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EXTENDING FUNCTORS FROM THE CATEGORY OF STRICT MORPHISMS OF INVERSE SYSTEMS TO THE ASSOCIATED PRO-CATEGORY WITH APPLICATIONS TO THE FIRST DERIVED LIMIT

By

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Abstract. We show that functors on the category of strict morphisms of inverse systems which are indexed by arbitrary cofiltered small categories have at most one extension to the associated pro-category and give conditions characterizing the existence of extensions. This is applied to provide a concrete extension of the first derived limit to the category of pro-groups.

1. Introduction

To any category **C** one can associate the category of inverse systems **inv-C** and the pro-category **pro-C**. A good reference is [8]. In the most general form inverse systems are indexed by cofiltered small categories. Many authors restrict to directed preordered sets as index categories which is a substantial simplification. The justification is the following reindexing principle which "improves" inverse systems: For each inverse system **X** indexed by a cofiltered small category there exists an isomorphism $\mathbf{f} : \mathbf{X} \to \mathbf{X}'$ in **pro-C** such that $\mathbf{X}'$ is indexed by a cofinite directed ordered set. Cofiniteness enables induction on the number of predecessors which is an essential technique in many proofs.

Working with these more special inverse systems is sufficient for most purposes. There are, however, questions where this approach appears inappropriate. Many important constructions for inverse systems (e.g. derived limits) are primarily not concerned with morphisms, but typically have natural continuations to functors living on the subcategory $\mathbf{lev\text{-}C} \subset \mathbf{inv\text{-}C}$ of level morphisms. Finding

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pro-extensions to functors living on **pro-C** is a highly non-trivial task and it is not expedient to restrict to any sort of special inverse systems.

As a further challenge some interesting functors F on **lev-C** (e.g. the derived limits) have a completely natural extension to functors F_{str} living on the subcategory **str-C** $\subset$ **inv-C** of *strict morphisms* which are the most elementary morphisms of **inv-C** in that they satisfy evident strict commutativity requirements (see Section 2). Here are some obvious questions.

(1) Does F_{str} have a pro-extension? More precisely, under what conditions does there exist a pro-extension?
(2) Are pro-extensions of F_{str} unique?
(3) If we have directly constructed a pro-extension of F from **lev-C** to **pro-C**, we get an induced extension F' of F to **str-C**. Does F' agree with the natural extension F_{str}?

In this paper we develop the machinery to address these questions. We generalize some classical results for inverse systems indexed by directed preordered sets to arbitrary inverse systems. In particular we show that in the realm of cofinite index categories all pro-morphisms can be represented by strict morphisms which is a basic prerequisite for most proofs. In Section 5 we focus on level morphisms and show that for a cofinite A the canonical functor $\Pi : \mathbf{C}^A \to \mathbf{pro\text{-}C}_A$ is a localization at a certain class of level morphisms which means in particular that functors on $\mathbf{C}^A$ have at most one pro-extension to $\mathbf{pro\text{-}C}_A$. In Section 6 we show that functors on **str-C** have at most one pro-extension to **pro-C** (which answers question (2) in the affirmative) and give criteria for their existence (which answers question (1)). In Section 8 we apply this to the first derived limit $\varprojlim^1$ and show that it has a unique pro-extension from **str-G** to **pro-G** ($\mathbf{G}$ = category of groups). In Section 9 we briefly discuss the abelian case and show that all derived limits $\varprojlim^n$ have a unique pro-extension from **str-AG** to **pro-AG** ($\mathbf{AG}$ = category of abelian groups) which generalizes previous results by Watanabe [10] and Mardešić [9].

For the derived limits the existence of pro-extensions from **lev-C** ($\mathbf{C} = \mathbf{G}, \mathbf{AG}$) to **pro-C** is well-known. For $n = 1$ and $\mathbf{C} = \mathbf{G}$ this is based on the topological description of $\varprojlim^1$ via the homotopy limit on **pro-SS**[1]; see e.g. [4]. For $\mathbf{C} = \mathbf{AG}$ the functors $\varprojlim^n$ occur as the right derived functors of $\varprojlim : \mathbf{pro\text{-}AG} \to \mathbf{AG}$ and are thus uniquely determined by this property. All this is based on completely

[1] This entails a certain vagueness because the homotopy limit depends on the precursory construction of a closed model structure on **pro-SS** ($\mathbf{SS}$ = category of simplicial sets). In the literature one can find various different constructions; see e.g [7].

natural "systemic" constructions, but it is a priori not clear how the resulting pro-extensions of $\varprojlim^n$ from **lev-C** to **pro-C** are related to the natural extensions of $\varprojlim^n$ from **lev-C** to **str-C**, i.e. we do not get answers to questions (1) and (3). In [10] and [9] one finds positive answers for the abelian case and directed preordered index categories; we generalize this to arbitrary index categories. In the non-abelian case the questions have never been addressed so far. We answer question (1) in the affirmative (Theorem 8.1); concerning question (3) we have partial results (Theorem 8.3).

2. Pro-categories

We recapitulate the basic definitions (cf. [8]). Let $\mathscr{S}$ denote the category of small categories (whose morphisms are functors) and $\mathscr{P}$ the category of preordered sets and increasing functions. Each preordered set A can be regarded as small category whose objects are the elements of A and whose morphisms are given by $mor(\alpha_1, \alpha_2) = \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq \alpha_2\}$. Doing so, the morphisms of $\mathscr{P}$ turn out to be functors between small categories. In that way we identify $\mathscr{P}$ with a full subcategory of $\mathscr{S}$. To each $A \in \mathscr{S}$ we associate $o(A) \in \mathscr{P}$ by setting $o(A) = ob(A)$ and $\alpha_1 \geq \alpha_2$ if there exists a morphism $u : \alpha_1 \to \alpha_2$. We call $\geq$ the *induced preordering on* A. To emphasize the role of u we also write $\alpha_1 \geq_u \alpha_2$.

For any two objects $A, B \in \mathscr{S}$ let $[B, A]$ denote the set of all functions $\varphi : ob(B) \to ob(A)$.

Let $\mathscr{C} \subset \mathscr{S}$ denote the full subcategory of cofiltered small categories and $\mathscr{D} \subset \mathscr{P}$ the full subcategory of directed preordered sets.

The objects of **inv-C** and **pro-C** are all functors $\mathbf{X} : A \to \mathbf{C}$, where A is any element of $\mathscr{C}$. Each such $\mathbf{X}$ is called an inverse system in $\mathbf{C}$ indexed by A. We also write $\mathbf{X} = (X_\alpha = \mathbf{X}(\alpha), p_u = \mathbf{X}(u))_{\alpha \in ob(A), u \in mor(A)}$.

Given $\mathbf{X} = (X_\alpha, p_u)_{\alpha \in ob(A), u \in mor(A)}$ and $\mathbf{Y} = (Y_\beta, q_v)_{\beta \in ob(B), v \in mor(B)}$, the morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of **inv-C** are all systems $\mathbf{f} = (\varphi, (f_\beta)_{\beta \in B})$ with $\varphi \in [B, A]$ and $f_\beta \in \mathbf{C}(X_{\varphi(\beta)}, Y_\beta)$ such that the following holds: For each morphism $v : \beta_1 \to \beta_2$ in B there exist $\alpha \in A$ and morphisms $u_i : \alpha \to \varphi(\beta_i)$ in A such that $f_{\beta_2} \circ p_{u_2} = q_v \circ f_{\beta_1} \circ p_{u_1}$. We refer to φ as the *index function* of $\mathbf{f}$ and denote it by $ind(\mathbf{f})$. Two morphisms $\mathbf{f_i} = (\varphi_i, (f_\beta^i)) : \mathbf{X} \to \mathbf{Y}$ are called equivalent ($\mathbf{f_1} \sim \mathbf{f_2}$) if each $\beta \in B$ admits $\alpha \in A$ and morphisms $u_i : \alpha \to \varphi_i(\beta)$ such that $f_\beta^1 \circ p_{u_1} = f_\beta^2 \circ p_{u_2}$. The morphisms of **pro-C** are the equivalence classes of morphisms in **inv-C** with respect to $\sim$. The canonical functor mapping each morphism to its equivalence class is denoted by $\Pi : \textbf{inv-C} \to \textbf{pro-C}$.

We give $[B, A]$ the structure of a category by defining a morphism $\tau : \psi \to \varphi$ to be a collection of morphisms $\tau_\beta : \psi(\beta) \to \varphi(\beta)$. The induced preordering on $[B, A]$ is denoted by $\geq$.

Given a morphism $\mathbf{f}$ of $\mathbf{inv\text{-}C}(\mathbf{X}, \mathbf{Y})$ and a morphism $\tau : \psi \to ind(\mathbf{f})$ in $[B, A]$, we define $\mathbf{f}^\tau = (\psi, f_\beta \circ p_{\tau_\beta})$ which is a morphism of $\mathbf{inv\text{-}C}(\mathbf{X}, \mathbf{Y})$ such that $\mathbf{f}^\tau \sim \mathbf{f}$. This endows $\mathbf{inv\text{-}C}(\mathbf{X}, \mathbf{Y})$ with the structure of category: A morphism $\tau : \mathbf{g} \to \mathbf{f}$ is a morphism $\tau : ind(\mathbf{g}) \to ind(\mathbf{f})$ in $[B, A]$ such that $\mathbf{g} = \mathbf{f}^\tau$. The induced preordering on $\mathbf{inv\text{-}C}(\mathbf{X}, \mathbf{Y})$ is denoted by $\geq$.

The following is an obvious consequence of the axiom of choice.

Proposition 2.1. *For* $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{inv\text{-}C}(\mathbf{X}, \mathbf{Y})$ *the following are equivalent*:

(1) $\mathbf{f}_1 \sim \mathbf{f}_2$

(2) *There exist* $\psi \in [B, A]$ *and morphisms* $\tau_i : \psi \to ind(\mathbf{f}_i)$ *such that* $\mathbf{f}_1^{\tau_1} = \mathbf{f}_2^{\tau_2}$.

(3) *There exists* $\mathbf{g} \geq \mathbf{f}_1, \mathbf{f}_2$.

For each $A \in \mathscr{C}$ we have the category $\mathbf{C}^A$ whose objects are the inverse systems indexed by A and whose morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ are the natural transformations between the functors $\mathbf{X}, \mathbf{Y} : A \to \mathbf{C}$. There is natural identification of $\mathbf{C}^A$ with a subcategory of $\mathbf{inv\text{-}C}$: Each natural transformation $\mathbf{f} = (f_\alpha)$ can be regarded as a morphism of $\mathbf{inv\text{-}C}$ by writing $\mathbf{f} = (id_A, (f_\alpha))$. The wide subcategory[2] of $\mathbf{inv\text{-}C}$ given as the union of all $\mathbf{C}^A$, $A \in \mathscr{C}$, will be denoted by $\mathbf{lev\text{-}C}$. Its morphisms are called *level morphisms*.

For each functor $\varphi : B \to A$ in $\mathscr{C}$ we obtain a functor

$$\varphi^* : \mathbf{C}^A \to \mathbf{C}^B, \quad \varphi^*(\mathbf{X}) = \mathbf{X} \circ \varphi, \quad \varphi^*(\mathbf{f})_\beta = f_{\varphi(\beta)}.$$

If $\psi : C \to B$ is another functor, we have $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

The category $\mathbf{str\text{-}C}$ of *strict morphisms*[3] is defined as follows. Its objects are all inverse systems in $\mathbf{C}$. For $\mathbf{X} \in \mathbf{C}^A$ and $\mathbf{Y} \in \mathbf{C}^B$ we set

$$\mathbf{str\text{-}C}(\mathbf{X}, \mathbf{Y}) = \{\mathbf{f} = (\varphi, \mathbf{f}^*) \mid \varphi \in \mathscr{C}(B, A), \mathbf{f}^* \in \mathbf{C}^B(\varphi^*(\mathbf{X}), \mathbf{Y})\}.$$

Composition of morphisms is defined by

$$(\psi, \mathbf{g}^*) \circ (\varphi, \mathbf{f}^*) = (\varphi \circ \psi, \mathbf{g}^* \circ \psi^*(\mathbf{f}^*)).$$

Obviously $\mathbf{str\text{-}C}$ is a wide subcategory of $\mathbf{inv\text{-}C}$ such that $\mathbf{lev\text{-}C} \subset \mathbf{str\text{-}C}$.

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[2] A subcategory $\mathbf{K}' \subset \mathbf{K}$ is *wide* if it contains all objects of $\mathbf{K}$.

[3] For inverse systems indexed by directed preordered sets this concept goes back to [5, Ch. VIII] under the name "map of inverse systems".

The set $\mathscr{C}(B,A)$ of functors $B \to A$ inherits the structure of a category from $[B,A]$. Let $\mathscr{C}_{nat}(B,A) \subset \mathscr{C}(B,A)$ denote the wide subcategory whose morphisms are natural transformations. The induced preordering on $\mathscr{C}_{nat}(B,A)$ is denoted by $\succeq$. If $A \in \mathscr{D}$, then $\mathscr{C}_{nat}(B,A) = \mathscr{C}(B,A)$.

This endows $\mathbf{str\text{-}C}(\mathbf{X},\mathbf{Y})$ with the structure of category: A morphism $\tau : \mathbf{g} \to \mathbf{f}$ is a morphism $\tau : ind(\mathbf{g}) \to ind(\mathbf{f})$ in $\mathscr{C}_{nat}(B,A)$ such that $\mathbf{g} = \mathbf{f}^\tau$. The induced preordering on $\mathbf{str\text{-}C}(\mathbf{X},\mathbf{Y})$ is denoted by $\succeq$. Clearly $\mathbf{g} \succeq \mathbf{f}$ implies $\mathbf{g} \geq \mathbf{f}$ in $\mathbf{inv\text{-}C}(\mathbf{X},\mathbf{Y})$. Note that $\mathbf{f}^\tau$ is a morphism of $\mathbf{str\text{-}C}(\mathbf{X},\mathbf{Y})$ provided $\tau : \psi \to ind(\mathbf{f})$ is a morphism in $\mathscr{C}_{nat}(B,A)$.

On $\mathbf{str\text{-}C}(\mathbf{X},\mathbf{Y})$ we define $\mathbf{f}_1 \triangleq \mathbf{f}_2$ if there exists $\mathbf{g} \in \mathbf{str\text{-}C}(\mathbf{X},\mathbf{Y})$ such that $\mathbf{g} \succeq \mathbf{f}_1, \mathbf{f}_2$. $\triangleq$ generates an equivalence relation $\equiv$ which is compatible with composition so that we obtain a quotient category $\mathbf{qstr\text{-}C} = \mathbf{str\text{-}C}/\equiv$ and a commutative diagram

$$\begin{array}{ccc} \mathbf{str\text{-}C} & \hookrightarrow & \mathbf{inv\text{-}C} \\ \downarrow & & \downarrow \\ \mathbf{qstr\text{-}C} & \xrightarrow[\iota]{} & \mathbf{pro\text{-}C} \end{array}$$

where the vertical arrows are the quotient functors.

3. An Alternative Representation of Pro-morphisms between Cofinitely Indexed Inverse Systems

For each preordered set A we define $\alpha \sim \alpha'$ if $\alpha \geq \alpha'$ and $\alpha' \geq \alpha$. The quotient set $p(A) = A/\sim$ becomes an ordered set[4] by defining $[\alpha] \geq [\alpha']$ if $\alpha \geq \alpha'$.

As the *skeleton* of $A \in \mathscr{C}$ we denote the ordered set $s(A) = p(o(A))$. The canonical function $\sigma_A : A \to s(A)$ is a morphism in $\mathscr{C}(A, s(A))$. A morphism $\psi \in [B,A]$ resp. $\psi \in \mathscr{C}(B,A)$ is called *skeletal* if it has the form $\psi = \hat{\psi} \circ \sigma_B$, where $\hat{\psi} \in [s(B),A]$ resp. $\hat{\psi} \in \mathscr{C}(s(B),A)$.

An *internal diagram* Δ in a category $\mathbf{C}$ consists of a set V of objects of $\mathbf{C}$ and a set E of morphisms between these objects. A *cone* over Δ consists of an object c of $\mathbf{C}$ and a family of morphisms $\gamma_v : c \to v$, $v \in V$, such that for all morphisms $e : v \to v'$ in E, $e \circ \gamma_v = \gamma_{v'}$. If c is not an object of Δ, we use the wording *outer cone*. The following is well-known.

[4] As an *ordering* on a set we understand an antiysmmetric preordering.

PROPOSITION 3.1. *Let A be a small category. Then $A \in \mathscr{C}$ if and only if each finite internal diagram in A has a cone.*

A small category B is called *cofinite* if for each $\beta \in B$ there exist only finitely many morphisms with domain β. By $\mathscr{C}(cfnt) \subset \mathscr{C}$ resp. $\mathscr{D}(cfnt) \subset \mathscr{D}$ we denote the full subcategories having as objects all cofinite $B \in \mathscr{C}$ resp. $B \in \mathscr{D}$.

A function $\xi \in [B, A]$ is called *weakly cofinal* if for all $\alpha \in A$ there exist $\beta \in B$ and a morphism $u : \xi(\beta) \to \alpha$ in A. A functor $\varphi \in \mathscr{C}(B, A)$ is called

(1) *equalizing* if for all $\beta \in B$ and all morphisms $u_1, u_2 : \varphi(\beta) \to \alpha$ in A there exists a morphism $v : \beta' \to \beta$ in B such that $u_1\varphi(v) = u_2\varphi(v)$,

(2) *cofinal* if it is weakly cofinal and equalizing.

If $\xi' \geq \xi$ in $[B, A]$ and ξ is weakly cofinal, then also ξ' is weakly cofinal. If $A \in \mathscr{P}$, then each functor $\varphi : B \to A$ is equalizing; thus φ is cofinal if and only if it is weakly cofinal.

Let $\mathscr{C}_{eq}$ denote the wide subcategory of $\mathscr{C}$ whose morphisms are all equalizing functors. This yields a wide subcategory $\mathbf{str}_{\mathrm{eq}}$**-C** of **str-C** whose morphisms have index functors in $\mathscr{C}_{eq}$. The relations $\triangleq$ and $\equiv$ on **str-C** can be modified in the obvious way to produce relations on $\mathbf{str}_{\mathrm{eq}}$**-C** which are denoted by the same symbols.

LEMMA 3.2. *Let $B \in \mathscr{C}(cfnt)$ and $A \in \mathscr{C}$. For $i = 1, \ldots, n$ let be given functions $\alpha_i, \alpha_i' : mor(B) \to ob(A)$, $\lambda_i, v_i, v_i' : mor(B) \to POW(mor(A))$[5], such that for each $v : \beta \to \beta'$*

- *$\lambda_i(v)$ is a finite set of morphisms $\alpha_i(v) \to \alpha_i'(v)$,*
- *$v_i(v)$ is a finite set of morphisms $\alpha_1(v) \to \alpha_i(v)$,*
- *$v_i'(v)$ is a finite set of morphisms $\alpha_1'(v) \to \alpha_i'(v)$.*

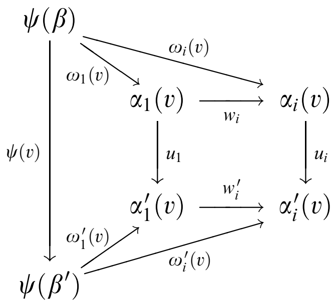
Then there exist

(1) *a skeletal functor $\psi : B \to A$*

(2) *for $i = 1, \ldots, n$ functions $\omega_i, \omega_i' : mor(B) \to mor(A)$ such that for all $v : \beta \to \beta'$, $\omega_i(v)$ is a morphism $\psi(\beta) \to \alpha_i(v)$ and $\omega_i'(v)$ is a morphism $\psi(\beta') \to \alpha_i'(v)$*

with the following property: For all $v : \beta \to \beta'$, all $i = 1, \ldots, n$, all $u_i \in \lambda_i(v)$, $w_i \in v_i(v)$ and $w_i' \in v_i'(v)$ the following diagram commutes with the possible exception of the right inner square:

[5] The symbol POW denotes powerset.



If for some i one has $\alpha_i(v) = \chi_i(\beta)$, $\alpha'_i(v) = \chi_i(\beta')$ *with a function* $\chi_i : ob(B) \to ob(A)$ *("functional case"), one can find a morphism* $\tau_i : \psi \to \chi_i$ *in* $[B, A]$ *such that one can take* $\omega_i(v) = (\tau_i)_\beta$, $\omega'_i(v) = (\tau_i)_{\beta'}$. *In case* χ_i *is a functor and* $\lambda_i(v) = \{\chi_i(v)\}$, *then* τ_i *is necessarily a natural transformation.*

Moreover, if A is cofinite, then ψ *can be chosen to be equalizing.*

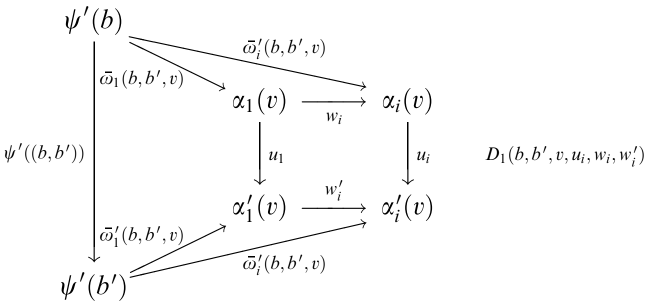
NB If $\lambda_1(v) = \emptyset$, $\lambda_i(v) = \emptyset$, $v_i(v) = \emptyset$ *or* $v'_i(v) = \emptyset$, *then in the above diagram it is understood that corresponding arrow* u_1, u_i, w_i *or* w'_i *is omitted. The consequence is that the corresponding commutativity assertion falls away.*

Proof. Let $T(B) = \{(b, b', v) \in ob(s(B) \times s(B)) \times mor(B) \mid \sigma_B(v) = (b, b')\}$ and $P(B) = \{(b, \beta) \in ob(s(B) \times B) \mid \sigma_B(\beta) = b\}$. We construct

(1) a functor $\psi' : s(B) \to A$

(2) for each $(b, b', v) \in T(B)$ morphisms $\bar{\omega}_i(b, b', v) : \psi'(b) \to \alpha_i(v)$, $\bar{\omega}'_i(b, b', v) : \psi'(b') \to \alpha'_i(v)$ resp. in the functional case for each $(b, \beta) \in P(B)$ a morphism $\bar{\tau}_i(b, \beta) : \psi'(b) \to \chi_i(\beta)$

such that for all $(b, b', v) \in T(B)$ and all $u_i \in \lambda_i(v)$, $w_i \in v_i(v)$, $w'_i \in v'_i(v)$ the following diagram commutes with the possible exception of the right inner square:



In the functional case we consider all $(b, \beta) \in P(B)$ instead of all $(b, b', v) \in T(B)$ and replace in the above diagram $\bar{\omega}_i(b, b', v)$ by $\bar{\tau}_i(b, \beta)$ and $\bar{\omega}'_i(b, b', v)$ by $\bar{\tau}_i(b', \beta')$.

This is clearly equivalent to the lemma. The right square subdiagram will be denoted by $D_2(b, b', v, u_i, w_i, w_i')$. Removing from $D_1(b, b', v, u_i, w_i, w_i')$ the object in the upper left corner and the three morphisms starting there yields a diagram denoted as $D_3(b, b', v, u_i, w_i, w_i')$. For a fixed $b \in s(B)$ there are only finitely many diagrams having the form $D_j(b, b', v, u_i, w_i, w_i')$. Let $D_j^*(b, b', v, u_i, w_i, w_i')$ denote the internal diagram canonically associated to $D_j(b, b', v, u_i, w_i, w_i')$.

Let $pr(b)$ denote the set of predecessors of b, i.e. of all b' such that $b \geq b'$. Then $pr(b) \supset pr(b')$ if and only if $b \geq b'$. Since B' is ordered, we have moreover $pr(b) = pr(b')$ if and only if $b = b'$.

Let $k(b) =$ number of predecessors of b. Assume $b \geq b'$. Then clearly $k(b) \geq k(b')$, and $k(b) = k(b')$ if and only $b = b'$. In particular, for $b, b' \in s(B)$ with $k(b) = k(b')$, we either have $b = b'$ or b, b' are not comparable with respect to $\geq$.

We construct the necessary objects and morphisms by induction over $k(b)$.

For $k(b) = 1$ let Δ be the union of the finitely many internal diagrams having the form $D_2^*(b, b, v, u_i, w_i, w_i')$. Choose a cone (μ, w_α) over Δ and set $\psi'(b) = \mu$, $\psi'((b, b)) = id$ and $\bar{\omega}_i(b, b, v) = w_{\alpha_i(v)}$, $\bar{\omega}_i'(b, b, v) = w_{\alpha_i'(v)}$. In the special case based on a function χ_i we set $\bar{\tau}_i(b, \beta) = w_{\chi_i(\beta)}$.

Assume we have constructed the components for all b with $k(b) \leq m$. If A is cofinite, assume moreover that for all pairs (b, b') such that $b \geq b'$ and $k(b') < k(b) \leq m$ the following holds: For any two morphisms $u_1, u_2 : \psi(b') \to \alpha$ in A one has $u_1\psi((b, b')) = u_2\psi((b, b'))$.

Consider b^* with $k(b^*) = m + 1$. Let Δ be the union of the finitely many internal diagrams having the form $D_1^*(b, b', v, u_i, w_i, w_i')$ with $b' \leq b < b^*$, $D_3^*(b^*, b', v, u_i, w_i, w_i')$ with $b' < b^*$ and $D_1^*(b^*, b^*, v, u_i, w_i, w_i')$. If A is cofinite add all (finitely many) morphisms $u : \psi'(b) \to \alpha$ in A where $b < b^*$.

Choose a cone (μ, w_α) over Δ and set $\psi'(b^*) = \mu$, $\psi'((b^*, b^*)) = id$ and, for $b < b^*$, $\psi'((b^*, b)) = w_{\psi'(b)}$ $\bar{\omega}_i(b^*, b, v) = w_{\alpha_i(v)}$, $\bar{\omega}_i'(b^*, b, v) = w_{\alpha_i'(v)}$. In the special case based on a function χ_i we set $\bar{\tau}_i(b^*, \beta^*) = w_{\chi_i(\beta^*)}$. □

Corollary 3.3. *Let $B \in \mathscr{C}(cfnt)$ and $A \in \mathscr{C}$. Then for each* $\mathbf{f} \in \mathbf{inv\text{-}C}(\mathbf{X}, \mathbf{Y})$ *there exists* $\mathbf{g} \in \mathbf{str\text{-}C}(\mathbf{X}, \mathbf{Y})$ *such that* $\mathbf{g} \geq \mathbf{f}$ *(so that* $[\mathbf{g}] = [\mathbf{f}]$ *in* **pro-C**). *The index functor of* $\mathbf{g}$ *can be chosen to be skeletal. If A is cofinite, then it can moreover be chosen to be equalizing. If we are given $\xi \in [B, A]$, we can achieve $ind(\mathbf{g}) \geq \xi$. In case $\xi \in \mathscr{C}(B, A)$, we can achieve $ind(\mathbf{g}) \succeq \xi$.*

PROOF. For each morphism $v : \beta \to \beta'$ in B there exist $\alpha(v) \in A$ and morphisms $\nu(v) : \alpha(v) \to \varphi(\beta)$, $\nu'(v) : \alpha(v) \to \varphi(\beta')$ such that the following diagram commutes:

$$\begin{array}{ccccc} X_{\alpha(v)} & \xrightarrow{p_{\nu(v)}} & X_{\varphi(\beta)} & \xrightarrow{f_\beta} & Y_\beta \\ {\scriptstyle p_{id}=id}\downarrow & & & & \downarrow{\scriptstyle q_v} \\ X_{\alpha(v)} & \xrightarrow[p_{\nu'(v)}]{} & X_{\varphi(\beta')} & \xrightarrow[f_{\beta'}]{} & Y_{\beta'} \end{array}$$

Now apply Lemma 3.2 with $\alpha_1(v) = \alpha_1'(v) = \alpha(v)$, $\alpha_2(v) = \varphi(\beta)$, $\alpha_2'(v) = \varphi(\beta')$, $\alpha_3(v) = \xi(\beta)$, $\alpha_3'(v) = \xi(\beta')$ and $\lambda_1(v) = \{id\}$, $\lambda_2(v) = \emptyset$. If $\xi \in \mathscr{C}(B, A)$ set $\lambda_3(v) = \{\xi(v)\}$, otherwise $\lambda_3(v) = \emptyset$. Moreover, let $\nu_2(v) = \{\nu(v)\}$, $\nu_2'(v) = \{\nu'(v)\}$ and $\nu_1(v) = \nu_1'(v) = \nu_3(v) = \nu_3'(v) = \emptyset$.

This yields a skeletal functor $\psi : B \to A$ and morphisms $\tau : \psi \to \varphi$, $\tau' : \psi \to \xi$; in the functorial case τ' is a natural transformation. Set $\mathbf{g} = \mathbf{f}^\tau$. □

COROLLARY 3.4. *Let $B \in \mathscr{C}(cfnt)$ and $A \in \mathscr{C}$ and let $\mathbf{f}_1, \mathbf{f}_2 \in$ **str-C**$(\mathbf{X}, \mathbf{Y})$. Then $\mathbf{f}_1 \sim \mathbf{f}_2$ in **inv-C**$(\mathbf{X}, \mathbf{Y})$ if and only if there exists $\mathbf{g} \in$ **str-C**$(\mathbf{X}, \mathbf{Y})$ such that $\mathbf{g} \succeq \mathbf{f}_1, \mathbf{f}_2$. The index functor of $\mathbf{g}$ can be chosen to be skeletal. If A is cofinite, then it can moreover be chosen to be equalizing. If we are given $\xi \in \mathscr{C}(B, A)$ such that $ind(\mathbf{f}_i) \succeq_{\tau_i} \xi$, we can achieve $\mathbf{g} \succeq_{\sigma_i} \mathbf{f}_i$ such that $\tau_1\sigma_1 = \tau_2\sigma_2$.*

PROOF. The "if"-part is obvious. Conversely, let $\mathbf{f}_1 \sim \mathbf{f}_2$. Then there exist $\psi \in [B, A]$ and morphisms $\tau_1 : \psi \to ind(\mathbf{f}_1)$, $\tau_2 : \psi \to ind(\mathbf{f}_2)$ such $\mathbf{f}_1^{\tau_1} = \mathbf{f}_2^{\tau_2} = \mathbf{g}$. A suitable application of Lemma 3.2 yields the assertion. □

COROLLARY 3.5. *In **str-C**$_{\mathscr{C}(cfnt)}$ the following are equivalent:*

(1) $\mathbf{f}_1 \sim \mathbf{f}_2$

(2) $\mathbf{f}_1 \triangleq \mathbf{f}_2$

(3) $\mathbf{f}_1 \equiv \mathbf{f}_2$

*The same holds in **str**$_{\text{eq}}$-**C**$_{\mathscr{C}(cfnt)}$.*

We therefore obtain the following alternative representation of pro-morphisms between cofinitely indexed inverse systems.

THEOREM 3.6. *$\imath$: **qstr-C**$_{\mathscr{C}(cfnt)} \to$ **pro-C**$_{\mathscr{C}(cfnt)}$ and $\imath$: **qstr**$_{\text{eq}}$-**C**$_{\mathscr{C}(cfnt)} \to$ **pro-C**$_{\mathscr{C}(cfnt)}$ are category isomorphisms.*

4. Reindexers

For each inverse system $\mathbf{X}$ indexed by $A \in \mathscr{C}$ and each functor $\varphi : B \to A$ with domain $B \in \mathscr{C}$ we obtain a canonical morphism

$$\mathbf{r}(\mathbf{X}, \varphi) = (\varphi, \mathbf{id}_{\varphi^*(\mathbf{X})}) : \mathbf{X} \to \varphi^*(\mathbf{X})$$

in **str-C**. For each morphism $\mathbf{f} = (\varphi, \mathbf{f}^*) : \mathbf{X} \to \mathbf{Y}$ in **str-C** we thus have a canonical decomposition

$$\mathbf{f} = \mathbf{f}^* \circ \mathbf{r}(\mathbf{X}, \varphi).$$

The $\mathbf{r}(\mathbf{X}, \varphi)$ constitute a natural transformation

$$\mathbf{r}(-, \varphi) : id \to \varphi^*$$

between the functors $id : \mathbf{C}^A \to \mathbf{C}^A \subset \textbf{str-C}$ and $\varphi^* : \mathbf{C}^A \to \mathbf{C}^B \subset \textbf{str-C}$.

Moreover, if φ splits as $\varphi = \psi \circ \chi$ with functors $\chi : B \to C$ and $\psi : C \to A$, then $\mathbf{f}$ splits as

$$\mathbf{f} = \mathbf{f}_{(\psi, \chi)} \circ \mathbf{r}(\mathbf{X}, \psi),$$

where $\mathbf{f}_{(\psi,\chi)} = (\chi, \mathbf{f}^*) : \psi^*(\mathbf{X}) \to \mathbf{Y}$ (note that $\chi^*(\psi^*(\mathbf{X})) = (\psi \circ \chi)^*(\mathbf{X}) = \varphi^*(\mathbf{X})$).

If $\psi : C \to B$ is another functor, then clearly

$$\mathbf{r}(\varphi^*(\mathbf{X}), \psi) \circ \mathbf{r}(\mathbf{X}, \varphi) = \mathbf{r}(\mathbf{X}, \varphi \circ \psi).$$

For a functor $\psi : B \to A$ and a natural transformation $\tau : \psi \to \varphi$ let

$$\mathbf{i}(\mathbf{X}, \tau) = (id, (p_{\tau_\beta})) : \psi^*(\mathbf{X}) \to \varphi^*(\mathbf{X}).$$

This is a morphism in $\mathbf{C}^B$ such that

$$\mathbf{i}(\mathbf{X}, \tau) \circ \mathbf{r}(\mathbf{X}, \psi) = \mathbf{r}(\mathbf{X}, \varphi)^\tau.$$

If $\chi : B \to A$ is a functor and $\omega : \chi \to \psi$ is a natural transformation, we have

$$\mathbf{i}(\mathbf{X}, \tau) \circ \mathbf{i}(\mathbf{X}, \omega) = \mathbf{i}(\mathbf{X}, \tau \circ \omega).$$

In the special case $B = A$ we may take $\varphi = id$. For each functor $\psi : A \to A$ and natural transformation $\tau : \psi \to id$ we get the level morphism

$$\mathbf{i}(\mathbf{X}, \tau) : \psi^*(\mathbf{X}) \to id^*(\mathbf{X}) = \mathbf{X}.$$

Definition 4.1. A morphism having the form $\mathbf{r}(\mathbf{X}, \varphi)$ for some $\mathbf{X}$ and some cofinal functor $\varphi : B \to A$ is called a *reindexer* (or more precisely a *reindexer over* φ).

Proposition 4.2. *Each reindexer* $\mathbf{r}(\mathbf{X}, \varphi)$ *induces an isomorphism in* **pro-C**.

Proof. A pair (τ, ψ) consisting of a function $\psi : A \to B$ and a morphism $\tau : \varphi \circ \psi \to id$ in $[A, A]$ is called an *associate*[6] of φ. Choose any associate and define

$$\mathbf{k}(\mathbf{X}, \varphi; \tau, \psi) = (\psi, (p_{\tau_\alpha} : \varphi^*(\mathbf{X})_{\psi(\alpha)} = X_{\varphi(\psi(\alpha))} \to X_\alpha)).$$

Let $u : \alpha_1 \to \alpha_2$ be a morphism in A. There exist $\beta \in B$ and morphisms $v_i : \beta \to \psi(\alpha_i)$. Since φ is equalizing, there exist $\beta' \in B$ and a morphism $w : \beta' \to \beta$ such that $(u \circ \tau_{\alpha_1} \circ \varphi(v_1)) \circ \varphi(w) = (\tau_{\alpha_2} \circ \varphi(v_2)) \circ \varphi(w)$. Set $w_i = v_i \circ w : \beta' \to \psi(\alpha_i)$. Then $(u \circ \tau_{\alpha_1}) \circ \varphi(w_1) = \tau_{\alpha_2} \circ \varphi(w_2)$. This implies that $\mathbf{k}(\mathbf{X}, \varphi; \tau, \psi) \in \mathbf{inv\text{-}C}(\varphi^*(\mathbf{X}), \mathbf{X})$.

We have $\mathbf{k}(\mathbf{X}, \varphi; \tau, \psi) \circ \mathbf{r}(\mathbf{X}, \varphi) = (\varphi \circ \psi, (p_{\tau_\alpha} : X_{\varphi(\psi(\alpha))} \to X_\alpha)) = \mathbf{id}^\tau \sim \mathbf{id}$ and $\mathbf{r}(\mathbf{X}, \varphi) \circ \mathbf{k}(\mathbf{X}, \varphi; \tau, \psi) = (\psi \circ \varphi, (p_{\tau_{\varphi(\beta)}} : \varphi^*(\mathbf{X})_{\psi(\varphi(\beta))} = X_{\varphi(\psi(\varphi(\beta)))} \to X_{\varphi(\beta)} = \varphi^*(\mathbf{X})_\beta)$. There exist $\beta' \in B$ and morphisms $v : \beta' \to \beta$, $v' : \beta' \to \psi(\varphi(\beta))$. Since φ is equalizing, there exist $\beta'' \in B$ and $w : \beta'' \to \beta'$ such that $(\tau_{\varphi(\beta)} \circ \varphi(v')) \circ \varphi(w) = \varphi(v) \circ \varphi(w)$. This implies $p_{\tau_{\varphi(\beta)}} \circ p_{\varphi(v' \circ w))} = p_{\varphi(v \circ w)}$ which shows $\mathbf{r}(\mathbf{X}, \varphi) \circ \mathbf{k}(\mathbf{X}, \varphi; \tau, \psi) \sim \mathbf{id}$. □

Remark 4.3. The reindexers $\mathbf{r}(\mathbf{X}, \varphi)$ with cofinal functors $\varphi \in \mathscr{C}(A, A)$ such that $\varphi \succeq id$ have a distinctive feature: The inverse isomorphism in **pro-C** is represented by a *level morphism*. In fact, any natural transformation $\tau : \varphi \to id$ yields the associate (id, τ) of φ. Then $\mathbf{k}(\mathbf{X}, \varphi; \tau, id) = \mathbf{i}(\mathbf{X}, \tau)$ is a level morphism. Note also that each functor $\varphi \succeq id$ is automatically weakly cofinal.

Example 4.4. Let $\mathbf{X}$ be an inverse system indexed by $A \in \mathscr{C}$ and $A' \subset A$ be a cofinal subcategory which means that the inclusion functor $\imath : A' \to A$ is cofinal. Then $\imath^*(\mathbf{X})$ is the cofinal subsystem of $\mathbf{X}$ indexed by A' and $\mathbf{r}(\mathbf{X}, \imath)$ is a reindexer.

Example 4.5. Let $A_1, A_2 \in \mathscr{C}$ and $\pi^i : A_1 \times A_2 \to A_i$ the projection functor (which is cofinal). Each reindexer over such a π^i is called a *projection reindexer*.

Example 4.6. This example is taken from [6, Proposition 8.1.6] where it appears in dual form; see also [8, Ch. I, §1.4, Theorems 2 and 4]. For $A \in \mathscr{C}$ let $P(A)$ be the set of finite internal diagrams Δ in A having a unique initial object $\mu_A(\Delta)$. An initial object of Δ is an object $\mu \in \Delta$ such that

[6] This concept is defined for any $\varphi \in [B, A]$. Associates of φ exist if and only if φ is weakly cofinal.

(1) For each object $\alpha \in \Delta$ there exist exactly one morphism $u_\alpha : \mu \to \alpha$ in Δ.
(2) $u_\mu = id$.
(3) For each morphism $u : \alpha \to \alpha'$ in Δ, $u \circ u_\alpha = u_{\alpha'}$.

$P(A)$ is ordered by inclusion; it is cofinite but in general not directed. The initial object function $\mu_A : ob(P(A)) \to ob(A)$ extends to a functor $\mu_A : P(A) \to A$ (the morphisms in $P(A)$ are the pairs (Δ_1, Δ_0) with $\Delta_1 \supset \Delta_0$, and we let $\mu_A(\Delta_1, \Delta_0)$ be the unique morphism in Δ_1 from $\mu_A(\Delta_1)$ to $\mu_A(\Delta_0) \in \Delta_0 \subset \Delta_1$).

For $A \in \mathscr{C}^{nmx}$ = full subcategory of $\mathscr{C}$ whose objects do not have maximal elements it turns out that

- $P(A) \in \mathscr{D}(ord, cfnt)$ = full subcategory of $\mathscr{D}$ whose objects are ordered cofinite sets.
- $\mu_A : P(A) \to A$ is a cofinal functor.

We remark that the same is true for $A \in \mathscr{D}(ord)$ = full subcategory of $\mathscr{D}$ whose objects are ordered sets. In this case $P(A)$ is nothing else than the set of finite internal diagrams Δ in A having a maximal element $\mu_A(\Delta)$ (which is automatically unique).

For each inverse system $\mathbf{X}$ indexed by $A \in \mathscr{C}^{nmx}$, we define $P(\mathbf{X}) = (\mu_A)^*(\mathbf{X})$. As the *standard cofinite reindexer* we denote

$$\mu_{\mathbf{X}} = \mathbf{r}(\mathbf{X}, \mu_A) : \mathbf{X} \to P(\mathbf{X}).$$

To deal with arbitrary $A \in \mathscr{C}$, [6] uses the cofinal projection functor $\pi_A : A \times \mathbf{N} \to A$. Define $P'(A) = P(A \times \mathbf{N})$ and $\mu'_A = \pi_A \circ \mu_{A \times \mathbf{N}} : P'(A) \to A$ which is a cofinal functor. As the *modified cofinite reindexer* we denote

$$\mu'_{\mathbf{X}} = \mathbf{r}(\mathbf{X}, \mu'_A) : \mathbf{X} \to (\mu'_A)^*(\mathbf{X}) = P'(\mathbf{X}).$$

It would be desirable if the association $A \mapsto P(A)$ had a continuation to a functor $P : \mathscr{C}^{nmx} \to \mathscr{D}(ord, cfnt)$. The natural definition of the induced $P(\varphi) : P(B) \to P(A)$ is of course $P(\varphi)(\Delta) = \varphi(\Delta)$, but in general $\varphi(\Delta) \notin P(A)$ when $\Delta \in P(B)$. We circumvent this problem by considering only *regular* functors[7] $\varphi : B \to A$ characterized by the property that $\varphi(\Delta) \in P(A)$ and $\varphi(\mu_B(\Delta)) = \mu_A(\varphi(\Delta))$ for all $\Delta \in P(B)$. Examples for such functors are all embeddings[8] $\varphi : B \to A$ and all $\varphi : B \to A$ such that A is ordered.

On the wide subcategory $\mathscr{C}^{nmx}_{reg} \subset \mathscr{C}^{nmx}$ whose morphisms are the regular functors we thus obtain a functor $P : \mathscr{C}^{nmx}_{reg} \to \mathscr{D}(ord, cfnt)$ and a natural transformation $\mu = (\mu_A) : P \to id$.

[7] This concept is defined for arbitrary $A, B \in \mathscr{C}$.
[8] This means that φ establishes a category isomorphism between A and a subcategory $A' \subset B$.

Let $E : \mathscr{C} \to \mathscr{C}^{nmx}$ be the functor defined by $E(A) = A \times \mathbf{N}$, $E(\varphi) = \varphi \times id_{\mathbf{N}}$ and $\mathscr{C}_{reg} \subset \mathscr{C}$ be the wide subcategory whose morphisms are the regular functors. We have $E(\mathscr{C}_{reg}) \subset \mathscr{C}^{nmx}_{reg}$ and define a functor

$$P' = P \circ E : \mathscr{C}_{reg} \to \mathscr{D}(ord, cfnt)$$

which comes together with the natural transformation $\mu' = (\mu'_A) : P' \to id$.

Let $\mathbf{str_{reg}\text{-}C}$ denote the wide subcategory of $\mathbf{str\text{-}C}$ whose morphisms have a regular index functor. Given $\mathbf{f} = (\varphi, \mathbf{f}^*) : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{str_{reg}\text{-}C}$, the index functor $\varphi \circ \mu'_B$ of $\mu'_{\mathbf{Y}} \circ \mathbf{f}$ splits as $\varphi \circ \mu'_B = \mu'_A \circ P'(\varphi)$. Hence

$$\mu'_{\mathbf{Y}} \circ \mathbf{f} = (\mu'_{\mathbf{Y}} \circ \mathbf{f})_{(\mu'_A, P'(\varphi))} \circ \mathbf{r}(\mathbf{X}, \mu'_A) = (\mu'_{\mathbf{Y}} \circ \mathbf{f})_{(\mu'_A, P'(\varphi))} \circ \mu'_{\mathbf{X}}.$$

We define

$$P'(\mathbf{f}) = (\mu'_{\mathbf{Y}} \circ \mathbf{f})_{(\mu'_A, P'(\varphi))} : P'(\mathbf{X}) \to P'(\mathbf{Y}).$$

It is easy to verify that this yields a functor

$$P' : \mathbf{str_{reg}\text{-}C} \to \mathbf{str\text{-}C}_{\mathscr{D}(ord, cfnt)}$$

coming together with the natural transformation $\mu' = (\mu'_{\mathbf{X}}) : P' \to id$.

5. Pro-extensions and Localization

We recall the concept of localization. For any functor $\Phi : \mathbf{K} \to \hat{\mathbf{K}}$ let $INV(\Phi)$ denote the class of all morphisms f in $\mathbf{K}$ such that $\Phi(f)$ is an isomorphism in $\hat{\mathbf{K}}$.

Definition 5.1. Let $\Phi : \mathbf{K} \to \hat{\mathbf{K}}$ be a functor.

(1) Let $F : \mathbf{K} \to \mathbf{L}$ be a functor. A functor $\hat{F} : \hat{\mathbf{K}} \to \mathbf{L}$ is called a Φ-*shift of* F if $\hat{F} \circ \Phi = F$.

(2) Let Σ be a class of morphisms of $\mathbf{K}$. Φ is said to be a *localization at* Σ if
   (a) $\Sigma \subset INV(\Phi)$
   (b) Each $F : \mathbf{K} \to \mathbf{L}$ satisfying $\Sigma \subset INV(F)$ has a unique Φ-shift.

For each full subcategory $\mathscr{F} \subset \mathscr{C}$ and each wide subcategory $\mathfrak{I} \subset \mathbf{inv\text{-}C}$ we denote by $\mathfrak{I}_{\mathscr{F}}$ resp. $\mathbf{pro\text{-}C}_{\mathscr{F}}$ the full subcategory of $\mathfrak{I}$ resp. $\mathbf{pro\text{-}C}$ having as objects all inverse systems indexed by some $A \in \mathscr{F}$. If $\mathscr{F}$ has only one object A, we simply write $\mathfrak{I}_A$ resp. $\mathbf{pro\text{-}C}_A$. The restriction of Π to $\mathfrak{I}_{\mathscr{F}}$ will again be denoted by

$$\Pi : \mathfrak{I}_{\mathscr{F}} \to \mathbf{pro\text{-}C}_{\mathscr{F}}.$$

Definition 5.2. Let $F : \mathfrak{J}_{\mathscr{F}} \to \mathbf{L}$ be a functor.

(1) A *pro-extension* of F is a Π-shift $\hat{F} : \textbf{pro-C}_{\mathscr{F}} \to \mathbf{L}$ of F.

(2) F satisfies the *shifting condition* if for all morphisms $\mathbf{f}'$, $\mathbf{f}$ of $\mathfrak{J}_{\mathscr{F}} \cap \textbf{str-C}_{\mathscr{F}}$ such that $\mathbf{f}' \succeq \mathbf{f}$ one has $F(\mathbf{f}') = F(\mathbf{f})$.

The focus of this paper are existence and uniqueness of pro-extensions. The existence of a pro-extension clearly implies the shifting condition. The following is an immediate consequence of Theorem 3.6.

Proposition 5.3. *Let $\mathscr{F} \subset \mathscr{C}(cfnt)$ be a full subcategory and $F : \textbf{str-C}_{\mathscr{F}} \to \mathbf{L}$ be a functor. Then the following are equivalent*:

(1) *F has a unique pro-extension.*

(2) *F has a pro-extension.*

(3) *F satisfies the shifting condition.*

In particular, F has at most one pro-extension.

Lemma 5.4. *Let $F : \mathbf{C}^A \to \mathbf{L}$ be a functor and $\hat{F} : \textbf{pro-C}_A \to \mathbf{L}$ a pro-extension of F. Then for each morphism $\mathbf{f} = (\varphi, \mathbf{f}^*) : \mathbf{X} \to \mathbf{Y}$ in $\textbf{str-C}_A$ with cofinal index functor $\varphi \succeq_\tau id$*

$$\hat{F}(\Pi(\mathbf{f})) = F(\mathbf{f}^*) \circ F(\mathbf{i}(\mathbf{X}, \tau))^{-1}.$$

NB $F(\mathbf{i}(\mathbf{X}, \tau)) = \hat{F}(\Pi(\mathbf{i}(\mathbf{X}, \tau)))$ is an isomorphism in $\mathbf{L}$ because $\Pi(\mathbf{i}(\mathbf{X}, \tau))$ is an isomorphism in $\textbf{pro-C}_A$.

Proof. $\hat{F}(\Pi(\mathbf{f})) = \hat{F}(\Pi(\mathbf{f}^*)) \circ \hat{F}(\Pi(\mathbf{r}(\mathbf{X}, \varphi))) = \hat{F}(\Pi(\mathbf{f}^*)) \circ \hat{F}(\Pi(\mathbf{i}(\mathbf{X}, \tau))^{-1}) = \hat{F}(\Pi(\mathbf{f}^*)) \circ \hat{F}(\Pi(\mathbf{i}(\mathbf{X}, \tau)))^{-1} = F(\mathbf{f}^*) \circ F(\mathbf{i}(\mathbf{X}, \tau))^{-1}$. $\square$

Theorem 5.5. *Let $A \in \mathscr{C}(cfnt)$. Then $\Pi : \mathbf{C}^A \to \textbf{pro-C}_A$ is a localization at the class $I(A)$ of all morphisms $\mathbf{i}(\mathbf{X}, \tau)$ where τ establishes a relation $\varphi \succeq_\tau id$ for some cofinal $\varphi \in \mathscr{C}(A, A)$. (NB A functor $\varphi \succeq id$ is cofinal if and only if it is equalizing.)*

Proof. By Proposition 5.3 it suffices to show that each functor $F : \mathbf{C}^A \to \mathbf{L}$ with $I(A) \subset INV(F)$ has a pro-extension.

We know that each morphism $\mathfrak{f} : \mathbf{X} \to \mathbf{Y}$ in $\textbf{pro-C}_A$ is represented by a morphism $\mathbf{f} = (\varphi, \mathbf{f}^*)$ in $\textbf{str-C}_A$ with an equalizing $\varphi \succeq_\tau id$. Define

$$\hat{F}(\mathfrak{f}) = F(\mathbf{f}^*) \circ F(\mathbf{i}(\mathbf{X}, \tau))^{-1}.$$

We show that this does not depend on the choice of the representative $\mathbf{f}$ and the choice of τ. Let $\mathbf{f}_i = (\varphi_i, \mathbf{f}_i^*)$ be representatives of $\mathfrak{f}$ such that $\varphi_i \succeq_{\tau_i} id$. There exists $\mathbf{g} = (\psi, \mathbf{g}^*)$ such that ψ is equalizing, $\mathbf{g} \succeq_{\sigma_i} \mathbf{f}_i$ and $\tau_1\sigma_1 = \tau_2\sigma_2 = \omega : \psi \to id$. The diagram

$$\begin{array}{ccccc}
\mathbf{X} & \xleftarrow{\mathbf{i}(\mathbf{X},\tau_i)} & \varphi^*(\mathbf{X}) & \xrightarrow{\mathbf{f}_i^*} & \mathbf{Y} \\
& {\scriptstyle \mathbf{i}(\mathbf{X},\omega)} \nwarrow & \uparrow {\scriptstyle \mathbf{i}(\mathbf{X},\sigma_i)} & \nearrow {\scriptstyle \mathbf{g}^*} & \\
& & \psi^*(\mathbf{X}) & &
\end{array}$$

commutes and we infer $F(\mathbf{f}_i^*) \circ F(\mathbf{i}(\mathbf{X}, \tau_i))^{-1} = F(\mathbf{g}^*) \circ F(\mathbf{i}(\mathbf{X}, \omega))^{-1}$.

We next show that $\hat{F}$ is a functor. It is trivial that $\hat{F}(\mathfrak{id}) = id$. Let $\mathfrak{g}$ be represented by $\mathbf{g} = (\psi, \mathbf{g}^*) : \mathbf{X} \to \mathbf{Z}$ with $\psi \succeq_\sigma id$. Define a natural transformation $\varphi^*(\sigma) : \varphi \circ \psi \to \varphi, \varphi^*(\sigma)_\alpha = \varphi(\sigma_\alpha)$. Then $\mathfrak{g} \circ \mathfrak{f}$ is represented by $\mathbf{g} \circ \mathbf{f} = (\varphi \circ \psi, (\mathbf{g} \circ \mathbf{f})^*)$, where $\varphi \circ \psi \succeq_{\tau \circ \varphi^*(\sigma)} id$. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
\mathbf{X} & \xleftarrow{\mathbf{i}(\mathbf{X},\tau)} & \varphi^*(\mathbf{X}) & \xrightarrow{\mathbf{f}^*} & \mathbf{Y} & \xleftarrow{\mathbf{i}(\mathbf{Y},\sigma)} & \psi^*(\mathbf{Y}) & \xrightarrow{\mathbf{g}^*} & \mathbf{Z} \\
& & & {\scriptstyle \mathbf{i}(\varphi^*(\mathbf{X}),\sigma)} \nwarrow & & \nearrow {\scriptstyle \psi^*(\mathbf{f}^*)} & & & \\
& {\scriptstyle \mathbf{i}(\mathbf{X},\tau\circ\varphi^*(\sigma))} \nwarrow & & & \psi^*((\varphi^*(\mathbf{X}))) & & & \nearrow {\scriptstyle (\mathbf{g}\circ\mathbf{f})^*} & \\
& & & & \| & & & & \\
& & & & (\varphi \circ \psi)^*(\mathbf{X}) & & & &
\end{array}$$

which shows that $\hat{F}(\mathfrak{g} \circ \mathfrak{f}) = \hat{F}(\mathfrak{g}) \circ \hat{F}(\mathfrak{f})$.

For level morphisms one has $\varphi = id$ and $\tau = id$ so that $\mathbf{i}(\mathbf{X}, id) = \mathbf{id}$ and $\mathbf{f}^* = \mathbf{f}$, hence $\hat{F} \circ \Pi = F$. $\square$

Let $\varphi \in \mathscr{C}(B, A)$ be cofinal. Define a functor

$$\bar{\varphi}^* : \textbf{pro-C}_A \to \textbf{pro-C}_B$$

as follows: For the objects set $\bar{\varphi}^*(\mathbf{X}) = \varphi^*(\mathbf{X})$, for the morphisms $\mathfrak{f} : \mathbf{X} \to \mathbf{Y}$ set $\bar{\varphi}^*(\mathfrak{f}) = \Pi(\mathbf{r}(\mathbf{Y}, \varphi))\mathfrak{f}\Pi(\mathbf{r}(\mathbf{X}, \varphi))^{-1}$. Then by construction

(1) The following diagram commutes:

$$\begin{array}{ccc}
\mathbf{C}^A & \xrightarrow{\varphi^*} & \mathbf{C}^B \\
{\scriptstyle \Pi} \downarrow & & \downarrow {\scriptstyle \Pi} \\
\textbf{pro-C}_A & \xrightarrow[\bar{\varphi}^*]{} & \textbf{pro-C}_B
\end{array}$$

(2) The $\Pi(\mathbf{r}(\mathbf{X}, \varphi))$ constitute a natural isomorphism $id \to \bar{\varphi}^*$.

Given a cofinal $\varphi \in \mathscr{C}(cfnt)(B, A)$, we call a functor $F : \textbf{str-C}_{\mathscr{C}(cfnt)} \to \mathbf{L}$ *admissible with respect to* φ if all reindexers over φ are contained in $INV(F)$. It is called *strongly admissible with respect to* φ if in addition $F|_{\textbf{str-C}_A}$, $F|_{\textbf{str-C}_B}$ have pro-extensions $F_A : \textbf{pro-C}_A \to \mathbf{L}$, $F_B : \textbf{pro-C}_B \to \mathbf{L}$. Note that these are unique by Theorem 5.5.

LEMMA 5.6. *Let* $F : \textbf{str-C}_{\mathscr{C}(cfnt)} \to \mathbf{L}$ *be strongly admissible with respect to* φ. *Then the* $F(\mathbf{r}(\mathbf{X}, \varphi))$ *constitute a natural isomorphism* $F_A \to F_B\bar{\varphi}^*$.

PROOF. Define a functor $F'_A : \textbf{pro-C}_A \to \mathbf{L}$ by $F'_A(\mathbf{X}) = F(\mathbf{X})$ and $F'_A(\mathfrak{f}) = F(\mathbf{r}(\mathbf{Y}, \varphi))^{-1} F_B\bar{\varphi}^*(\mathfrak{f}) F(\mathbf{r}(\mathbf{X}, \varphi))$ for $\mathfrak{f} : \mathbf{X} \to \mathbf{Y}$. For level morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ we have $F'_A([\mathbf{f}]) = F(\mathbf{r}(\mathbf{Y}, \varphi))^{-1} F_B([\varphi^*(\mathbf{f})]) F(\mathbf{r}(\mathbf{X}, \varphi)) = F(\mathbf{r}(\mathbf{Y}, \varphi))^{-1} F(\varphi^*(\mathbf{f})) F(\mathbf{r}(\mathbf{X}, \varphi))$ $= F(\mathbf{r}(\mathbf{Y}, \varphi))^{-1} F(\varphi^*(\mathbf{f})\mathbf{r}(\mathbf{X}, \varphi)) = F(\mathbf{r}(\mathbf{Y}, \varphi))^{-1} F(\mathbf{r}(\mathbf{Y}, \varphi)\mathbf{f}) = F(\mathbf{f}) = F_A([\mathbf{f}])$. By the uniqueness of pro-extensions of functors living on $\mathbf{C}^A$ we see that $F'_A = F_A$. □

LEMMA 5.7. *Let* $\mathscr{F} \subset \mathscr{C}$ *and* $\mathscr{G}, \mathscr{H} \subset \mathscr{C}(cfnt)$ *be full subcategories such that* $\mathscr{G}, \mathscr{H} \subset \mathscr{F}$, $F : \textbf{str-C}_{\mathscr{F}} \to \mathbf{L}$ *be a functor and* $G : \textbf{pro-C}_{\mathscr{G}} \to \mathbf{L}$ *resp.* $H : \textbf{pro-C}_{\mathscr{H}} \to \mathbf{L}$ *be pro-extensions of* $F|_{\textbf{str-C}_{\mathscr{G}}}$ *resp.* $F|_{\textbf{str-C}_{\mathscr{H}}}$. *Let* $A_1, A_2, A_3 \in \mathscr{F}$, $A'_1, A'_2, A'_3 \in \mathscr{G}$ *and* $\varphi_i : A'_i \to A_i$ *be cofinal functors such that F is admissible with respect to* φ_1, φ_2, φ_3. *For each pro-morphism* $\mathfrak{f} : \mathbf{X_1} \to \mathbf{X_2}$ *between inverse systems* $\mathbf{X_i}$ *indexed by* A_i *define a morphism* $E(F, G, \varphi_1, \varphi_2)(\mathfrak{f}) : F(\mathbf{X_1}) \to F(\mathbf{X_2})$ *in* $\mathbf{L}$ *by*

$$E(F, G, \varphi_1, \varphi_2)(\mathfrak{f}) = F(\mathbf{r}(\mathbf{X_2}, \varphi_2))^{-1} G([\mathbf{r}(\mathbf{X_2}, \varphi_2)]\mathfrak{f}[\mathbf{r}(\mathbf{X_1}, \varphi_1)]^{-1}) F(\mathbf{r}(\mathbf{X_1}, \varphi_1)).$$

(1) *For any morphism* $\mathbf{f} : \mathbf{X_1} \to \mathbf{X_2}$ *in* $\textbf{str-C}$ *which admits a morphism* $\mathbf{f}' : \varphi_1^*(\mathbf{X_1}) \to \varphi_2^*(\mathbf{X_2})$ *in* $\textbf{str-C}$ *such that* $\mathbf{r}(\mathbf{X_2}, \varphi_2)\mathbf{f} = \mathbf{f}'\mathbf{r}(\mathbf{X_1}, \varphi_1)$ *one has*

$$E(F, G, \varphi_1, \varphi_2)([\mathbf{f}]) = F(\mathbf{f}).$$

(2) *If* $\mathfrak{g} : \mathbf{X_2} \to \mathbf{X_3}$ *is pro-morphism and* $\mathbf{X_3}$ *is indexed by* A_3, *then*

$$E(F, G, \varphi_2, \varphi_3)(\mathfrak{g}) E(F, G, \varphi_1, \varphi_2)(\mathfrak{f}) = E(F, G, \varphi_1, \varphi_3)(\mathfrak{g}\mathfrak{f}).$$

(3) *If* $A'_1 = A'_2 = B$ *and* $\psi : B' \to B$ *is a cofinal functor such that* $B' \in \mathscr{H}$ *and* F *is admissible with respect to* ψ, *then*

$$E(F, G, \varphi_1, \varphi_2)(\mathfrak{f}) = E(F, H, \varphi_1\psi, \varphi_2\psi)(\mathfrak{f}).$$

PROOF. (1) follows from

$$
\begin{aligned}
&E(F, G, \varphi_1, \varphi_2)([\mathbf{f}])\\
&\quad = F(\mathbf{r}(\mathbf{X_2}, \varphi_2))^{-1} G([\mathbf{r}(\mathbf{X_2}, \varphi_2)][\mathbf{f}][\mathbf{r}(\mathbf{X_1}, \varphi_1)]^{-1}) F(\mathbf{r}(\mathbf{X_1}, \varphi_1))\\
&\quad = F(\mathbf{r}(\mathbf{X_2}, \varphi_2))^{-1} G([\mathbf{f}']) F(\mathbf{r}(\mathbf{X_1}, \varphi_1)) = F(\mathbf{r}(\mathbf{X_2}, \varphi_2))^{-1} F(\mathbf{f}') F(\mathbf{r}(\mathbf{X_1}, \varphi_1))\\
&\quad = F(\mathbf{r}(\mathbf{X_2}, \varphi_2))^{-1} F(\mathbf{r}(\mathbf{X_2}, \varphi_2)) F(\mathbf{f}) = F(\mathbf{f}).
\end{aligned}
$$

(2) is obvious and (3) follows from Lemma 5.6. □

THEOREM 5.8. *Let $F : \mathbf{str\text{-}C}_{\mathscr{C}(cfnt)} \to \mathbf{L}$ be a functor such that all projection reindexers in $\mathbf{str\text{-}C}_{\mathscr{C}(cfnt)}$ are contained in $INV(F)$ and each $F|_{\mathbf{str\text{-}C}_A}$ has a pro-extension $F_A : \mathbf{pro\text{-}C}_A \to \mathbf{L}$ (which is unique by Theorem 5.5). Then F has a unique pro-extension.*

PROOF. By Proposition 5.3 it suffices to prove the existence of a pro-extension. We use Lemma 5.7.

For a morphism $\mathfrak{f} : \mathbf{X_1} \to \mathbf{X_2}$ in $\mathbf{pro\text{-}C}_{\mathscr{C}(cfnt)}$, where $\mathbf{X_i}$ is indexed by A_i, define

$$\hat{F}(\mathfrak{f}) = E(F, F_{A_1 \times A_2}, \pi^1_{A_1, A_2}, \pi^2_{A_1, A_2})(\mathfrak{f}) : F(\mathbf{X_1}) \to F(\mathbf{X_2})$$

where $\pi^i_{A_1, A_2} : A_1 \times A_2 \to A_i$ denotes the projection which is cofinal. Note that $\mathscr{G} \subset \mathscr{C}$ is the full subcategory having the one object $A_1 \times A_2$.

Claim 1: For $A_1 = A_2 = A$ we have $\hat{F}([\mathbf{f}]) = F(\mathbf{f})$ for any morphism $\mathbf{f} : \mathbf{X_1} \to \mathbf{X_2}$ in $\mathbf{C}^A$.

PROOF. Let $\tau : A \times A \to A \times A$ be the functor exchanging coordinates. Set $\mathbf{f}' = \mathbf{r}((\pi^1_{A,A})^*(\mathbf{X_2}), \tau)(\pi^1_{A,A})^*(\mathbf{f}) \in \mathbf{str\text{-}C}((\pi^1_{A,A})^*(\mathbf{X_1}), \tau^*((\pi^1_{A,A})^*(\mathbf{X_2})))$. Since $\pi^1_{A,A}\tau = \pi^2_{A,A}$, we have

$$
\begin{aligned}
\mathbf{f}'\mathbf{r}(\mathbf{X}, \pi^1_{A,A}) &= \mathbf{r}((\pi^1_{A,A})^*(\mathbf{Y}), \tau)(\pi^1_{A,A})^*(\mathbf{f})\mathbf{r}(\mathbf{X}, \pi^1_{A,A})\\
&= \mathbf{r}((\pi^1_{A,A})^*(\mathbf{Y}), \tau)\mathbf{r}(\mathbf{Y}, \pi^1_{A,A})\mathbf{f} = \mathbf{r}(\mathbf{Y}, \pi^2_{A,A})\mathbf{f}.
\end{aligned}
$$

Lemma 5.7 proves Claim 1. □

Claim 2: The above definition yields a functor $\hat{F} : \mathbf{pro\text{-}C}_{\mathscr{C}(cfnt)} \to \mathbf{L}$.

PROOF. Claim 1 shows that $\hat{F}([\mathbf{id}]) = id$. Let $\mathfrak{g} : \mathbf{X_2} \to \mathbf{X_3}$ in $\mathbf{pro\text{-}C}_{\mathscr{C}(cfnt)}$. We show that $\hat{F}(\mathfrak{g}\mathfrak{f}) = \hat{F}(\mathfrak{g})\hat{F}(\mathfrak{f})$. Let $\pi^{ij} : A_1 \times A_2 \times A_3 \to A_i \times A_j$ and $\rho^i : A_1 \times A_2 \times$

$A_3 \to A_i$ denote the projection functors which are cofinal. Using Lemma 5.7 we see that

$$\hat{F}(\mathfrak{f}) = E(F, F_{A_1 \times A_2 \times A_3}, \rho^2, \rho^1)(\mathfrak{f}),$$

$$\hat{F}(\mathfrak{g}) = E(F, F_{A_1 \times A_2 \times A_3}, \rho^3, \rho^2)(\mathfrak{f}),$$

$$\hat{F}(\mathfrak{g}\mathfrak{f}) = E(F, F_{A_1 \times A_2 \times A_3}, \rho^3, \rho^1)(\mathfrak{f}).$$

This shows that $\hat{F}(\mathfrak{g}\mathfrak{f}) = \hat{F}(\mathfrak{g})\hat{F}(\mathfrak{f})$. □

Claim 3: Let $\mathbf{X}$ be an inverse system indexed by A, $\varphi \in \mathscr{C}(B, A)$ be any (not necessarily cofinal) functor and $\mathbf{r} = \mathbf{r}(\mathbf{X}, \varphi) : \mathbf{X} \to \varphi^*(\mathbf{X})$ the induced morphism. Then $\hat{F}([\mathbf{r}]) = F(\mathbf{r})$.

Proof. Define $\psi = (\varphi \times id_B)\Delta_B \pi^2_{A,B} : A \times B \to A \times B$ where $\Delta_B : B \to B \times B$ is the diagonal functor. Then $\varphi\pi^2_{A,B} = \pi^1_{A,B}\psi$. With $\mathbf{s} = \mathbf{r}((\pi^1_{A,B})^*(\mathbf{X}), \psi) : (\pi^1_{A,B})^*(\mathbf{X}) \to \psi^*((\pi^1_{A,B})^*(\mathbf{X})) = (\pi^2_{A,B})^*(\varphi^*(\mathbf{X}))$ we obtain $\mathbf{r}(\varphi^*(\mathbf{X}), \pi^2_{A,B})\mathbf{r} = \mathbf{sr}(\mathbf{X}, \pi^1_{A,B})$. Lemma 5.7 shows $\hat{F}([\mathbf{r}]) = F(\mathbf{r})$. □

Claims 1–3 prove $\hat{F}([\mathbf{f}]) = F(\mathbf{f})$ for all morphisms in $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{C}(cfnt)}$ since we have $\mathbf{f} = \mathbf{f}^*\mathbf{r}(\mathbf{X}, ind(\mathbf{f}))$ with a level morphism $\mathbf{f}^*$. □

Remark 5.9. Theorem 5.8 can be generalized to functors $F : \mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}} \to \mathbf{L}$ where $\mathscr{F} \subset \mathscr{C}(cfnt)$ is a full subcategory such that $A \times B \in \mathscr{F}$ whenever $A, B \in \mathscr{F}$.

An interesting question is whether $\Pi : \mathbf{str}\text{-}\mathbf{C}_{\mathscr{C}(cfnt)} \to \mathbf{pro}\text{-}\mathbf{C}_{\mathscr{C}(cfnt)}$ is a localization at reindexers in $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{C}(cfnt)}$. We conjecture that it is not. A first indication is

Proposition 5.10. *Let $\mathscr{F} \subset \mathscr{C}^{nmx}$ be a full subcategory such that all $A \in \mathscr{F}$ are totally preordered with respect to the induced preordering. Then $\Pi : \mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}} \to \mathbf{pro}\text{-}\mathbf{C}_{\mathscr{F}}$ is not a localization at reindexers in $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}}$.*

Proof. Let $\mathbf{Z}_2$ denote the category having one object $*$ and two morphisms 0, 1 which are composed by $1 \circ 1 = 1$ and $0 \circ \mu = \mu \circ 0 = 0$. Define a functor $\Theta : \mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}} \to \mathbf{Z}_2$ by setting for each morphism $\mathbf{f} = (\varphi, \mathbf{f}^*)$

$$\Theta(\mathbf{f}) = \begin{cases} 0 & \varphi \text{ is not weakly cofinal} \\ 1 & \varphi \text{ is weakly cofinal} \end{cases}$$

That this is in fact a functor can be seen as follows. Let $\varphi \in \mathscr{C}(B, A)$, $\psi \in \mathscr{C}(C, B)$. It is obvious that if φ is not weakly cofinal, then $\varphi \circ \psi$ is not weakly cofinal, and if φ, ψ are weakly cofinal, then $\varphi \circ \psi$ is weakly cofinal. We claim that if ψ is not weakly cofinal and B is totally preordered and A has no maximal element, then $\varphi \circ \psi$ is not weakly cofinal. There exists $\beta_0 \in B$ such that $B(\psi(\gamma), \beta_0) = \varnothing$ for all $\gamma \in C$. Since B is totally preordered, we have $\beta_0 \geq \psi(\gamma)$ for all $\gamma \in C$. Moreover $\varphi(\beta_0)$ is not a maximal element so that we can find $\alpha_0 \in A$ such that $A(\varphi(\beta_0), \alpha_0) = \varnothing$. Choose $\alpha_1 \geq \alpha_0, \varphi(\beta_0)$. If $\varphi \circ \psi$ were weakly cofinal, we could find $\gamma_0 \in C$ such that $\varphi(\psi(\gamma_0)) \geq \alpha_1$. But then $\varphi(\beta_0) \geq \varphi(\psi(\gamma_0)) \geq \alpha_1 \geq \alpha_0$ which is a contradiction.

Assume $\Pi : \mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}} \to \mathbf{pro}\text{-}\mathbf{C}_{\mathscr{F}}$ were a localization at reindexers in $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}}$. Since all these reindexers are contained in $INV(\Theta)$, there is a unique functor $\Theta' : \mathbf{pro}\text{-}\mathbf{C}_{\mathscr{F}} \to \mathbf{Z}_2$ such that $\Theta' \circ \Pi = \Theta$. This implies that all morphisms in $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}}$ which induce isomorphisms in $\mathbf{pro}\text{-}\mathbf{C}_{\mathscr{F}}$ ("Π-isomorphisms") are contained in $INV(\Theta)$. But this is not true because there exist Π-isomorphisms having no weakly cofinal index function. For example, choose any object X of $\mathbf{C}$, any $A \in \mathscr{F}$ and any constant functor $\varphi : A \to A$. Let $[X]_A$ be the inverse system indexed by A such that all $X_\alpha = X$ and all bondings are identities. Then $(\varphi, (f_\alpha = id_X)) : [X]_A \to [X]_A$ is a Π-isomorphism not contained in $INV(\Theta)$. □

6. Pro-extensions of Functors on str-C

Theorem 6.1. *Let $F : \mathbf{str}\text{-}\mathbf{C} \to \mathbf{L}$ be a functor. Then the following are equivalent*:

(1) *F has a unique pro-extension.*

(2) *F has a pro-extension.*

(3) *F satisfies the shifting condition and $INV(F)$ contains all standard cofinite reindexers.*

(4) *$F|_{\mathbf{str}\text{-}\mathbf{C}_{\mathscr{D}(ord,\, cfnt)}}$ has a pro-extension* (*which is unique by Proposition* 5.3) *and $INV(F)$ contains all modified cofinite reindexers.*

In particular, *F has at most one pro-extension.*

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (4)$: By Proposition 5.3 $F|_{\mathbf{str}\text{-}\mathbf{C}_{\mathscr{D}(ord,\, cfnt)}}$ has a pro-extension. To complete the proof it suffices to show that all reindexers having the form $\mathbf{r} = \mathbf{r}(\mathbf{X}, \pi_A) : \mathbf{X} \to \pi_A^*(\mathbf{X})$ are contained in $INV(F)$. Define a functor $\iota : A \to A \times \mathbf{N}$, $\iota(\alpha) = (\alpha, 1)$. Let $\mathbf{s} = \mathbf{r}(\pi_A^*(\mathbf{X}), \iota) : \pi_A^*(\mathbf{X}) \to \iota^*(\pi_A^*(\mathbf{X})) = \mathbf{X}$. We have $\mathbf{s} \circ \mathbf{r} = \mathbf{id}$ and $\mathbf{id} \succeq \mathbf{r} \circ \mathbf{s}$. This implies that $F(\mathbf{r})$ is an isomorphism whose inverse is $F(\mathbf{s})$.

(4) $\Rightarrow$ (1): Let $\bar{F} : \textbf{pro-C}_{\mathscr{D}(ord, cfnt)} \to \mathbf{L}$ be a pro-extension of $F|_{\textbf{str-C}_{\mathscr{D}(ord, cfnt)}}$. Define $\hat{F}$ by $\hat{F}(\mathbf{X}) = F(\mathbf{X})$ for the objects; for the morphisms $\mathfrak{f} : \mathbf{X_1} \to \mathbf{X_2}$ set (cf. Lemma 5.7)

$$\hat{F}(\mathfrak{f}) = E(F, \bar{F}, \mu'_{A_1}, \mu'_{A_2})(\mathfrak{f}).$$

It is obvious that $\hat{F}$ is a functor. We show that it is a pro-extension of F.

Let $\mathbf{f}$ be a morphism of $\textbf{str}_{\textbf{reg}}\textbf{-C}$. Then $\mathbf{r}(\mathbf{X_2}, \mu'_{A_2})\mathbf{f} = P'(\mathbf{f})\mathbf{r}(\mathbf{X_1}, \mu'_{A_1})$ so that $\hat{F}([\mathbf{f}]) = F(\mathbf{f})$.

For an arbitrary morphism $\mathbf{f} = (\varphi, \mathbf{f}^*)$ in $\textbf{str-C}$ we split $\varphi = \pi \circ \tilde{\varphi}$ where $\pi : A \times B \to A$ denotes projection and $\tilde{\varphi} : B \to A \times B$, $\tilde{\varphi}(\beta) = (\varphi(\beta), \beta)$. This induces a splitting

$$\mathbf{f} = \mathbf{f}_{(\pi, \tilde{\varphi})} \circ \mathbf{r}(\mathbf{X}, \pi).$$

$\mathbf{r} = \mathbf{r}(\mathbf{X}, \pi)$ is a reindexer, hence $[\mathbf{r}]$ is an isomorphism in $\textbf{pro-C}$ so that $\hat{F}([\mathbf{r}])$ is an isomorphism.

$\mathbf{f}_{(\pi, \tilde{\varphi})} = (\tilde{\varphi}, \mathbf{f}^*)$ is a morphism of $\textbf{str}_{\textbf{reg}}\textbf{-C}$ since $\tilde{\varphi}$ is an embedding. Choose any $\beta_0 \in B$ and define a functor $\iota : A \to A \times B$, $\iota(\alpha) = (\alpha, \beta_0)$. We have $\pi \circ \iota = id$, thus $\iota^*(\pi^*(\mathbf{X})) = \mathbf{X}$. Letting $\mathbf{s} = \mathbf{r}(\pi^*(\mathbf{X}), \iota) : \pi^*(\mathbf{X}) \to \mathbf{X}$, we obtain $\mathbf{s} \circ \mathbf{r} = \mathbf{id}$ so that $[\mathbf{s}]$ is the inverse isomorphism to $[\mathbf{r}]$. Since ι is an embedding, $\mathbf{s}$ is a morphism of $\textbf{str}_{\textbf{reg}}\textbf{-C}$ so that $F(\mathbf{s}) = \hat{F}([\mathbf{s}])$ which is an isomorphism. We have $F(\mathbf{s}) \circ \hat{F}([\mathbf{r}]) = \hat{F}([\mathbf{s}]) \circ \hat{F}([\mathbf{r}]) = \hat{F}([\mathbf{id}]) = F(\mathbf{id}) = F(\mathbf{s}) \circ F(\mathbf{r})$, hence $\hat{F}([\mathbf{r}]) = F(\mathbf{r})$. This yields

$$\hat{F}([\mathbf{f}]) = \hat{F}([\mathbf{f}_{(\pi, \tilde{\varphi})}]) \circ \hat{F}([\mathbf{r}]) = F(\mathbf{f}_{(\pi, \tilde{\varphi})}) \circ F(\mathbf{r}) = F(\mathbf{f}).$$

Finally let $G : \textbf{pro-C} \to \mathbf{L}$ be any pro-extension of F. Then $G' = G|_{\textbf{pro-C}_{\mathscr{D}(ord, cfnt)}}$ is a pro-extension of $F|_{\textbf{str-C}_{\mathscr{D}(ord, cfnt)}}$ whence $G' = \bar{F}$ by Proposition 5.3. We infer

$$\begin{aligned} G(\mathfrak{f}) &= G([\mathbf{r}(\mathbf{X_2}, \mu'_{A_2})])^{-1} G([\mathbf{r}(\mathbf{X_2}, \mu'_{A_2})]\mathfrak{f}[\mathbf{r}(\mathbf{X_1}, \mu'_{A_1})]^{-1}) G([\mathbf{r}(\mathbf{X_1}, \mu'_{A_1})]) \\ &= F(\mathbf{r}(\mathbf{X_2}, \mu'_{A_2}))^{-1} \bar{F}([\mathbf{r}(\mathbf{X_2}, \mu'_{A_2})]\mathfrak{f}[\mathbf{r}(\mathbf{X_1}, \mu'_{A_1})]^{-1}) F(\mathbf{r}(\mathbf{X_1}, \mu'_{A_1})) = \hat{F}(\mathfrak{f}). \quad \square \end{aligned}$$

Theorem 6.1 is an extension of Proposition 5.3. The price we have to pay in (3) is the additional condition that $INV(F)$ contains all standard cofinite reindexers; but note that since F satisfies the shifting condition, each standard cofinite reindexer based on a $\mu_A : A \to P(A)$ with $A \in \mathscr{C}(cfnt)$ is automatically contained in $INV(F)$.

REMARK 6.2. Theorem 6.1 can be generalized to functors $F : \textbf{str-C}_{\mathscr{F}} \to \mathbf{L}$ where $\mathscr{F} \subset \mathscr{C}$ is a full subcategory such that $\mathscr{D}(ord, cfnt) \subset \mathscr{F}$ and $A \times B \in \mathscr{F}$ whenever $A, B \in \mathscr{F}$. Such an $\mathscr{F}$ will be called *admissible*.

7. Extending Functors from lev-C to str-C

Throughout this section let $\mathscr{F} \subset \mathscr{C}$ be an admissible full subcategory and $F : \mathbf{lev}\text{-}\mathbf{C}_{\mathscr{F}} \to \mathbf{L}$ be a fixed functor. A necessary criterion for the existence of a pro-extension is the existence of an extension of F to $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}} \supset \mathbf{lev}\text{-}\mathbf{C}_{\mathscr{F}}$. Since functors on $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}}$ have at most one pro-extension, we have a 1-1-correspondence between pro-extensions of F and extensions of F to $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}}$ which itself have a pro-extension. A characterization of pro-extensible functors on $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}}$ was given in Theorem 6.1.

For each extension $\tilde{F}$ of F to $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}}$ and each morphism $\varphi : B \to A$ in $\mathscr{F}$ we obtain a natural transformation

$$\Lambda(\varphi) = \Lambda_{\tilde{F}}(\varphi) : F|_{\mathbf{C}^A} \to F|_{\mathbf{C}^B} \circ \varphi^*, \quad \Lambda(\varphi)_{\mathbf{X}} = \tilde{F}(\mathbf{r}(\mathbf{X}, \varphi))$$

such that

(7.1) $\quad \Lambda(\varphi \circ \psi)_{\mathbf{X}} = \Lambda(\psi)_{\varphi^*(\mathbf{X})} \circ \Lambda(\varphi)_{\mathbf{X}} \quad$ for all $\varphi \in \mathscr{F}(B, A)$, $\psi \in \mathscr{F}(C, B)$.

Any collection $\Lambda = (\Lambda(\varphi))_{\varphi \in mor(\mathscr{F})}$ assigning to each morphism $\varphi : B \to A$ in $\mathscr{F}$ a natural transformation $\Lambda(\varphi) : F|_{\mathbf{C}^A} \to F|_{\mathbf{C}^B} \circ \varphi^*$ such that (7.1) is satisfied will be called an *extensor* for F.

Given an extensor, define for each morphism $\mathbf{f} = (\varphi, \mathbf{f}^*) : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}}$

$$F_\Lambda(\mathbf{f}) = F(\mathbf{f}^*) \circ \Lambda(\varphi)_{\mathbf{X}} : F(\mathbf{X}) \to F(\mathbf{Y}).$$

This yields a functor $F_\Lambda : \mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}} \to \mathbf{L}$ which is an extension of F. Moreover we have $F_{\Lambda_{\tilde{F}}} = \tilde{F}$ and $\Lambda_{F_\Lambda} = \Lambda$. This means that there is 1-1-correspondence between extensions $\tilde{F}$ of F and extensors Λ for F.

Examples of extensors occur in the context of the homotopy limit (see e.g. [1, Ch. XI §3.2], [4, §4.3] although (7.1) has not been considered there). A necessary condition for F_Λ having a pro-extension is

(7.2) $\quad \Lambda(\varphi)$ is a natural isomorphism whenever φ is a cofinal functor.

This reflects the fact that a necessary condition for the existence of a pro-extension of a functor $\tilde{F}$ on $\mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}}$ is

(7.3) $\qquad \tilde{F}(\mathbf{r})$ is an isomorphism whenever $\mathbf{r}$ is a reindexer.

It is not known to the author whether this condition is sufficient (this would imply that $\Pi : \mathbf{str}\text{-}\mathbf{C}_{\mathscr{F}} \to \mathbf{pro}\text{-}\mathbf{C}_{\mathscr{F}}$ is a localization at reindexers which appears doubtful in the light of Proposition 5.10).

This indicates that the concrete construction of the homotopy limit on **Ho**(**pro-SS**) in [4, §4.3] contains a gap[9]. In [4] one finds an explicit construction of a functor $Ex^{\infty}: \mathbf{lev\text{-}SS}_{\mathscr{D}(ord,cfnt)} \to \mathbf{Ho}((\mathbf{pro\text{-}SS})_f)$ which is claimed to have a pro-extension to $\mathbf{pro\text{-}SS}_{\mathscr{D}(ord,cfnt)}$. An extensor Λ for Ex^{∞} is constructed in [4, (4.3.3)]; it satisfies (7.2). What is missing is the verification of (7.1) and a proof of either that (7.3) is sufficient for the existence of a pro-extension or that Ex^{∞}_{Λ} satisfies the shifting condition. Fortunately this gap is not dramatic because the universal construction of the homotopy limit on **Ho**(**pro-C**) in [4, §4.2] is correct.

8. The First Derived Limit on pro-G

We begin by reviewing the definition of the first derived limit of an inverse system $\mathbf{X}: A \to \mathbf{G}$ given by Bousfield and Kan [1, Ch. XI, §6.5] as the cohomotopy set $\pi^1(\Pi^*\mathbf{X})$ of the cosimplicial replacement $\Pi^*\mathbf{X}$ of $\mathbf{X}$. The latter is defined for inverse systems in arbitrary categories **C** with products. It consists of objects $\Pi^n\mathbf{X} \in \mathbf{C}$, $n \geq 0$, and coface and codeneracy morphisms. With $A_n = \{\mathbf{u} = (a_0 \xleftarrow{u_1} a_1 \xleftarrow{u_2} \cdots \xleftarrow{u_{n-1}} a_{n-1} \xleftarrow{u_n} a_n) \mid a_i \in ob(A), u_i \in mor(A)\}$ we have

$$\Pi^n\mathbf{X} = \prod_{\mathbf{u} \in A_n} \mathbf{X}_{\mathbf{u}}, \quad \mathbf{X}_{\mathbf{u}} = \mathbf{X}(a_0) = X_{a_0}.$$

This construction produces a functor $\Pi^*: \mathbf{lev\text{-}C} \to c\mathbf{C}$ = category of cosimplicial objects in **C** (see [1]). π^1 is a functor from $c\mathbf{G}$ to the category $\mathbf{Set_0}$ of pointed sets and Bousfield and Kan define

$$\varprojlim{}^1 = \pi^1 \circ \Pi^*: \mathbf{lev\text{-}G} \to \mathbf{Set_0}.$$

Π^* has a straightforward extension to **str-C**. In fact, each $\mathbf{f} = (\varphi, (f_b)) \in \mathbf{str\text{-}C}(\mathbf{X}, \mathbf{Y})$ induces a cosimplicial morphism

$$\Pi^*\mathbf{f}: \Pi^*\mathbf{X} \to \Pi^*\mathbf{Y}$$

which consists of the unique morphisms $\Pi^n\mathbf{f}$ making the following diagrams commute for all $\mathbf{v} = (b_0 \xleftarrow{v_1} b_1 \xleftarrow{v_2} \cdots \xleftarrow{v_n} b_n) \in B_n$:

$$\begin{array}{ccc} \Pi^n\mathbf{X} & \xrightarrow{\Pi^n\mathbf{f}} & \Pi^n\mathbf{Y} \\ \downarrow{\scriptstyle \pi_{\varphi(\mathbf{v})}} & & \downarrow{\scriptstyle \pi_{\mathbf{v}}} \\ \mathbf{X}_{\varphi(\mathbf{v})} & \xrightarrow[f_{b_0}]{} & \mathbf{Y}_{\mathbf{v}} \end{array}$$

[9] Also the proof of [3, Theorem 4.1] contains a gap. The "naturality properties" of the homotopy limit do not apply to diagrams which commute in the pro-category.

Hence the original $\varprojlim^1 : \mathbf{lev\text{-}G} \to \mathbf{Set_0}$ from [1] has a natural extension

$$\varprojlim^1 = \pi^1 \circ \Pi^* : \mathbf{str\text{-}G} \to \mathbf{Set_0}.$$

Boiling down the definition of π^1 to $\Pi^*\mathbf{X}$ gives us explicit formulae. Set

$$Z\Pi^1\mathbf{X} = \{(x_u) \in \Pi^1\mathbf{X} \mid \forall (u_1, u_2) \in A_2 : p_{u_1}(x_{u_2})x_{u_1u_2}^{-1}x_{u_1} = e\}$$

and define an operation of $\Pi^0\mathbf{X}$ on $Z\Pi^1\mathbf{X}$ by

$$(g_a) \cdot (x_u) = (g_{a_0}x_u p_u(g_{a_1})^{-1})$$

where $u : a_1 \to a_0$. Then $\pi^1(\Pi^*\mathbf{X}) = \varprojlim^1 \mathbf{X}$ is orbit set of this operation. For the morphisms we have

$$\varprojlim^1(\varphi, (f_b))([(x_u)]) = [(f_{b_0}(x_{\varphi(v)})]$$

where $v : b_1 \to b_0$.

$\varprojlim^1 : \mathbf{lev\text{-}G} \to \mathbf{Set_0}$ has a topological description based on the homotopy limit (see [3], [4]). On **lev-G** one has a natural isomorphism

$$\varprojlim^1 \approx \pi_0 \circ \mathrm{holim} \circ \mathrm{Ho} \circ \Phi^{lev}$$

where $\mathrm{holim} : \mathbf{Ho}(\mathbf{lev\text{-}SS}) \to \mathbf{Ho}(\mathbf{SS})$ is the homotopy limit, $\mathrm{Ho} : \mathbf{lev\text{-}SS} \to \mathbf{Ho}(\mathbf{lev\text{-}SS})$ the quotient functor, $\Phi : \mathbf{G} \to \mathbf{SS}$ a suitably defined functor and $\Phi^{lev} : \mathbf{lev\text{-}G} \to \mathbf{lev\text{-}SS}$ the canonically induced functor. There exists an extension[10] of holim to $\mathbf{Ho}(\mathbf{pro\text{-}SS})$; this induces a pro-extension of $\varprojlim^1$. Unfortunately the extension of holim is not concrete enough to understand what the "topological" pro-extension of $\varprojlim^1$ does with non-level morphisms. As a compensation we establish the purely algebraic

Theorem 8.1. *$\varprojlim^1 : \mathbf{str\text{-}G} \to \mathbf{Set_0}$ has a unique pro-extension $\varprojlim^1 : \mathbf{pro\text{-}G} \to \mathbf{Set_0}$.*

For the proof we need a modified description of $\varprojlim^1 \mathbf{X}$. Let

$$\tilde{A}_n = \{\mathbf{u} \in A_n \mid a_0, \dots, a_n \text{ are } n+1 \text{ distinct objects}\},$$

$$\tilde{\Pi}^n\mathbf{X} = \prod_{\mathbf{u} \in \tilde{A}_n} \mathbf{X}_\mathbf{u},$$

$$Z\tilde{\Pi}^1\mathbf{X} = \{(x_u) \in \tilde{\Pi}^1\mathbf{X} \mid \forall (u_1, u_2) \in \tilde{A}_2 : p_{u_1}(x_{u_2})x_{u_1u_2}^{-1}x_{u_1} = e\}.$$

[10] It is adjoint to the inclusion $\mathbf{Ho}(\mathbf{SS}) \to \mathbf{Ho}(\mathbf{pro\text{-}SS})$ and in that sense unique up to natural isomorphism.

The canonical projection $\tilde{\pi} : \Pi^1\mathbf{X} \to \tilde{\Pi}^1\mathbf{X}$ restricts to $\tilde{\pi} : Z\Pi^1\mathbf{X} \to Z\tilde{\Pi}^1\mathbf{X}$. Moreover, the operation of $\Pi^0\mathbf{X}$ on $Z\Pi^1\mathbf{X}$ obviously restricts to an operation of $\Pi^0\mathbf{X}$ on $Z\tilde{\Pi}^1\mathbf{X}$ defined by the same formula as above.

LEMMA 8.2. *If $A \in \mathscr{C}^{nmx}$, then $\tilde{\pi} : Z\Pi^1\mathbf{X} \to Z\tilde{\Pi}^1\mathbf{X}$ is a bijection such that $\tilde{\pi}((g_a) \cdot (x_u)) = (g_a) \cdot \tilde{\pi}((x_u))$. If $(g_a) \cdot \tilde{\pi}((x_u)) = \tilde{\pi}((x'_u))$, then $(g_a) \cdot (x_u) = (x'_u)$. Therefore $\tilde{\pi}$ induces a bijection $\varprojlim^1 \mathbf{X} \to Z\tilde{\Pi}^1\mathbf{X}/\Pi^0\mathbf{X}$.*

PROOF. It is an easy exercise to show that if A does not have maximal elements, then each diagram in A has an outer cone. Let A'_1 denote the complement of $\tilde{A}_1$ in A_1.

For $u : a_1 \to a_0$ we define an internal diagram $(u) = (a_0, a_1; u, id_{a_0}, id_{a_1})$. Let $(b; v_0 : b \to a_0, v_1 : b \to a_1)$ be an outer cone for (u); note that $v_0 = v_1$ if $a_0 = a_1$. Any such outer cone will be called a *resolution* of u. We have $v_0, v_1 \in \tilde{A}_1$ and for all $(x_u) \in Z\Pi^1\mathbf{X}$ as well as for all $(x_u) \in Z\tilde{\Pi}^1\mathbf{X}$ the following holds:

$$x_u = x_{v_0} p_u(x_{v_1})^{-1} \tag{8.1}$$

This is true because $(u, v_1) \in A_2$ (resp. $(u, v_1) \in \tilde{A}_2$ for $(x_u) \in Z\tilde{\Pi}^1\mathbf{X}$) so that $e = p_u(x_{v_1})x_{uv_1}^{-1}x_u = p_u(x_{v_1})x_{v_0}^{-1}x_u$.

Claim 1: $\tilde{\pi} : Z\Pi^1\mathbf{X} \to Z\tilde{\Pi}^1\mathbf{X}$ is injective.

PROOF. Let $(x_u) \in Z\Pi^1\mathbf{X}$. Then (8.1) shows that the coordinates x_u for $u \in A'_1$ are uniquely determined by the coordinates x_v with $v \in \tilde{A}_1$. $\square$

Claim 2: For $(x_v) \in Z\tilde{\Pi}^1\mathbf{X}$ and $u \in A'_1$, $u : a \to a$, define $x_u = x_v p_u(x_v)^{-1}$ where $(b; v : b \to a)$ is a resolution of (u). This definition is independent on the choice of the resolution and thus produces a canonical extension function $\iota : Z\tilde{\Pi}^1\mathbf{X} \to \Pi^1\mathbf{X}$ (i.e. with $\tilde{\pi}\iota = id$).

PROOF. Let $(b'; v' : b' \to a)$ be another resolution. Choose an outer cone $(c; w : c \to b, w' : c \to b', s : c \to a)$ for $(a, b, b'; v, v')$. We have $(v, w) \in \tilde{A}_2$ so that $e = p_v(x_w)x_{vw}^{-1}x_v = p_v(x_w)x_s^{-1}x_v$. Since $p_u p_v = p_{uv} = p_v$ we obtain

$$x_s p_u(x_s)^{-1} = x_v p_v(x_w) p_u(p_v(x_w)^{-1}x_v^{-1}) = x_v p_u(x_v)^{-1}.$$

Similarly $x_s p_u(x_s)^{-1} = x_{v'} p_u(x_{v'})^{-1}$ which proves the claim. $\square$

Claim 3: $\iota(Z\tilde{\Pi}^1\mathbf{X}) \subset Z\Pi^1\mathbf{X}$ so that $\tilde{\pi} : Z\Pi^1\mathbf{X} \to Z\tilde{\Pi}^1\mathbf{X}$ is bijective.

Proof. Let $(x_u) = \iota((x_v))$. For $(u_0, u_1) \in A_2$ choose an outer cone $(b; v_i : b \to a_i)$ for $(a_0, a_1, a_2; u_0, u_1, u_0u_1, id_{a_0}, id_{a_1}, id_{a_2})$. This yields resolutions for u_0, u_1, u_0u_1. Using (8.1) resp. the definition in Claim 2 we obtain

$$p_{u_0}(x_{u_1})x_{u_0u_1}^{-1}x_{u_0} = p_{u_0}(x_{v_1}p_{u_1}(x_{v_2})^{-1})(x_{v_0}p_{u_0u_1}(x_{v_2})^{-1})^{-1}x_{v_0}p_{u_0}(x_{v_1})^{-1} = e. \qquad \square$$

Claim 4: $\tilde{\pi}((g_a)\cdot(x_u)) = (g_a)\cdot\tilde{\pi}((x_u))$. This is obvious.
Claim 5: If $(g_a)\cdot\tilde{\pi}((x_u)) = \tilde{\pi}((x'_u))$, then $(g_a)\cdot(x_u) = (x'_u)$.

Proof. For $v \in \tilde{A}_1$, $v : a_1 \to a_0$, we have $g_{a_0}x_vp_v(g_{a_1})^{-1} = x'_v$. For an arbitrary $u \in A_1$, $u : a_1 \to a_0$, choose a resolution $(b; v_i : b \to a_i)$. Then

$$\begin{aligned} g_{a_0}x_up_u(g_{a_1})^{-1} &= g_{a_0}x_{v_0}p_u(x_{v_1})^{-1}p_u(g_{a_1})^{-1} \\ &= g_{a_0}x_{v_0}p_{v_0}(g_b)^{-1}p_{v_0}(g_b)p_u(x_{v_1})^{-1}p_u(g_{a_1})^{-1} = x'_{v_0}p_u(x'_{v_1}) = x'_u \end{aligned}$$

where we used $p_{v_0} = p_up_{v_1}$. $\square$

$\square$

Proof of Theorem 8.1. We apply Theorem 6.1 by showing that the conditions in 6.1 (3) are satisfied.

(a) Let $\mathbf{f} = (\varphi, (f_b))$ and $\mathbf{g} = (\psi, (g_b))$ be morphisms $\mathbf{X} \to \mathbf{Y}$ such that $\mathbf{g} \succeq_\tau \mathbf{f}$. We show that $\varprojlim^1 \mathbf{g} = \varprojlim^1 \mathbf{f}$.

We have

$$\varprojlim^1 \mathbf{g}([(x_u)]) = [(g_{b_0}(x_{(\psi(v)}))] = [(f_{b_0}p_{\tau_{b_0}}(x_{(\psi(v)}))].$$

For the pairs $(\tau_{b_0}, \psi(v)), (\varphi(v), \tau_{b_1}) \in A_2$ we obtain

$$p_{\tau_{b_0}}(x_{(\psi(v)})x_{\tau_{b_0}\psi(v)}^{-1}x_{\tau_{b_0}} = e = p_{\varphi(v)}(x_{\tau_{b_1}})x_{\varphi(v)\tau_{b_1}}^{-1}x_{\varphi(v)}.$$

$\varphi(v)\tau_{b_1} = \tau_{b_0}\psi(v)$ implies

$$p_{\tau_{b_0}}(x_{\psi(v)}) = x_{\tau_{b_0}}^{-1}x_{\tau_{b_0}\psi(v)} = x_{\tau_{b_0}}^{-1}x_{\varphi(v)}p_{\varphi(v)}(x_{\tau_{b_1}}).$$

Thus

$$\begin{aligned} f_{b_0}p_{\tau_{b_0}}(x_{(\psi(v)}) &= f_{b_0}(x_{\tau_{b_0}}^{-1})f_{b_0}(x_{\varphi(v)})f_{b_0}p_{\varphi(v)}(x_{\tau_{b_1}}) \\ &= f_{b_0}(x_{\tau_{b_0}})^{-1}f_{b_0}(x_{\varphi(v)})q_vf_{b_1}(x_{\tau_{b_1}}). \end{aligned}$$

Setting $z_b = f_b(x_{\tau_b})^{-1}$ we see that

$$(f_{b_0}p_{\tau_{b_0}}(x_{(\psi(v)})) = (z_b)\cdot(f_{b_0}(x_v)).$$

This means

$$\varprojlim{}^1 \mathbf{g}([(x_u)]) = [(z_b)\cdot(f_{b_0}(x_u))] = [(f_{b_0}(x_u))] = \varprojlim{}^1 \mathbf{f}([(x_u)]).$$

(b) Let $\mathbf{r} = \mathbf{r}(\mathbf{X}, \mu_A) : \mathbf{X} \to \mu_A^* \mathbf{X}$ be the standard cofinite reindexer for an inverse system $\mathbf{X}$ over $A \in \mathscr{C}^{nmx}$. Writing $\mu = \mu_A$ we have explicitly

$$\mu^* \mathbf{X} = (X_\Delta = X_{\mu\Delta}, p_{(\Delta_1,\Delta_0)} = p_{\mu(\Delta_1,\Delta_0)}),$$

$$\mathbf{r} = (\mu, id_{X_{\mu\Delta}} : X_{\mu\Delta} \to X_\Delta),$$

$$\varprojlim{}^1 \mathbf{r}([(x_u)]) = [(x_{\mu(\Delta_1,\Delta_0)})].$$

Adapting the technique used in [10] for the case of an inverse system $\mathbf{X}$ of abelian groups indexed by an ordered set A, we shall construct a function

$$s^1 : Z\Pi^1 \mu^* \mathbf{X} \to Z\Pi^1 \mathbf{X}$$

which will induce an inverse for $\varprojlim{}^1 \mathbf{r}$. For $v \in \tilde{A}_1$, $v : a_1 \to a_0$, let $(v)_i \subset (v)$ denote the diagram $(a_i; id_{a_i})$. Then $(v), (v)_i \in P(A)$. Note that if $v \in A_1'$, then $(v) \notin P(A)$ unless $v = id_a$. For $(y_{(\Delta_1,\Delta_0)}) \in Z\Pi^1 \mu^* \mathbf{X}$ set

$$\bar{y}_v = y_{((v),(v)_0)} p_v(y_{((v),(v)_1)})^{-1}. \tag{8.2}$$

We observe that also $\bar{y}_{id_a}$ is well-defined and yields $\bar{y}_{id_a} = e$. Define

$$\tilde{s}^1((y_{(\Delta_1,\Delta_0)})) = (\bar{y}_v) \in \tilde{\Pi}^1 \mathbf{X}. \tag{8.3}$$

We show that $\tilde{s}^1((y_{(\Delta_1,\Delta_0)})) \in Z\tilde{\Pi}^1 \mathbf{X}$, i.e. $p_{v_1}(\bar{y}_{v_2}) \bar{y}_{v_1 v_2}^{-1} \bar{y}_{v_1} = e$ for all $(v_1, v_2) \in \tilde{A}_2$, where $v_1 : a_1 \to a_0$, $v_2 : a_2 \to a_1$. For $i, j \in \{0,1,2\}$ define diagrams $\underline{i} = (a_i; id_{a_i})$, $\underline{ij} = (a_i, a_j; v_{ij}, id_{a_i}, id_{a_j})$ where $i < j$ and $v_{01} = v_1$, $v_{12} = v_2$, $v_{02} = v_1 v_2$, $\underline{012} = (a_0, a_1, a_2; v_1, v_2, v_1 v_2, id_{a_1}, id_{a_2}, id_{a_2})$. Since $(y_{(\Delta_1,\Delta_0)}) \in Z\Pi^1 \mu^* \mathbf{X}$ we obtain 6 equations

1) $p_{v_0}(y_{(\underline{012},\underline{01})}) y_{(\underline{012},\underline{0})}^{-1} y_{(\underline{01},\underline{0})} = e$ (note $p_{(\underline{01},\underline{0})} = p_{v_0}$)
2) $p_{v_0 v_1}(y_{(\underline{012},\underline{02})}) y_{(\underline{012},\underline{0})}^{-1} y_{(\underline{02},\underline{0})} = e$ (note $p_{(\underline{02},\underline{0})} = p_{v_0 v_1}$)
3) $p_{v_1}(y_{(\underline{012},\underline{12})}) y_{(\underline{012},\underline{1})}^{-1} y_{(\underline{12},\underline{1})} = e$ (note $p_{(\underline{12},\underline{1})} = p_{v_1}$)
4) $y_{(\underline{012},\underline{01})} y_{(\underline{012},\underline{1})}^{-1} y_{(\underline{01},\underline{1})} = e$ (note $p_{(\underline{01},\underline{1})} = p_{id_{a_1}} = id$)
5) $y_{(\underline{012},\underline{02})} y_{(\underline{012},\underline{2})}^{-1} y_{(\underline{02},\underline{2})} = e$ (note $p_{(\underline{02},\underline{2})} = p_{id_{a_2}} = id = id$)
6) $y_{(\underline{012},\underline{12})} y_{(\underline{012},\underline{2})}^{-1} y_{(\underline{12},\underline{2})} = e$ (note $p_{(\underline{12},\underline{2})} = p_{id_{a_2}} = id$)

From 3)–6) we derive

3′) $p_{v_0 v_1}(y_{(\underline{012},\underline{12})}) p_{v_0}(y_{(\underline{012},\underline{1})})^{-1} p_{v_0}(y_{(\underline{12},\underline{1})}) = e$
4′) $p_{v_0}(y_{(\underline{012},\underline{01})}) p_{v_0}(y_{(\underline{012},\underline{1})})^{-1} p_{v_0}(y_{(\underline{01},\underline{1})}) = e$
5′) $p_{v_0 v_1}(y_{(\underline{012},\underline{02})}) p_{v_0 v_1}(y_{(\underline{012},\underline{2})})^{-1} p_{v_0 v_1}(y_{(\underline{02},\underline{2})}) = e$
6′) $p_{v_0 v_1}(y_{(\underline{012},\underline{12})}) p_{v_0 v_1}(y_{(\underline{012},\underline{2})})^{-1} p_{v_0 v_1}(y_{(\underline{12},\underline{2})}) = e$

From 1) and 4′), 2) and 5′), 3′) and 6′) we infer

1″) $y^{-1}_{(\underline{012},\underline{0})}y_{(\underline{01},\underline{0})} = p_{v_0}(y_{(\underline{012},\underline{1})})^{-1}p_{v_0}(y_{(\underline{01},\underline{1})})$

2″) $y^{-1}_{(\underline{012},\underline{0})}y_{(\underline{02},\underline{0})} = p_{v_0v_1}(y_{(\underline{012},\underline{2})})^{-1}p_{v_0v_1}(y_{(\underline{02},\underline{2})})$

3″) $p_{v_0}(y_{(\underline{012},\underline{1})})^{-1}p_{v_0}(y_{(\underline{12},\underline{1})}) = p_{v_0v_1}(y_{(\underline{012},\underline{2})})^{-1}p_{v_0v_1}(y_{(\underline{12},\underline{2})})$

In 1″) we replace $y^{-1}_{(\underline{012},\underline{0})}$ via 2″) and obtain

4″) $p_{v_0v_1}(y_{(\underline{012},\underline{2})})^{-1}p_{v_0v_1}(y_{(\underline{02},\underline{2})})y^{-1}_{(\underline{02},\underline{0})}y_{(\underline{01},\underline{0})} = p_{v_0}(y_{(\underline{012},\underline{1})})^{-1}p_{v_0}(y_{(\underline{01},\underline{1})})$

In 4″) we replace $p_{v_0v_1}(y_{(\underline{012},\underline{2})})^{-1}$ via 3″) and obtain

5″) $p_{v_0}(y_{(\underline{012},\underline{1})})^{-1}p_{v_0}(y_{(\underline{12},\underline{1})})p_{v_0v_1}(y_{(\underline{12},\underline{2})})^{-1}p_{v_0v_1}(y_{(\underline{02},\underline{2})})y^{-1}_{(\underline{02},\underline{0})}y_{(\underline{01},\underline{0})} =$
$p_{v_0}(y_{(\underline{012},\underline{1})})^{-1}p_{v_0}(y_{(\underline{01},\underline{1})})$

which produces

6″) $p_{v_0}(y_{(\underline{12},\underline{1})})p_{v_0v_1}(y_{(\underline{12},\underline{2})})^{-1}p_{v_0v_1}(y_{(\underline{02},\underline{2})})y^{-1}_{(\underline{02},\underline{0})}y_{(\underline{01},\underline{0})}p_{v_0}(y_{(\underline{01},\underline{1})})^{-1} = e$

We have $p_{v_0}(y_{(\underline{12},\underline{1})})p_{v_0v_1}(y_{(\underline{12},\underline{2})})^{-1} = p_{v_0}(y_{(\underline{12},\underline{1})}p_{v_1}(y_{(\underline{12},\underline{2})})^{-1}) = p_{v_0}(\bar{y}_{v_1})$, $p_{v_0v_1}(y_{(\underline{02},\underline{2})})y^{-1}_{(\underline{02},\underline{0})} = \bar{y}^{-1}_{v_0v_1}$, $y_{(\underline{01},\underline{0})}p_{v_0}(y_{(\underline{01},\underline{1})})^{-1} = \bar{y}_{v_0}$, thus

$$p_{v_0}(\bar{y}_{v_1})\bar{y}^{-1}_{v_0v_1}\bar{y}_{v_0} = e.$$

Using Lemma 8.2 we obtain the desired function $s^1 : Z\Pi^1\mu^*\mathbf{X} \to Z\Pi^1\mathbf{X}$ as $s^1 = \tilde{\pi}^{-1}\tilde{s}^1$. Note that for all $u \in \bar{A}_1 = \tilde{A}_1 \cup \{id_a \mid a \in A\}$

$$s^1((y_{(\Delta_1,\Delta_0)}))_u = \bar{y}_u = y_{((u),(u)_0)}p_u(y_{((u),(u)_1)})^{-1}. \tag{8.4}$$

This is true for $u = id_a$ simply because $s^1((y_{(\Delta_1,\Delta_0)})) \in Z\Pi^1\mathbf{X}$. Now define

$$s^0 : \Pi^0\mu^*\mathbf{X} \to \Pi^0\mathbf{X}, \quad s^0((g_\Delta))_a = g_{(a)}$$

where (a) denotes the diagram $(a; id_a)$. We have

$$\tilde{s}^1(g \cdot y) = s^0(g) \cdot \tilde{s}^1(y)$$

since

$$\begin{aligned}
\tilde{s}^1(g \cdot y)_v &= \tilde{s}^1((g_{\Delta_0}y_{(\Delta_1,\Delta_0)}p_{(\Delta_1,\Delta_0)}(g_{\Delta_1})^{-1}))_v \\
&= g_{(v)_0}y_{((v)(v)_0)}p_{((v),(v)_0)}(g_{(v)})^{-1}p_v(g_{(v)_1}y_{((v)(v)_1)}p_{((v),(v)_1)}(g_{(v)})^{-1})^{-1} \\
&= g_{(v)_0}y_{((v)(v)_0)}p_v(g_{(v)})^{-1}p_v(g_{(v)})p_v(y_{((v)(v)_1)})^{-1}p_v(g_{(v)_1})^{-1} \\
&= g_{(v)_0}y_{((v)(v)_0)}p_v(y_{((v)(v)_1)})^{-1}p_v(g_{(v)_1})^{-1} = g_{(v)_0}\bar{y}_vp_v(g_{(v)_1})^{-1} \\
&= (s^0(g) \cdot \tilde{s}^1(y))_v.
\end{aligned}$$

Using again Lemma 8.2 we see that

$$s^1(g \cdot y) = s^0(g) \cdot s^1(y)$$

so that s^1 induces

$$\sigma : \varprojlim{}^1 \mu^* \mathbf{X} \to \varprojlim{}^1 \mathbf{X}.$$

We have

$$\sigma(\varprojlim{}^1 \mathbf{r}([(x_u)]) = \sigma([(x_{\mu(\Delta_1,\Delta_0)})]) = [s^1((x_{\mu(\Delta_1,\Delta_0)}))].$$

For $v \in \tilde{A}_1$ we have

$$s^1((x_{\mu(\Delta_1,\Delta_0)}))_v = x_{\mu((v),(v)_0)} p_v(x_{\mu((v),(v)_1)})^{-1} = x_v$$

so that

$$\tilde{\pi}(s^1((x_{\mu(\Delta_1,\Delta_0)}))) = \tilde{\pi}((x_u))$$

whence $s^1((x_{\mu(\Delta_1,\Delta_0)})) = (x_u)$. This proves

$$\sigma \circ \varprojlim{}^1 \mathbf{r} = id.$$

For all $(\Delta_1, \Delta_0) \in P(A)_2$ we have $\mu(\Delta_1, \Delta_0) \in \bar{A}_1$ so that by 8.4

$$\varprojlim{}^1 \mathbf{r}(\sigma([(y_{(\Delta_1,\Delta_0)})])) = \varprojlim{}^1 \mathbf{r}([s^1((y_{(\Delta_1,\Delta_0)}))]) = [(s^1((y_{(\Delta_1,\Delta_0)}))_{\mu(\Delta_1,\Delta_0)})]$$

$$= [(\bar{y}_{\mu(\Delta_1,\Delta_0)})].$$

Setting

$$z_\Delta = y_{(\Delta,(\mu\Delta))} \in X_{(\mu\Delta)} = X_{\mu\Delta} = \mu^*(\mathbf{X})_\Delta, \tag{8.5}$$

we shall show

$$(z_\Delta) \cdot (y_{(\Delta_1,\Delta_0)}) = ((\bar{y}_{\mu(\Delta_1,\Delta_0)})) \tag{8.6}$$

which proves

$$\varprojlim{}^1 \mathbf{r} \circ \sigma = id$$

and thus shows that $\varprojlim{}^1 \mathbf{r}$ is a bijection, i.e. an isomorphism in $\mathbf{Set_0}$.

We set $a_i = \mu\Delta_i$ and $u = \mu(\Delta_1, \Delta_0) : a_1 \to a_0$. Then (8.6) means explicitly

$$y_{(\Delta_0,(a_0))} y_{(\Delta_1,\Delta_0)} p_{(\Delta_1,\Delta_0)}(y_{(\Delta_1,(a_1))})^{-1} = y_{((u),(a_0))} p_u(y_{((u),(a_1))})^{-1}. \tag{8.7}$$

This will be verified by transforming it into equivalent equations. For $(a_i) \subset (u) \subset \Delta_1$ we obtain

$$y_{(\Delta_1,(u))} y^{-1}_{(\Delta_1,(a_1))} y_{((u),(a_1))} = p_{((u),(a_1))}(y_{(\Delta_1,(u))}) y^{-1}_{(\Delta_1,(a_1))} y_{((u),(a_1))} = e,$$

$$p_u(y_{(\Delta_1,(u))}) y^{-1}_{(\Delta_1,(a_0))} y_{((u),(a_0))} = p_{((u),(a_0))}(y_{(\Delta_1,(u))}) y^{-1}_{(\Delta_1,(a_0))} y_{((u),(a_0))} = e.$$

Inserting $y_{((u),(a_i))}$ into (8.7) yields (note $p_{(\Delta_1,\Delta_0)} = p_u$)

$$y_{(\Delta_0,(a_0))} y_{(\Delta_1,\Delta_0)} p_u\big(y_{(\Delta_1,(a_1))}\big)^{-1}$$
$$= y_{(\Delta_1,(a_0))} p_u\big(y_{(\Delta_1,(u))}\big)^{-1} p_u\big(y_{(\Delta_1,(u))}\big) p_u\big(y_{(\Delta_1,(a_1))}\big)^{-1}$$

which is equivalent to

(8.8) $$y_{(\Delta_0,(a_0))} y_{(\Delta_1,\Delta_0)} = y_{(\Delta_1,(a_0))}.$$

For $(a_0) \subset \Delta_0 \subset \Delta_1$ we obtain

$$y_{(\Delta_1,\Delta_0)} y^{-1}_{(\Delta_1,(a_0))} y_{(\Delta_0,(a_0))} = p_{(\Delta_0,(a_0))}\big(y_{(\Delta_1,\Delta_0)}\big) y^{-1}_{(\Delta_1,(a_0))} y_{(\Delta_0,(a_0))} = e$$

which proves (8.8). □

An immediate consequence of Lemma 5.4 is

Theorem 8.3. *Let* $A \in \mathscr{C}(cfnt)$. *Then* $\varprojlim^1 : \mathbf{G}^A \to \mathbf{Set_0}$ *has a unique pro-extension* $\varprojlim^1 : \textbf{pro-G}_A \to \mathbf{Set_0}$.

Whether our functor $\varprojlim^1 : \textbf{pro-G} \to \mathbf{Set_0}$ coincides with the holim-based pro-extension remains open. The question is complicated by the dependency of holim on the choice of a closed model structure on **pro-SS** (cf. [7]). However, our $\varprojlim^1$ has the following characteristic feature.

Theorem 8.4. *Let* $\mathscr{S}$ *denote the category whose objects are short exact sequences* $0 \to \mathbf{A}' \xrightarrow{\mathfrak{f}'} \mathbf{A} \xrightarrow{\mathfrak{f}} \mathbf{A}'' \to 0$ *in* **pro-G** *and whose morphisms* $\gamma : (0 \to \mathbf{A}' \xrightarrow{\mathfrak{f}'} \mathbf{A} \xrightarrow{\mathfrak{f}} \mathbf{A}'') \to (\mathbf{0} \to \mathbf{B}' \xrightarrow{\mathfrak{g}'} \mathbf{B} \xrightarrow{\mathfrak{g}} \mathbf{B}'')$ *are triples* $\gamma = (\gamma', \gamma, \gamma'')$ *of morphisms* $\gamma' : \mathbf{A}' \to \mathbf{B}'$, $\gamma : \mathbf{A} \to \mathbf{B}$, $\gamma'' : \mathbf{A}'' \to \mathbf{B}''$ *in* **pro-G** *such that the following diagram commutes*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{A}' & \xrightarrow{\mathfrak{f}'} & \mathbf{A} & \xrightarrow{\mathfrak{f}} & \mathbf{A}'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\gamma'} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma''} & & \\ 0 & \longrightarrow & \mathbf{B}' & \xrightarrow[\mathfrak{g}']{} & \mathbf{B} & \xrightarrow[\mathfrak{g}]{} & \mathbf{B}'' & \longrightarrow & 0 \end{array}$$

For $i = 1,2,3$ *let* $Comp_i : \mathscr{S} \to$ **pro-G** *be the functor selecting the i-th component of short exact sequences and morphisms between such sequences.*

There exists a natural transformation $\delta : \varprojlim \circ Comp_3 \to \varprojlim^1 \circ Comp_1$ *such that the following sequence is exact for each* $A \in \mathscr{S}$:

$$0 \longrightarrow \varprojlim \mathbf{A}' \xrightarrow{\varprojlim \mathfrak{f}'} \varprojlim \mathbf{A} \xrightarrow{\varprojlim \mathfrak{f}} \varprojlim \mathbf{A}'' \xrightarrow{\delta} \varprojlim^1 \mathbf{A}' \xrightarrow{\varprojlim^1 \mathfrak{f}'} \varprojlim^1 \mathbf{A} \xrightarrow{\varprojlim^1 \mathfrak{f}} \varprojlim^1 \mathbf{A}''$$

PROOF. We do not go into details. The same arguments as in [9, §15.2] reduce the general case to short exact sequences of level morphisms where everything is well-known. Only [9, Lemma 15.12] requires a new proof. This is a routine exercise requiring to use the construction of $\delta : \varprojlim \circ Comp_3 \to \varprojlim^1 \circ Comp_1$ as presented e.g. in [8, Ch. II, §6.2, Proof of Theorem 8]. $\square$

We finally consider the case $A = \mathbf{N}$ where

$$Z\Pi^1\mathbf{X} = \{(x_{(n,m)}) \mid \forall n \le m \le p : p_{(m,n)}(x_{(m,p)})x_{(n,p)}^{-1}x_{(n,m)} = e\}$$

with $x_{(n,m)} \in X_n$. Define

$$\Theta : Z\Pi^1\mathbf{X} \to \Pi^0\mathbf{X} = \prod_{i=1}^{\infty} X_i, \quad \Theta((x_{(n,m)}))_i = x_{(i,i+1)}.$$

It is known that Θ is a bijection. The action of $\Pi^0\mathbf{X}$ on $Z\Pi^1\mathbf{X}$ transforms via Θ into an action of $\Pi^0\mathbf{X}$ on $\Pi^0\mathbf{X}$ given by

$$((g_i)\cdot(x_i'))_n = g_n x_n' p_{(n+1,n)}(g_{n+1})^{-1}.$$

This yields the well-known elementary description of $\varprojlim^1 \mathbf{X}$ for inverse sequences from [1, Ch. IX, §2.1]. We denote it as $LIM^1\mathbf{X}$. It comes as a functor $LIM^1 : \mathbf{G}^{\mathbf{N}} \to \mathbf{Set_0}$: Each level morphism $\mathbf{f} = (f_i) : \mathbf{X} \to \mathbf{Y}$ induces $\Pi^0\mathbf{f} : \Pi^0\mathbf{X} \to \Pi^0\mathbf{Y}$, $\Pi^0\mathbf{f}((x_i')) = (f_i(x_i'))$, which induces $LIM^1\mathbf{f} : LIM^1\mathbf{X} \to LIM^1\mathbf{Y}$, $LIM^1\mathbf{f}([x']) = [\Pi^0\mathbf{f}(x')]$.

For a morphism $\mathbf{f} = (\varphi, (f_i))$ in $\mathbf{str}\text{-}\mathbf{G_N}$ let us define $\hat{\Pi}^0\mathbf{f} = \Theta_{\mathbf{Y}}(Z\Pi^1\mathbf{f})\Theta_{\mathbf{X}}^{-1} : \Pi^0\mathbf{X} \to \Pi^0\mathbf{Y}$. Explicitly

$$\hat{\Pi}^0\mathbf{f}((x_i')) = \left(\prod_{j=\varphi(i)}^{\varphi(i+1)-1} f_i p_i^j(x_j')\right).$$

For level morphisms we have $\hat{\Pi}^0\mathbf{f} = \Pi^0\mathbf{f}$. This implies

(1) LIM^1 extends naturally to $\mathbf{str}\text{-}\mathbf{G_N}$ by setting

$$LIM^1_{str}\mathbf{f}([x']) = [\hat{\Pi}^0\mathbf{f}(x')] = \left[\left(\prod_{j=\varphi(i)}^{\varphi(i+1)-1} f_i p_i^j(x_j')\right)\right].$$

(2) Θ induces a natural isomorphism $\Theta' : \varprojlim^1 \to LIM^1_{str}$ between functors on $\mathbf{str}\text{-}\mathbf{G_N}$.

We conclude that LIM^1 has a unique pro-extension $LIM^1 : \mathbf{pro}\text{-}\mathbf{G_N} \to \mathbf{Set_0}$ which coincides with the unique pro-extension of LIM^1_{str}.

9. The Derived Limits on pro-AG

Let $\varprojlim^n : \mathbf{lev\text{-}AG} \to \mathbf{AG}$ be the n-th derived limit functor which can be represented as $\varprojlim^n = \pi^n \circ \Pi^*$ where π^n is the n-th cohomotopy group[11] on $c\mathbf{AG}$ (cf. [1, Ch. XI, §6]). The natural extension of Π^* to **str-AG** generates a natural extension $\varprojlim^n : \mathbf{str\text{-}AG} \to \mathbf{AG}$. In [10] and [9] it is proved that $\varprojlim^n : \mathbf{str\text{-}AG}_{\mathscr{D}} \to \mathbf{AG}$ has a pro-extension $\varprojlim^n : \mathbf{pro\text{-}AG}_{\mathscr{D}} \to \mathbf{AG}$. Using the methods of this paper and the technique of [10], [9] one can prove the stronger

THEOREM 9.1. $\varprojlim^n : \mathbf{str\text{-}AG} \to \mathbf{AG}$ *has a unique pro-extension* $\varprojlim^n : \mathbf{pro\text{-}AG} \to \mathbf{AG}$.

The crucial and difficult part of the proof is to show that each standard cofinite reindexer **r** induces an isomorphism $\varprojlim^n \mathbf{r}$. This was proved for arbitrary reindexers in [2, Lemma 6.3]. Moreover we have

THEOREM 9.2. *The functors* $\varprojlim^n : \mathbf{pro\text{-}AG} \to \mathbf{AG}$ *of Theorem* 9.1 *are the right derived functors of* $\varprojlim : \mathbf{pro\text{-}AG} \to \mathbf{AG}$.

This has been proved in [9] for the case of directed preordered index categories by showing that the functors in question form a universal connected sequence of functors whose connecting homomorphisms come from the short exact sequence $\Pi^*(\mathscr{S})$ of cochain complexes associated to any short exact sequence $\mathscr{S}$ in **str-AG**. The same proof applies in the general case.

REMARK 9.3. In their role as right derived functors the $\varprojlim^n : \mathbf{pro\text{-}AG} \to \mathbf{AG}$ are up to natural isomorphism uniquley determined by $\varprojlim : \mathbf{pro\text{-}AG} \to \mathbf{AG}$, but this does not mean eo ipso that each individual $\varprojlim^n : \mathbf{str\text{-}AG} \to \mathbf{AG}$ (let alone $\varprojlim^n : \mathbf{lev\text{-}AG} \to \mathbf{AG}$) has a unique pro-extension.

We conclude with

THEOREM 9.4. *Let* $A \in \mathscr{C}(cfnt)$. *Then* $\varprojlim^n : \mathbf{AG}^A \to \mathbf{AG}$ *has a unique pro-extension* $\varprojlim^n : \mathbf{pro\text{-}AG}_A \to \mathbf{AG}$.

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[11] For $G \in c\mathbf{AG}$, $\pi^n(G)$ is defined as the n-th cohomology group of G considered as a cochain complex with coboundaries $\delta^n = \sum_i (-1)^i d^i$, d^i the cofaces of G.

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