B. Teissier, Variétés polaires I, Invent. Math. **40** (1977), 267–292.

[48] B. Teissier, Variétés polaires II, Lecture Notes in Math., **961** (1982), Springer, Berlin, 314–491.

[49] H. Terao, The bifurcation set and logarithmic vector fields, Math. Ann. **263** (1983), 313–321.

[50] J. Wahl, Automorphisms and deformations of quasi-homogeneous singularities. In: Sympos. Pure. Math. **40-2**. Amer. Math. Soc., Providence, RI (1983), 613–624.

Katsusuke Nabeshima
Graduate School of Technology
Industrial and Social Sciences
Tokushima University
2-1, Minamijosanjima, Tokushima, Japan
E-mail: nabeshima@tokushima-u.ac.jp

Shinichi Tajima
Graduate School of Science and Technology
Niigata University
8050, Ikarashi 2-no-cho, Nishi-ku Niigata, Japan
E-mail: tajima@emeritus.niigata-u.ac.jp