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THE HYPERGEOMETRIC FUNCTION FOR THE ROOT SYSTEM OF TYPE A WITH A CERTAIN DEGENERATE PARAMETER

By

Nobukazu SHIMENO and Yuichi TAMAOKA

Abstract. We express explicitly the Heckman-Opdam hypergeometric function for the root system of type A with a certain degenerate parameter in terms of the Lauricella hypergeometric function.

1. Introduction

Radial parts of zonal spherical functions on real semisimple Lie groups give a class of multivariable hypergeometric functions ([9], [12]). In rank one cases they are expressed by the Gauss hypergeometric function ([12], [14]). Heckman and Opdam develop the theory of hypergeometric functions associated with root systems by generalizing zonal spherical functions ([10], [11], [21]).

On the other hand, generalizations of the classical hypergeometric functions of one-variable include hypergeometric series given by Appell, Lauricella, and Kampé de Fériet, and the hypergeometric function of matrix argument ([1, 13, 15], [3]).

It is of interest to identify these different approaches to hypergeometric functions in several variables. Sekiguchi [25, 26] shows that the zonal spherical function on $SL(n, \mathbf{R})$ with a certain degenerate parameter can be written by the Lauricella hypergeometric function F_D. Tamaoka [29] shows that the Jack polynomial with a certain degenerate parameter can be written by F_D (see Theorem 4.1 of this paper).

Beerends [2, Theorem 5.4] shows that the hypergeometric function associated with the root system of type BC with a certain degenerate parameter can be

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written by the generalized Kampé de Fériet function. Beerends and Opdam [3, Theorem 4.2] give a precise relation between the hypergeometric function of matrix argument and the hypergeometric function associated with the root system of type BC with a certain degenerate parameter.

In the present paper we express explicitly the hypergeometric function associated with the root system of type A with a certain degenerate parameter in terms of the Lauricella hypergeometric function F_D (Theorem 3.2). This result can be regarded as a generalization of a result of Sekiguchi [25, 26] for a zonal spherical function on $SL(n, \mathbf{R})$ and that of Tamaoka [29] for the Jack polynomial. The result is already indicated in [28] without proof and might be known for specialists. However, as far as we know, a precise representation formula has never been obtained in the literature.

This paper is organized as follows. In §2 we review the hypergeometric function associated with the root system of type A and define a certain degenerate parameter. In §3 we prove our main theorem by introducing a system of differential equations of second order. We remark on the case of the Jack polynomial in §4 and relate our second order differential operators with the Cherednik operators in §5.

2. Hypergeometric function for the root system of type A_{n-1}

In a series of papers starting from [10], Heckman and Opdam develop the theory of the hypergeometric function associated with a root system. In this section we review the hypergeometric function for the root system of type A_{n-1}. We refer to [11] and [21] for details.

Let n be a positive integer greater than 1 and equip $\mathbf{R}^n$ with the standard inner product $(\,,\,)$. Consider the subspace $\mathfrak{a}$ of $\mathbf{R}^n$ defined by

$$\mathfrak{a} = \{t \in \mathbf{R}^n : t_1 + \cdots + t_n = 0\}.$$

Let $\mathfrak{a}^*$ denote the space of real-valued linear functions on $\mathfrak{a}$. We identify $\mathfrak{a}^*$ with $\mathfrak{a}$ by the inner product $(\,,\,)$. Let e_i denote the element of $\mathbf{R}^n$ with i-th entry 1 and all the other entries 0. We consider the root system R of type A_{n-1},

$$R = \{e_j - e_i : 1 \leq i \neq j \leq n\}.$$

The Weyl group W of R is isomorphic to the symmetric group S_n.

Let $\mathfrak{a}_{\mathbf{C}}^*$ denote the space of complex-valued linear functions on $\mathfrak{a}$. By the correspondence $\mathfrak{a}_{\mathbf{C}}^* \ni \lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n \mapsto (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$, we have the following identification:

$$\mathfrak{a}_{\mathbf{C}}^* \simeq \{(\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n : \lambda_1 + \cdots + \lambda_n = 0\}.$$

The Weyl group $W \simeq S_n$ acts on $\mathfrak{a}_{\mathbf{C}}^*$ by $w\lambda = (\lambda_{w^{-1}(1)}, \dots, \lambda_{w^{-1}(n)})$ ($w \in W$, $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$).

Let R_+ denote the set of positive roots defined by

$$R_+ = \{e_j - e_i : 1 \le i < j \le n\}.$$

Let P_+ denote the set of dominant integral weights

$$\begin{aligned} P_+ &= \{\lambda \in \mathfrak{a}_{\mathbf{C}}^* : (\lambda, \alpha^\vee) \in \mathbf{Z}_+ \ \ (\alpha \in R_+)\} \\ &\simeq \{(\lambda_1, \dots, \lambda_n) : \lambda_1 + \cdots + \lambda_n = 0, \ \lambda_j - \lambda_i \in \mathbf{Z}_+ \ \ (1 \le i < j \le n)\}. \end{aligned}$$

Here $\mathbf{Z}_+$ denotes the set of non-negative integers and $\alpha^\vee$ the coroot of α defined by $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. For $k \in \mathbf{C}$, define

$$\rho(k) = \frac{k}{2} \sum_{\alpha \in R_+} \alpha = \left(-\frac{n-1}{2}k, -\frac{n-3}{2}k, \dots, \frac{n-1}{2}k\right). \tag{2.1}$$

Put

$$A = \exp \mathfrak{a} = \{z \in \mathbf{R}_{>0}^n : z_1 \cdots z_n = 1\} \subset \mathbf{R}_{>0}^n$$

and $\vartheta_i = z_i \dfrac{\partial}{\partial z_i}$ $(1 \le i \le n)$. The Weyl group $W \simeq S_n$ acts on A by

$$w(z_1, \dots, z_n) = (z_{w^{-1}(1)}, \dots, z_{w^{-1}(n)}).$$

We employ "GL"-picture for convenience. We consider a function ϕ on $\mathbf{R}_{>0}^n$ and impose the differential equation

$$(\vartheta_1 + \cdots + \vartheta_n)\phi = 0$$

to give a function on A. For $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ write $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$.

Define the differential operator $L(k)$ by

$$L(k) = \sum_{i=1}^n \vartheta_i^2 + k \sum_{1 \le i < j \le n} \frac{z_i + z_j}{z_i - z_j} (\vartheta_i - \vartheta_j). \tag{2.2}$$

Let $S(\mathfrak{a}_{\mathbf{C}})$ denote the symmetric algebra of $\mathfrak{a}_{\mathbf{C}}$ and $S(\mathfrak{a}_{\mathbf{C}})^W$ its subalgebra consisting of the W-invariant elements. It is known that there exist a commutative algebra $\mathbf{D}(k)$ of W-invariant differential operators containing $L(k)$ and an algebra isomorphism $\gamma : \mathbf{D}(k) \to S(\mathfrak{a}_{\mathbf{C}})^W$. In particular, we have $\gamma(L(k))(\lambda) = (\lambda, \lambda) - (\rho(k), \rho(k))$.

Remark 2.1. If $k = 1/2$, then $\mathbf{D}(k)$ is the set of the radial parts of invariant differential operators on $SL(n, \mathbf{R})/SO(n)$. For general k, the commutative algebra $\mathbf{D}(k)$ was first constructed by Sekiguchi [25, 27] giving a set of generators explicitly. In §5, we review a construction of $\mathbf{D}(k)$ by the Cherednik operators.

Let Q be the $\mathbf{Z}$-span of R and Q_+ the $\mathbf{Z}_+$-span of R_+. There exists a unique solution $\varphi(z) = \Phi(\lambda, k; z)$ on $A_+ := \{z \in A : z_1 < z_2 < \cdots < z_n\}$ for

$$L(k)\varphi = ((\lambda, \lambda) - (\rho(k), \rho(k)))\varphi \tag{2.3}$$

of the form

$$\Phi(\lambda, k; z) = \sum_{\mu \in Q_+} \Gamma_\mu(\lambda, k) z^{\lambda - \rho(k) - \mu}, \quad \Gamma_0(\lambda, k) = 1, \tag{2.4}$$

where the coefficients $\Gamma_\mu(\lambda, k)$ are rational functions in λ with possible poles at the hyperplane H_μ for some $\mu < 0$, with

$$H_\mu = \{\lambda \in \mathfrak{a}_{\mathbf{C}}^* : (2\lambda + \mu, \mu) = 0\}.$$

Moreover, $\varphi = \Phi(\lambda, k)$ also satisfies

$$D\varphi = \gamma(D)(\lambda)\varphi \quad (D \in \mathbf{D}(k)). \tag{2.5}$$

From [11, Proposition 4.2.5] the apparent simple pole of $\Phi(\lambda, k)$ along H_μ is removable unless $\mu = n\alpha$ for some $n \in -\mathbf{Z}_+$ and $\alpha \in R_+$. We call $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ generic if $(\lambda, \alpha^\vee) \notin \mathbf{Z}$ for all $\alpha \in R$. $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{a}_{\mathbf{C}}^*$ is generic if and only if $\lambda_i - \lambda_j \notin \mathbf{Z}$ $(1 \leq i < j \leq n)$. If $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ is generic, then

$$\{\Phi(w\lambda, k) : w \in W\}$$

forms a basis of the solution space of (2.5) on A_+ ([11, Corollary 4.2.6]).

Let $\Gamma(\cdot)$ denote the Gamma function. Let $\tilde{c}(\lambda, k)$ and $c(\lambda, k)$ denote the meromorphic functions defined by

$$\tilde{c}(\lambda, k) = \prod_{1 \leq i < j \leq n} \frac{\Gamma(\lambda_j - \lambda_i)}{\Gamma(\lambda_j - \lambda_i + k)}, \tag{2.6}$$

$$c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)}. \tag{2.7}$$

For $k = 1/2, 1$, or 2, $c(\lambda, k)$ agrees with Gindikin-Karpelevich's product formula for Harish-Chandra's c-function on $G/K = SL(n, \mathbf{K})/SO(n, \mathbf{K})$, where $\mathbf{K} = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$, respectively.

The denominator in (2.7) is given explicitly as follows.

$$\tilde{c}(\rho(k), k) = \prod_{1 \le i < j \le n} \frac{\Gamma((j-i)k)}{\Gamma((j-i+1)k)} = \prod_{j=2}^{n} \left(\frac{\Gamma(k)}{\Gamma(jk)} \right).$$

Let S denote the set of poles of $1/\tilde{c}(\rho(k), k)$, that is

$$S = \{k \in \mathbf{C} \backslash \mathbf{Z}_{<0} : jk \in \mathbf{Z}_{<0} \text{ for some } j = 2, 3, \dots, n\}. \tag{2.8}$$

For $k \in \mathbf{C} \backslash S$ and generic λ, define

$$F(\lambda, k) = \sum_{w \in W} c(w\lambda, k) \Phi(w\lambda, k). \tag{2.9}$$

Heckman and Opdam proved that $\varphi(z) = F(\lambda, k; z)$ extends to an entire function of $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, $k \in \mathbf{C} \backslash S$ and z in a tubular neighborhood of A in $A_{\mathbf{C}}$ and is a unique W-invariant real analytic solution of (2.5) on A such that $\phi(\mathbf{1}) = 1$ ([11, Part I, Chapter 4], [21, §6.3], [19, Corollary 4.8]). Here we put $\mathbf{1} = (1, \dots, 1) \in A$.

If $k = 1/2, 1$, or 2, then $F(\lambda, k)$ is the restriction to A of the zonal spherical function on $G/K = SL(n, \mathbf{K})/SO(n, \mathbf{K})$ with $\mathbf{K} = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$, respectively. Here A is the maximally split abelian subgroup of G with the Cartan decomposition $G = KAK$.

If $k \ge 0$ and $\mu \in P_+$, then from [11, (4.4.10)] we have

$$F(\mu + \rho(k), k) = c(\mu + \rho(k), k) P(\mu, k), \tag{2.10}$$

where $P(\mu, k)$ is the Jacobi polynomial of Heckman and Opdam.

Let n be an integer greater than 1. For $v \in \mathbf{C}$ define $\lambda(v, k) \in \mathfrak{a}_{\mathbf{C}}^*$ by

$$\lambda(v, k) = \left(-\frac{v}{n}, \dots, -\frac{v}{n}, \frac{(n-1)v}{n} \right) + \rho(k). \tag{2.11}$$

It follows from (2.7) that

$$c(\lambda(v, k), k) = \frac{\Gamma(nk)\Gamma(v + k)}{\Gamma(k)\Gamma(v + nk)}. \tag{2.12}$$

For $v \in \mathbf{Z}_+$ it can be written by the shifted factorial

$$c(\lambda(v, k), k) = \frac{(k)_v}{(nk)_v}. \tag{2.13}$$

Here the shifted factorial is defined by $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \in \mathbf{Z}_{>0}$.

Notice that $\lambda(v,k)$ is generic if and only if

$$pk \notin \mathbf{Z} \ (1 \le p \le n-2) \quad \text{and} \quad v + qk \notin \mathbf{Z} \ (1 \le q \le n-1). \tag{2.14}$$

Let W_Θ denote the permutation group of $\{1, \ldots, n-1\}$ and W^Θ denote the set of representatives of minimal length for the coset $W_\Theta \backslash W$. That is, W^Θ consists of the elements $w_1 = e$,

$$w_2 = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ 1 & 2 & \cdots & n-2 & n & n-1 \end{pmatrix}, \ldots, w_n = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & 1 & 2 & \cdots & n-1 \end{pmatrix}.$$

Here w_i $(1 \le i \le n)$ is the element of S_n of minimal length such that

$$w_i(n+1-i) = n.$$

We have the following proposition for the hypergeometric function with the degenerate parameter $\lambda(v,k)$.

PROPOSITION 2.2. (i) *Assume* $k \in \mathbf{C} \backslash S$ *and* $\lambda(v,k)$ *is generic* ((2.8), (2.14)). *Then*

$$F(\lambda(v,k),k;z) = \sum_{i=1}^{n} c(w_i\lambda(v,k),k)\Phi(w_i\lambda(v,k),k;z) \quad (z \in A_+).$$

(ii) *Assume* $v \in \mathbf{Z}_+$ *and* $qk \notin \mathbf{Z}_{<0}$ *for any* $1 \le q \le n$. *Then*

$$F(\lambda(v,k),k;z) = \frac{(k)_v}{(nk)_v}\Phi(\lambda(v,k),k;z) \quad (z \in A).$$

PROOF. First we prove (i). If $n = 2$, then (i) is just (2.9). Assume $n > 2$. Then $k \neq 0$ by (2.14). If $w \in W \backslash W^\Theta$, then there exists l $(1 \le l \le n-2)$ such that $w(l) > w(l+1)$. Then $w\lambda(v,k)_j - w\lambda(v,k)_i = \lambda(v,k)_l - \lambda(v,k)_{l+1} = -k$ for $j = w(l)$ and $i = w(l+1)$. By the definition (2.6) and (2.7) of the c-function, $c(w\lambda(v,k),k) = 0$ unless $w \in W^\Theta$. Hence (i) is proved.

It follows from (2.7) that

$$c(w_i\lambda(v,k),k) = \frac{\Gamma(nk)\Gamma(v+ik)\Gamma(-v-(i-1)k)}{\Gamma(k)\Gamma(v+nk)\Gamma(-v)} \quad (2 \le i \le n).$$

Thus $c(w_i\lambda(v,k),k) = 0$ $(2 \le i \le n)$ for $v \in \mathbf{Z}_+$ and the equality of (ii) holds on A_+ under the assumption of (i). Notice that $\lambda(v,k) - \rho(k) \in P_+$ if and only if

$\nu \in \mathbf{Z}_+$. Thus we have $\Phi(\lambda(\nu,k),k) = P(\lambda(\nu,k) - \rho(k),k)$ and (ii) follows from (2.10) and (2.12) by analytic continuation. $\square$

In §3 we will show that $F(\lambda(\nu,k),k)$ can be written by the Lauricella hypergeometric function F_D.

3. A system of hypergeometric differential equations of second order

Let n be an integer greater than 1 and k a complex number. Let $z = (z_1, z_2, \dots, z_n)$ denote a variable in $\mathbf{R}^n$ and put $\vartheta_i = z_i \dfrac{\partial}{\partial z_i}$ $(1 \le i \le n)$. For $1 \le i < j \le n$ define the differential operator Δ_{ij} by

$$\Delta_{ij} = \vartheta_i\vartheta_j - \frac{k}{2}\frac{z_i + z_j}{z_i - z_j}(\vartheta_i - \vartheta_j) + \left(\frac{\nu}{n} + \frac{k}{2}\right)(\vartheta_i + \vartheta_j). \tag{3.1}$$

We consider the system of differential equations

$$(\vartheta_1 + \cdots + \vartheta_n)\varphi = 0, \tag{3.2}$$

$$\Delta_{ij}\varphi = -\frac{\nu(\nu + nk)}{n^2}\varphi \quad (1 \le i < j \le n). \tag{3.3}$$

By summing up (3.3) for $1 \le i < j \le n$ and using (3.2) we have

$$\left(\sum_{1 \le i < j \le n} \vartheta_i\vartheta_j - \frac{k}{2}\frac{z_i + z_j}{z_i - z_j}(\vartheta_i - \vartheta_j)\right)\varphi = -\frac{(n-1)\nu(\nu + nk)}{2n}\varphi. \tag{3.4}$$

The differential operator in the left hand side of (3.4) is equivalent to $-\frac{1}{2}$ times the second order hypergeometric differential operator $L(k)$ given by (2.2) and the coefficients of φ in the right hand side of (3.4) is $-\frac{1}{2}\{(\lambda(\nu,k),\lambda((\nu,k)) - (\rho(k),\rho(k))\}$. Hence (3.4) is equivalent to (2.3) with $\lambda = \lambda(\nu,k)$.

If $k = 1/2$, then (3.3) are radial parts of the differential equations satisfied by the zonal spherical function on $G = SL(n,\mathbf{R})$ expressed by the Poisson integral of a $SO(n)$-invariant section of a degenerate principal series representation on G/P_Θ, where P_Θ is a maximal parabolic subgroup of G whose Levi part is isomorphic to $GL(n-1,\mathbf{R})$. These differential equations on G are given by Oshima [22] using generalized Capelli operators in the universal enveloping algebra $U(\mathfrak{gl}(n,\mathbf{C}))$. Moreover, (3.4) is the radial part of the differential equation corresponding to the Casimir operator.

By the change of variables

$$y_i = \frac{z_i}{z_n} \quad (1 \le i \le n-1), \quad y_n = z_n, \tag{3.5}$$

we have

$$\vartheta_i = y_i \frac{\partial}{\partial y_i} \quad (1 \le i \le n-1), \quad \vartheta_1 + \cdots + \vartheta_n = y_n \frac{\partial}{y_n}.$$

Hence, (3.2) means that φ does not depend on y_n. By the change of variables, (3.1) for $1 \le i < j \le n$ become

$$\Delta_{ij} = \vartheta_i \vartheta_j - \frac{k}{2} \frac{y_i + y_j}{y_i - y_j} (\vartheta_i - \vartheta_j) + \left(\frac{v}{n} + \frac{k}{2} \right) (\vartheta_i + \vartheta_j) \quad (1 \le i < j \le n-1), \tag{3.6}$$

$$\Delta_{in} = \vartheta_i (\vartheta_1 + \cdots + \vartheta_{n-1}) - \frac{k}{2} \frac{y_i + 1}{y_i - 1} (\vartheta_i + \vartheta_1 + \cdots + \vartheta_{n-1}) + \left(\frac{v}{n} + \frac{k}{2} \right) (\vartheta_i - (\vartheta_1 + \cdots + \vartheta_{n-1})) \quad (1 \le i \le n-1). \tag{3.7}$$

Here we write $\vartheta_i = y_i \frac{\partial}{\partial y_i}$ $(1 \le i \le n-1)$ in (3.6) and (3.7). Putting

$$\varphi(y_1, \ldots, y_{n-1}) = (y_1 \cdots y_{n-1})^{-v/n} u(y_1, \ldots, y_{n-1}), \tag{3.8}$$

the differential equations (3.3) become the following differential equations for u:

$$\left(\vartheta_i \vartheta_j - k \frac{y_j \vartheta_i - y_i \vartheta_j}{y_i - y_j} \right) u = 0 \quad (1 \le i < j \le n-1), \tag{3.9}$$

$$\left(-\left(\vartheta_1 + \cdots + \vartheta_{n-1} - v + \frac{k}{y_i - 1} \right) \vartheta_i - \frac{k y_i}{y_i - 1} (\vartheta_1 + \cdots + \vartheta_{n-1} - v) \right) u = 0 \quad (1 \le i \le n-1), \tag{3.10}$$

which can be written in the following form:

$$y_i (\vartheta_i + k) \vartheta_j u = y_j (\vartheta_j + k) \vartheta_i u \quad (1 \le i < j \le n-1), \tag{3.11}$$

$$\vartheta_i (\vartheta_1 + \cdots + \vartheta_{n-1} - v - k) u = y_i (\vartheta_i + k)(\vartheta_1 + \cdots + \vartheta_{n-1} - v) u \quad (1 \le i \le n-1). \tag{3.12}$$

We recall the Lauricella hypergeometric function F_D of $n-1$ variables and the corresponding system E_D of differential equations of rank n. The Lauricella hypergeometric function F_D is the analytic continuation of the series

$$F_D(\alpha, \beta_1, \dots, \beta_{n-1}, \gamma;\, y_1, \dots, y_{n-1}) = \sum_{m_1, \dots, m_{n-1} \geq 0} \frac{(\alpha)_{m_1+\cdots+m_{n-1}} (\beta_1)_{m_1} \cdots (\beta_{n-1})_{m_{n-1}}}{(\gamma)_{m_1+\cdots+m_{n-1}} m_1! \cdots m_{n-1}!} y_1^{m_1} \cdots y_{n-1}^{m_{n-1}}, \tag{3.13}$$

where $\alpha, \beta_1, \dots, \beta_{n-1}, \gamma$ are complex constants with $\gamma \notin \mathbf{Z}_{<0}$. It satisfies the following system of differential equations:

$$\begin{cases} y_i(\vartheta_i + \beta_i)\vartheta_j F = y_j(\vartheta_j + \beta_j)\vartheta_i F \quad (1 \leq i < j \leq n-1), \\ \vartheta_i(\vartheta_1 + \cdots + \vartheta_{n-1} + \gamma - 1)F \\ \quad = y_i(\vartheta_i + \beta_i)(\vartheta_1 + \cdots + \vartheta_{n-1} + \alpha)F \quad (1 \leq i \leq n-1). \end{cases} \tag{E_D}$$

The system (E_D) is holonomic of rank n. If $\gamma \notin \mathbf{Z}_{<0}$, then the Lauricella hypergeometric function $F_D(\alpha, \beta_1, \dots, \beta_{n-1}, \gamma; y_1, \dots, y_{n-1})$ is the unique analytic solution of (E_D) such that $F(0) = 1$. We refer to [15], [13, §9.1], and [18] for details.

Equations (3.11) and (3.12) constitute (E_D) with

$$\alpha = -v, \quad \beta_1 = \cdots = \beta_{n-1} = k, \quad \gamma = -v - k + 1. \tag{3.14}$$

By the change of variables

$$x_i = 1 - y_i \quad (1 \leq i \leq n-1), \tag{3.15}$$

(3.11) give equations of the same form

$$x_i(\vartheta_i + k)\vartheta_j u = x_j(\vartheta_j + k)\vartheta_i u \quad (1 \leq i < j \leq n-1). \tag{3.16}$$

Here we write $\vartheta_i = x_i \frac{\partial}{\partial x_i}$ $(1 \leq i \leq n-1)$. By (3.12) and (3.16), we have the following equations.

$$\vartheta_i(\vartheta_1 + \cdots + \vartheta_{n-1} + nk - 1)u = x_i(\vartheta_i + k)(\vartheta_1 + \cdots + \vartheta_{n-1} - v)u \quad (1 \leq i \leq n-1). \tag{3.17}$$

Equations (3.16) and (3.17) constitute (E_D) with

$$\alpha = -v, \quad \beta_1 = \cdots = \beta_{n-1} = k, \quad \gamma = nk. \tag{3.18}$$

Consequently, if $nk \notin \mathbf{Z}_{<0}$, then

$$\varphi(y) = (y_1 \cdots y_{n-1})^{-v/n} F_D(-v, k, \dots, k, nk;\, 1 - y_1, \dots, 1 - y_{n-1}) \tag{3.19}$$

is the unique analytic solution for (3.2) and (3.3) satisfying $\varphi(1, \dots, 1) = 1$.

The symmetric group S_n acts on the variable $z = (z_1, \dots, z_n)$ as permutations, hence it acts on the variable $y = (y_1, \dots, y_{n-1})$.

Lemma 3.1. *The function φ given by* (3.19) *is S_n-invariant.*

Proof. Recall that $y_i = z_i/z_n$ for $1 \le i \le n-1$. The transposition $(i, i+1)$ $(1 \le i \le n-2)$ interchanges y_i and y_{i+1}. Since $\beta_1 = \cdots = \beta_{n-1} = k$, $\varphi(y)$ is invariant under the transposition $(i, i+1)$ $(1 \le i \le n-2)$.

By the transposition $(n-1, n)$, $y_1, \dots, y_{n-2}, y_{n-1}$ change to $y_1/y_{n-1}, \dots, y_{n-2}/y_{n-1}, 1/y_{n-1}$, respectively. It follows from the transformation formula ([15, p. 149])

$$F_D(\alpha, \beta_1, \dots, \beta_{n-1}, \gamma; x_1, \dots, x_{n-1})$$

$$= (1 - x_{n-1})^{-\alpha} \times F_D\Bigg(\alpha, \beta_1, \dots, \beta_{n-2}, \gamma - \beta_1 - \cdots - \beta_{n-1}, \gamma; \frac{x_{n-1} - x_1}{x_{n-1} - 1}, \dots, \frac{x_{n-1} - x_{n-2}}{x_{n-1} - 1}, \frac{x_{n-1}}{x_{n-1} - 1}\Bigg)$$

that $\varphi(y)$ is invariant under the transposition $(n-1, n)$. Since S_n is generated by the transpositions $(1,2), (2,3), \dots, (n-1, n)$, the lemma is proved. □

Now we state the main result of this paper, which asserts that the hypergeometric function of type A_{n-1} with parameter $\lambda(v, k)$ defined by (2.11) is written by F_D as in (3.19). In the case of $k = 1/2$, that is the case of the zonal spherical function on $SL(n, \mathbf{R})/SO(n)$, this theorem is given by Sekiguchi [25, 26].

Theorem 3.2. *Assume $k \in \mathbf{C} \backslash S$ and $v \in \mathbf{C}$. Then we have*

$$F(\lambda(v, k), k; z) = (y_1 \cdots y_{n-1})^{-v/n} F_D(-v, k, \dots, k, nk; 1 - y_1, \dots, 1 - y_{n-1}),$$

where $\lambda(v, k)$ and y_i are given by (2.11) *and* (3.5).

Proof. For $2 \le j \le n$, there exists a unique series solution $u_j(y)$ with the leading term $\prod_{i=n-j+2}^{n-1} y_i^{-k} y_{n-j+1}^{(j-1)k+v}$ for the system (E_D) with (3.14) that converges on a neighbourhood of $y = 0$ in $\{y \in \mathbf{R}^{n-1} : 0 < y_1 < y_2 < \cdots < y_{n-1}\}$. Moreover, the set of $u_1(y) := F_D(-v, k, \dots, k, -v - k + 1; y)$ and $u_j(y)$ $(2 \le j \le n)$ forms a basis of local solutions of E_D for generic k and v ([7, Section 3.3.1 (f)], [24, Section 1.5], [8, Section 5]).

For $1 \le i \le n$, let φ_i denote the solution of the system of equations (3.2) and (3.3) corresponding to u_i by (3.5) and (3.8). Then φ_i is a solution of (3.4) with the characteristic exponent $w_i \lambda(v, k) - \rho(k)$ and the leading coefficient 1. Thus $\varphi_i =$

$\Phi(w_i\lambda(v,k),k)$ and it is a solution of the hypergeometric system (2.5) with $\lambda = \lambda(v,k)$. Since $\{\varphi_i : 1 \leq i \leq n\}$ forms a basis of the solution space of the system of equations (3.2) and (3.3) for generic k and v, φ given by (3.19) is a solution of (2.5) with $\lambda = \lambda(v,k)$ for any $k \in \mathbf{C}\backslash S$ and $v \in \mathbf{C}$ by analytic continuation.

Thus φ is a S_n invariant solution of (2.5) with $\lambda = \lambda(v,k)$ that is real analytic and $\varphi(1,\dots,1) = 1$. Hence $\varphi(z) = F(\lambda(v,k),k,z)$ by the uniqueness of the hypergeometric function. □

Example 3.3. We give examples of A_1 and A_2. Assume $k \in \mathbf{C}\backslash S$.

First we consider the case of A_1. The Lauricella hypergeometric function F_D of one variable is the Gauss hypergeometric function ${}_2F_1$. From Proposition 2.2 and Theorem 3.2, it holds on A_+ that

$$\begin{aligned} F(\lambda(v,k),k;z) &= y_1^{-v/2}{}_2F_1(-v,k,2k;1-y_1) \\ &= \frac{\Gamma(2k)\Gamma(v+k)}{\Gamma(k)\Gamma(v+2k)} y_1^{-v/2}{}_2F_1(-v,k,-v-k+1;y_1) \\ &\quad + \frac{\Gamma(2k)\Gamma(-v+k)}{\Gamma(k)\Gamma(-v)} y_1^{v/2+k}{}_2F_1(v+2k,k,v+k+1;y_1). \end{aligned}$$

The above equalities are well-known formulae for the hypergeometric function of type A_1 and the Gauss hypergeometric function ([21, Example 6.3], [11, proof of Theorem 4.3.6], [6]). Note that $F(\lambda(v,k),k)$ can be written by the Jacobi function ([14])

$$\begin{aligned} F(\lambda(v,k),k;z) &= \phi^{(k-1/2,k-1/2)}_{2\sqrt{-1}(v+k)}\left(\frac{t}{2}\right) \\ &:= {}_2F_1\left(-v, v+2k, k+\frac{1}{2}; -\sinh^2\frac{t}{2}\right), \end{aligned} \tag{3.20}$$

where $z = (e^t, e^{-t})$ and $y_1 = e^{2t}$. If $v \in \mathbf{Z}_+$, then

$$F(\lambda(v,k),k;z) = \frac{(k)_v}{(2k)_v} y_1^{-v/2}{}_2F_1(-v,k,-v-k+1;y_1)$$

and from (3.20)

$$F(\lambda(v,k),k;z) = \frac{v!}{(2k)_v} C_v^{(k)}(\cosh t),$$

where $C_v^{(k)}$ denote the Gegenbauer polynomial ([6, §3.15.1]).

Next we consider the case of A_2. The Lauricella hypergeometric function F_D of two variables is the Appell hypergeometric function F_1. From Proposition 2.2 and Theorem 3.2, it holds on A_+ that

$$
\begin{aligned}
F(\lambda(\nu,k),k;z) &= (y_1y_2)^{-\nu/3}F_1(-\nu,k,k,3k;1-y_1,1-y_2) \\
&= \frac{\Gamma(3k)\Gamma(\nu+k)}{\Gamma(k)\Gamma(\nu+3k)}(y_1y_2)^{-\nu/3}F_1(-\nu,k,k,-\nu-k+1;y_1,y_2) \\
&\quad + \frac{\Gamma(3k)\Gamma(\nu+2k)\Gamma(-\nu-k)}{\Gamma(k)\Gamma(\nu+3k)\Gamma(-\nu)} \\
&\quad \times y_1^{-\nu/3}y_2^{2\nu/3+k}G_2\left(k,k,\nu+2k,-\nu-k;-\frac{y_1}{y_2},-y_2\right) \\
&\quad + \frac{\Gamma(3k)\Gamma(-\nu-2k)}{\Gamma(k)\Gamma(-\nu)} \\
&\quad \times (y_1^{-2}y_2)^{-\nu/3-k}F_1\left(\nu+3k,k,k,\nu+2k+1;\frac{y_1}{y_2},y_1\right).
\end{aligned}
$$

Here

$$G_2(\alpha,\alpha',\beta,\beta',x,y) = \sum_{m,n=0}^{\infty}(\alpha)_m(\alpha')_n(\beta)_{n-m}(\beta')_{m-n}\frac{x^my^n}{m!n!},$$

where $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$. The above equality involving F_1 and G_2 is a special case of [20, (19)]. If $\nu \in \mathbf{Z}_+$, then

$$F(\lambda(\nu,k),k;z) = \frac{(k)_\nu}{(3k)_\nu}(y_1y_2)^{-\nu/3}F_1(-\nu,k,k,-\nu-k+1;y_1,y_2)$$

or

$$P(\lambda(\nu,k)-\rho(k),k;z) = (y_1y_2)^{-\nu/3}F_1(-\nu,k,k,-\nu-k+1;y_1,y_2),$$

where $P(\mu,k;z)$ is the Jacobi polynomial of Heckman and Opdam.

4. The case of $\nu \in \mathbf{Z}_+$

In this section assume that $k > 0$ and $\nu \in \mathbf{Z}_+$. Put

$$\mu(\nu) = \left(-\frac{\nu}{n},\dots,-\frac{\nu}{n},\frac{(n-1)\nu}{n}\right). \tag{4.1}$$

Then it follows from Proposition 2.2 and Theorem 3.2 that

$$P(\mu(v), k; z)$$
$$= \frac{(nk)_v}{(k)_v}(y_1 \cdots y_{n-1})^{-v/n} F_D(-v, k, \dots, k, nk;\ 1 - y_1, \dots, 1 - y_{n-1})$$

and

$$P(\mu(v), k; z) = (y_1 \cdots y_{n-1})^{-v/n} F_D(-v, k, \dots, k, -v - k + 1;\ y_1, \dots, y_{n-1}),$$

where $P(\mu(v), k)$ is the Jacobi polynomial of Heckman and Opdam.

The Jacobi polynomial for the root system of type A_{n-1} is essentially the Jack polynomial. A partition λ of length equal or less than n is a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Define $|\lambda| = \sum_{i=1}^n \lambda_i$. For two partitions λ and μ we write $\mu \leq \lambda$ if $|\mu| = |\lambda|$ and $\sum_{i=1}^j \mu_j \leq \sum_{i=1}^j \lambda_i$ for all $j \geq 1$.

For a partition λ of length equal or less than n define the monomial symmetric function m_λ by

$$m_\lambda = \sum_{\alpha \in S_n \lambda} x^\alpha.$$

There exists a unique $P_\lambda^{(1/k)}$ that satisfies the following conditions:

$$P_\lambda^{(1/k)} = \sum_{\mu \leq \lambda} v_{\lambda\mu} m_\mu \quad v_{\lambda\mu} \in \mathbf{C}(k), \quad v_{\lambda\lambda} = 1, \tag{4.2}$$

$$L(k) P_\lambda^{(1/k)} = h(\lambda) P_\lambda^{(1/k)}, \quad h(\lambda) = \sum_{i=1}^n \lambda_i(\lambda_i + k(n + 1 - 2i)). \tag{4.3}$$

We call $P_\lambda^{(1/k)}(z)$ the Jack polynomial ([16, 17]). From [3, Proposition 3.3] we have

$$P_\lambda^{(1/k)}(z) = P(\pi(\lambda), k; z) \quad (z \in A) \tag{4.4}$$

for a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of length equal or less than n. Here $\pi(\lambda) \in P_+$ is given by

$$\pi(\lambda) = \sum_{i=1}^n \lambda_i e_i - \frac{1}{n}\left(\sum_{i=1}^n \lambda_i\right)\left(\sum_{i=1}^n e_i\right).$$

Thus we have the following result as a corollary of Theorem 3.2.

Theorem 4.1 (Tamaoka [29]). *Assume $k > 0$, $p, q \in \mathbf{Z}_+$ and $p \geq q$. Then for $z \in \mathbf{C}^n$*

$$P^{(1/k)}_{(p,q,\dots,q)}(z_1,\dots,z_n)$$
$$= \frac{(nk)_{p-q}}{(k)_{p-q}}\left(\prod_{i=1}^{n-1} z_i^q z_n^p\right) F_D\left(q-p, k,\dots,k, nk;\ 1-\frac{z_1}{z_n},\dots,1-\frac{z_{n-1}}{z_n}\right)$$
$$= \left(\prod_{i=1}^{n-1} z_i^q z_n^p\right) F_D\left(q-p, k,\dots,k, q-p-k+1;\ \frac{z_1}{z_n},\dots,\frac{z_{n-1}}{z_n}\right).$$

Remark 4.2. In [29], the second author proves Theorem 4.1 without using the Heckman-Opdam theory. He just used the characterization of the Jack polynomial by the conditions (4.2), (4.3) and properties of the Lauricella hypergeometric function.

In view of (2.10) and (4.4), Theorem 4.1 asserts that Theorem 3.2 holds for $k > 0$ and $\nu \in \mathbf{Z}_+$. We can deduce Theorem 3.2 from Theorem 4.1 in the same manner as the proof of [3, Theorem 4.2].

5. Cherednik operators

First we review the Cherednik operator in the GL_n case ([4, 21]). For $1 \leq i \leq n$, define the Cherednik operator T_i ([4, §3.5]) by

$$T_i = \vartheta_i + k\sum_{i<j}\frac{z_i}{z_i - z_j}(1-\sigma_{ij}) + k\sum_{i>j}\frac{z_j}{z_i - z_j}(1-\sigma_{ij}) + \rho(k)_i.$$

Here σ_{ij} is the permutation (ij) that acts as the transposition of the coordinates z_i and z_j. Notice that the choice of the positive system of R to define the Cherednik operators is opposite to that of [4, §3.5]. We take $-R_+$ as the positive system of R to define T_i.

The Cherednik operators satisfy the following relations:

$$[T_i, T_j] = 0 \quad (1 \leq i, j \leq n), \quad \sigma_i T_j = T_j \sigma_i \quad (j \neq i, i+1), \tag{5.1}$$

$$\sigma_i T_i - T_{i+1}\sigma_i = -k, \tag{5.2}$$

where $\sigma_i = \sigma_{ii+1}$ $(1 \leq i \leq n-1)$. (5.1) and (5.2) are the defining relations of the degenerate affine Hecke algebra $\mathbf{H} = \langle \mathbf{C}S_n, x_1, \dots, x_n\rangle$ of type GL_n, if we replace T_i by x_i in the above relations.

For $p \in S(\mathfrak{a}_{\mathbf{C}})$, put

$$T_p = p(T_1, \ldots, T_n).$$

Write

$$T_p = \sum_{w \in W} D_w^{(p)} w,$$

where $D_w^{(p)}$ $(w \in W)$ are differential operator on A and define the differential operator D_p on A by

$$D_p = \sum_{w \in W} D_w^{(p)}.$$

D_p is the differential operator that has the same restriction to symmetric functions as T_p.

For $1 \leq m \leq n$, let $e_m(x)$ denote the m-th elementary symmetric polynomial:

$$e_m(x) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} x_{i_1} \cdots x_{i_m}.$$

Then we have

$$D_{e_1} = \vartheta_1 + \cdots + \vartheta_n$$

and

$$D_{e_2} = \sum_{1 \leq i < j \leq n} \vartheta_i \vartheta_j - \frac{k}{2} \frac{z_i + z_j}{z_i - z_j} (\vartheta_i - \vartheta_j) - \frac{k^2}{4} \binom{n+1}{3},$$

which are differential operators in (3.2) and (3.4), respectively. For $p(x) = x_1^2 + \cdots + x_n^2$, $D_p = L(k) + (\rho(k), \rho(k))$, where $L(k)$ is defined in (2.2). The commutative algebra $\mathbf{D}(k)$ mentioned in §2 is given by

$$\mathbf{D}(k) = \{D_p : p \in S(\mathfrak{a}_{\mathbf{C}})^W\}$$

and has generators $\{L_{e_1}, \ldots, L_{e_n}\}$. Moreover, the algebra isomorphism $\gamma : \mathbf{D}(k) \to S(\mathfrak{a}_{\mathbf{C}})^W$ mentioned in §2 is defined by $D_p \mapsto p$.

The following proposition asserts that the differential operator Δ_{ij} given in §3 (3.3) is also related with the Cherednik operators.

PROPOSITION 5.1 ([28]). *For* $1 \le i < j \le n$, *define* $p_{ij} \in S(\mathfrak{a}_{\mathbf{C}})$ *by*

$$p_{ij}(x) = \left(x_i - \rho(k)_i + \frac{v}{n}\right)\left(x_j - \rho(k)_j + k + \frac{v}{n}\right).$$

Then we have

$$D_{p_{ij}} = \Delta_{ij} + \frac{v(v+nk)}{n^2}.$$

REMARK 5.2. Though the choice of the positive system to define the Cherednik operators is not essential, we choose as above because it matches better with the characteristic exponents $w_i\lambda(v,k) - \rho(k)$ $(1 \le i \le n)$ given in §2 and the indicial equation

$$\left(\mu_i - \rho(k)_i + \frac{v}{n}\right)\left(\mu_j - \rho(k)_j + k + \frac{v}{n}\right) = 0 \tag{5.3}$$

of (3.3) at infinity on A_+. If

$$v \neq -k, -2k, \dots, (-n+1)k,$$

then the set of common solutions for (5.3) for $1 \le i < j \le n$ is

$$\{w_1\lambda(v,k), \dots, w_n\lambda(v,k)\},$$

where $\lambda(v,k)$ and w_j are given in §2.

This fact can be regarded as a special case of [23, Theorem 9, Equation (27), Theorem 22]. Indeed, in [22, 23], Oshima constructed generators of annihilators of generalized Verma modules for $\mathfrak{gl}_n$ by using generalized Capelli operators. The deformation parameter ε in [23] corresponds to k in this paper. Using results in [23], some part of results in this paper can be generalized to the case of arbitrary $\Theta \subset \{1, 2, \dots, n\}$ as indicated in [28]. We will discuss in detail elsewhere.

REMARK 5.3. After we have finished our work, we noticed that the system of differential equations (3.2), (3.3) and its characteristic exponents are stated in [5]. [5, Theorem 3.3] asserts that the system (3.2), (3.3) is of rank n and its solutions are also solutions of the hypergeometric system (2.5) with $\lambda = \lambda(v,k)$ without proof.

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References

[1] Appell, P. and Kampé de Fériet, J., Fonctions hypergéométriques et hypersphériques: polynomes d'Hermite, Paris: Gauthier-Villars, 1926.

[2] Beerends, R. J., Some special values for the BC type hypergeometric function, Contemp. Math. **138** (1992), 27–49.

[3] Beerends, R. J. and Opdam, E. M., Certain hypergeometric series related to the root system BC, Trans. Amer. Math. Soc. **339** (1993), 581–609.

[4] Cherednik, I., Lectures on Knizhnik-Zamolodchikov equations and Hecke algebras, MSJ Memoirs **1**, Mathematical Society of Japan, 1998, 1–96.

[5] Couwenberg, W., Heckman, G. and Looijenga, E., On the geometry of the Calogero-Moser system, Indag. Math. N.S, **16** (2005), 443–459.

[6] Erdélyi, A. ed., Higher Transcendental Functions, Vol. 1, McGraw Hill, New York, 1953.

[7] Gel'fand, I. M., Zelevinsky, A. V. and Kapranov, M. M., Hypergeometric functions and toral manifolds, Funct. Anal. Appl. **23** (1989), 94–106.

[8] Goto, Y., Contiguity relations of Lauricella's F_D revisited, Tohoku Math. J. **69** (2017), 287–304.

[9] Harish-Chandra, Spherical functions on a semisimple Lie group I, Amer. J. Math. **80** (1958), 241–310.

[10] Heckman, G. J. and Opdam, E. M., Root systems and hypergeometric functions I, Comp. Math. **64** (1987), 329–352.

[11] Heckman, G. J., Hypergeometric and Spherical Functions, In: Harmonic Analysis and Special Functions on Symmetric Spaces, Perspect. Math., Academic Press, Boston, MA, 1994.

[12] Helgason, S., Groups and Geometric Analysis, Amer. Math. Soc., 2000, c1984.

[13] Iwasaki, K., Kimura, H., Shimomura, S. and Yoshida, M., From Gauss to Painlevé: A Modern Theory of Special Functions, Springer, 1991.

[14] Koornwinder, T., Jacobi functions and analysis on noncompact semisimple Lie groups. In: Special Functions: Group Theoretical Aspects and Applications, 1–85, Math. Appl., Reidel, Dordrecht, 1984.

[15] Lauricella, G., Sulle Funzioni ipergeometrche a piu variabili, Rend. Circ. Math. Palermo **7** (1893), 111–158.

[16] Macdonald, I. G., Commuting differential operators and zonal spherical functions, Algebraic groups, Utrecht 1986, 189–200, Lecture Notes in Math., **1271**, Springer, Berlin, 1987.

[17] Macdonald, I. G., Symmetric Functions and Hall Polynomials, 2nd ed., Oxford University Press, 1995.

[18] Mimachi, K. and Sasaki, T., Irreducibility and reducibility of Lauricella's system of differential equations E_D and the Jordan-Pochhammer differential equation E_{JP}, Kyushu J. Math. **66** (2012), 61–87.

[19] Oda, H. and Shimeno, N., Spherical functions for small K-types, arXiv:1710.02975.

[20] Olsson, P. O. M., Integration of the partial differential equations for the hypergeometric functions F_1 and F_D of two and more variables, J. Math. Phys. **5** (1964), 420–430.

[21] Opdam, E. M., Lecture notes on Dunkl operators for real and complex reflection groups, MSJ Memoirs **8**, Mathematical Society of Japan, 2000.

[22] Oshima, T., Generalized Capelli identities and boundary value problems for $GL(n)$, Structure of Solutions of Differential Equations, World Scientific, 1996, 307–335.

[23] Oshima T., A quantization of conjugacy classes of matrices, Adv. Math. **196** (2005), 124–146.

[24] Saito, M., Sturmfels, B. and Takayama, N., Gröbner Deformations of Hypergeometric Differential Equations, Springer-Verlag, Berlin, 2000.

[25] Sekiguchi, J., Zonal spherical functions on the symmetric space $SL(3,\mathbf{R})/SO(3)$ and related topics (in Japanese), Master Dissertation, Nagoya University, 1976.

[26] Sekiguchi, J., Zonal spherical functions on $SL(3,\mathbf{R})$ (in Japanese), RIMS Kokyuroku **266** (1976), 259–274.

[27] Sekiguchi, J., Zonal spherical functions on some symmetric spaces, Publ. RIMS. Kyoto Univ., **12** Suppl. (1977), 455–464.
[28] Shimeno, N., Factorization of invariant differential equations, Proceeding of the Symposium on Representation Theory held at Hadomisaki, Saga prefecture, Japan (1997), 54–58.
[29] Tamaoka, Y., Jack polynomials and Lauricella's hypergeometric series (in Japanese), Master Dissertation, Kwansei Gakuin University, 2018.

School of Science and Technology
Kwansei Gakuin University
2-1 Gakuen, Sanda, Hyogo 669-1337, Japan
E-mail: shimeno@kwansei.ac.jp

Graduate School of Science and Technology
Kwansei Gakuin University
2-1 Gakuen, Sanda, Hyogo 669-1337, Japan