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ON MULTIVALENT FUNCTIONS IN THE UNIT DISC

By

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Abstract. Let $\mathcal{A}(p)$ be the class of functions $f(z)$, analytic in $|z| < 1$ in the complex plane, of the form $f(z) = z^p + \cdots$. We study the question, that naturally rises, about the relation between the expressions $\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$ and $\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}$, when $f(z) \in \mathcal{A}(p)$. Some relations of this type imply that $f(z)$ is p-valent or p-valent starlike in $|z| < 1$.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. Let $\mathcal{A}$ denote the class of functions $f(z) \in \mathcal{H}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbf{D}). \tag{1.1}$$

A function $f(z)$ which is analytic in a domain $D \in \mathbf{C}$ is called p-valent in D if for every complex number w, the equation $f(z) = w$ have at most p roots in D and there will be a complex number w_0 such that the equation $f(z) = w_0$, has exactly p roots in D. Let $\mathcal{A}(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p}^{\infty} a_n z^n, \quad (z \in \mathbf{D}), \tag{1.2}$$

where $a_p \neq 0$ and $p \in \mathbf{N} = \{1, 2, \ldots\}$. In [10] S. Ozaki proved that if $f(z) \in \mathcal{A}(p)$ and is analytic in a convex domain $D \subset \mathbf{C}$ and for some real α we have

$$\Re\mathrm{e}\{\exp(i\alpha) f^{(p)}(z)\} > 0 \quad z \in D, \tag{1.3}$$

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then $f(z)$ is at most p-valent in D. Ozaki's condition is a generalization of the well known Noshiro-Warschawski univalence condition with $p = 1$, [3], [11]. Applying Ozaki's theorem, we find that if $f(z) \in \mathscr{A}(p)$ and

$$\Re e\{f^{(p)}(z)\} > 0 \quad (z \in \mathbf{D}), \tag{1.4}$$

then $f(z)$ is at most p-valent in $\mathbf{D}$. In [5, 454] it was proved that if $f(z) \in \mathscr{A}(p)$, $p \geq 2$, and

$$|\arg\{f^{(p)}(z)\}| < \frac{3\pi}{4} \quad (z \in \mathbf{D}), \tag{1.5}$$

then $f(z)$ is at most p-valent in $\mathbf{D}$. A function $f(z) \in \mathscr{A}(p)$ is said to be p-valently starlike of order α, $0 < \alpha < p$, if and only if

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathbf{D}), \tag{1.6}$$

and let us denote it as

$$f(z) \in \mathscr{S}_p(\alpha).$$

Further, a function $f(z) \in \mathscr{A}(p)$ is said p-valently convex of order α, $0 < \alpha < p$, if and only if

$$1 + \Re e\left\{\frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathbf{D}) \tag{1.7}$$

and let us denote it as

$$f(z) \in \mathscr{C}_p(\alpha).$$

LEMMA 1.1 ([1]). *Let $w(z)$ be non-constant and analytic function in the unit disc $\mathbf{D}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then $z_0 w'(z_0) = kw(z_0)$ and $k \geq 1$.*

LEMMA 1.2 ([6]). *Let p be analytic function in $\mathbf{D}$, with $p(0) = 1$. If there exists a point z_0, $|z_0| < 1$, such that $\Re e\{p(z)\} > 0$ for $|z| < |z_0|$ and $p(z_0) = \pm ia$ for some $a > 0$, then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi}, \quad \arg\{p(z_0)\} = \pm\frac{\pi}{2}$$

for some $k \geq (a + a^{-1})/2 \geq 1$.

2. Main Result

Let $f(z) \in \mathscr{A}(p)$, $p \geq 2$. Assume for a moment that $0 < \alpha < 1$ and

$$p(z) = \frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)} \neq \alpha \quad (z \in \mathbf{D}). \tag{2.1}$$

Then the function

$$w(z) = \frac{1 - p(z)}{\alpha - p(z)} \quad (z \in \mathbf{D})$$

is analytic in $\mathbf{D}$, $w(0) = 0$. If $|w(z)|$ attains its maximum value 1 on the circle $|z| = r < 1$ at the point z_0, then by Lemma 1.1, $z_0 w'(z_0) = kw(z_0)$ and $k \geq 1$. We have also

$$p(z) = \frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)} = \frac{1 - \alpha w(z)}{1 - w(z)}, \quad |w(z_0)| = 1 \quad \text{so } \Re\mathrm{e}\{2p(z_0) - 1\} = \alpha \tag{2.2}$$

which gives the following equation

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} = \frac{zp'(z)}{p(z)} + 2p(z) - 1,$$

Then $z_0 w'(z_0) = kw(z_0)$ and $k \geq 1$ follow that

$$\begin{aligned}
\Re\mathrm{e}\left\{\frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)}\right\} \qquad (2.3) \\
&= \Re\mathrm{e}\left\{-\frac{\alpha z_0 w'(z_0)}{1 - \alpha w(z_0)} + \frac{z_0 w'(z_0)}{1 - w(z_0)} + 2p(z_0) - 1\right\} \\
&= \Re\mathrm{e}\left\{k\frac{-\alpha w(z_0)}{1 - \alpha w(z_0)}\right\} + \Re\mathrm{e}\left\{\frac{kw(z_0)}{1 - w(z_0)}\right\} + \Re\mathrm{e}\{2p(z_0) - 1\} \\
&= \Re\mathrm{e}\left\{-\frac{k}{2}\frac{1 + \alpha w(z_0)}{1 - \alpha w(z_0)}\right\} + \frac{k}{2} + \Re\mathrm{e}\left\{\frac{kw(z_0)}{1 - w(z_0)}\right\} + \Re\mathrm{e}\{2p(z_0) - 1\} \\
&= \Re\mathrm{e}\left\{-\frac{k}{2}\frac{1 + \alpha w(z_0)}{1 - \alpha w(z_0)}\right\} + \frac{k}{2} - \frac{k}{2} + \alpha \\
&\leq -\frac{1 - \alpha}{2(1 + \alpha)} + \alpha \\
&= \frac{2\alpha^2 + 3\alpha - 1}{2(1 + \alpha)}.
\end{aligned}$$

Hence, if we would assume that

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \frac{2\alpha^2+3\alpha-1}{2(1+\alpha)}, \quad (z \in \mathbf{D}), \tag{2.4}$$

then (2.3) would imply that $|w(z)| < 1$ in $\mathbf{D}$ and by (2.2)

$$\Re\mathrm{e}\left\{\frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)}\right\} > \frac{1+\alpha}{2} \quad (z \in \mathbf{D}).$$

Furthermore condition (2.1) is not necessary. Inequality (2.4) implies (2.1). To show this we apply a result from [2, p. 26] which says that if $p(z) = 1 + \cdots$, $p(z) \not\equiv 1$, $z_1 \in \mathbf{D}$ and

$$\Re\mathrm{e}\{p(z_1)\} = \min\{\Re\mathrm{e}\{p(z)\} : |z| < |z_1|\}, \tag{2.5}$$

then $z_1 p'(z_1)$ is a negative real and

$$z_1 p'(z_1) \le -\frac{1}{2}\frac{|1-p(z_1)|^2}{\Re\mathrm{e}\{1-p(z_1)\}}.$$

From that, if we suppose that there exists $z_1 \in \mathbf{D}$ such that (2.5) holds with $\Re\mathrm{e}\{p(z_1)\} = \alpha$, then we have

$$\begin{aligned}
\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\bigg|_{z=z_1} &= \Re\mathrm{e}\left\{\frac{zp'(z)}{p(z)} + 2p(z) - 1\right\}\bigg|_{z=z_1} \\
&\le -\frac{1}{2\alpha}\frac{|1-p(z_1)|^2}{1-\alpha} + 2\alpha - 1 \\
&\le -\frac{1}{2\alpha}\frac{(1-\alpha)^2}{1-\alpha} + 2\alpha - 1 \\
&\le -\frac{1-\alpha}{2\alpha} + 2\alpha - 1 \\
&\le \frac{2\alpha^2+3\alpha-1}{2(1+\alpha)}
\end{aligned}$$

which negates (2.4). Therefore, we have the following theorem.

THEOREM 2.1. *Let $f(z) \in \mathscr{A}(p)$, $p \ge 2$, $0 < \alpha < 1$, and let it satisfy the following condition*

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \frac{2\alpha^2+3\alpha-1}{2(1+\alpha)}, \quad (z \in \mathbf{D}), \tag{2.6}$$

Then we have

$$\Re\mathrm{e}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > 1+\alpha, \quad (z \in \mathbf{D}). \tag{2.7}$$

For $p = 2$, $\alpha = 0$, we have

Corollary 2.2. *Let $f(z) \in \mathscr{A}(2)$, and let it satisfy the following condition*

$$\Re\mathrm{e}\left\{\frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad (z \in \mathbf{D}), \tag{2.8}$$

Then we have

$$\Re\mathrm{e}\left\{\frac{zf'(z)}{f(z)}\right\} > 1, \quad (z \in \mathbf{D}). \tag{2.9}$$

Putting

$$\beta = \frac{2\alpha^2 + 3\alpha - 1}{2(1+\alpha)}, \quad 0 \le \alpha < 1, \tag{2.10}$$

then $-1/2 \le \beta \le 1$ and Theorem 2.1 is equivalent to the following result.

Theorem 2.3. *Assume that $f(z) \in \mathscr{A}(p)$, $p \ge 2$, $-1/2 \le \beta \le 1$. If the function*

$$\frac{f^{(p-1)}(z)}{p!} = z + \cdots$$

is starlike of order β or

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \beta, \quad (z \in \mathbf{D}),$$

then

$$\Re\mathrm{e}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > \frac{1}{4}\{2\beta + 1 + \sqrt{4\beta^2 + 4\beta + 17}\}, \quad (z \in \mathbf{D}),$$

or

$$f(z) \in \mathscr{S}(p, p-1, \beta) \Rightarrow f(z) \in \mathscr{S}(p-1, p-2, \{2\beta + 1 + \sqrt{4\beta^2 + 4\beta + 17}\}/4),$$

where we use for simplification the notation

$$\mathscr{S}(k, k-1, \beta) = \left\{ f(z) \in \mathscr{A}(p) : \Re\mathrm{e}\left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > \beta \right\},$$

for an integer k, $0 < k \leq p$.

Now, we are going to improve the following theorem

THEOREM 2.4 ([4]). *Let* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ *be analytic in* $\mathbf{D}$ *and suppose that*

$$\Re\mathrm{e}\left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0, \quad (z \in \mathbf{D}).$$

Then we have

$$\Re\mathrm{e}\left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0, \quad (z \in \mathbf{D}),$$

for each k, $k = 1, 2, 3, \ldots, p$.

THEOREM 2.5. *Let* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $2 \leq p$, $0 < \alpha < 1$, *be analytic in* $\mathbf{D}$ *and suppose that*

$$\Re\mathrm{e}\left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > -\frac{2\alpha^2 - \alpha + 1}{2(1-\alpha)}, \quad (z \in \mathbf{D}). \tag{2.11}$$

Then we have

$$\Re\mathrm{e}\left\{ \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} \right\} > 1 + \alpha, \quad (z \in \mathbf{D}). \tag{2.12}$$

PROOF. Let us put

$$p(z) = \frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)} = \frac{1 - \alpha w(z)}{1 - w(z)}, \quad p(0) = 1, \tag{2.13}$$

where $-1 \leq \alpha \leq 0$, and $w(z)$ is analytic in $\mathbf{D}$ if we assume that

$$\frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)} \neq \alpha, \quad (z \in \mathbf{D}) \tag{2.14}$$

but (2.12) implies (2.14), the proof of this runs in the same way as for condition (2.1) in Theorem 2.1, so we not need add (2.14) to the hypothesis of Theorem 2.5.

Then it follows that

$$1+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}-\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}=\frac{zp'(z)}{p(z)}$$

or

$$\begin{aligned}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}&=\frac{zp'(z)}{p(z)}+2p(z)-1\\&=-\frac{\alpha zw'(z)}{1-\alpha w(z)}+\frac{zw'(z)}{1-w(z)}+\frac{2(1-\alpha w(z))}{1-w(z)}-1.\end{aligned}$$

Now, we prove that $|w(z)|<1$ in $\mathbf{D}$. If there exists a point z_0, $|z_0|<1$, such that

$$|w(z)|<1 \quad \text{for } |z|<|z_0|$$

and

$$|w(z_0)|=1$$

then from Jack's lemma [1], we have

$$\frac{z_0w'(z_0)}{w(z_0)}=k\geq 1$$

and so, we have

$$\begin{aligned}\frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)}&=\frac{z_0p'(z_0)}{p(z_0)}+2p(z_0)-1\\&=-\frac{k\alpha w(z_0)}{1-\alpha w(z_0)}+\frac{kw(z_0)}{1-w(z_0)}+\frac{2(1-\alpha w(z_0))}{1-w(z_0)}-1.\end{aligned}$$

Applying the same calculation as in [1], we have

$$\begin{aligned}\mathfrak{Re}\left\{\frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)}\right\}&\leq\mathfrak{Re}\left\{-\frac{k\alpha w(z_0)}{1-\alpha w(z_0)}+\frac{kw(z_0)}{1-w(z_0)}+\frac{2(1-\alpha w(z_0))}{1-w(z_0)}-1\right\}\\&\leq\frac{k}{2}-\frac{k(1+\alpha)}{2(1-\alpha)}-\frac{k}{2}+(1+\alpha)-1\\&\leq-\frac{1+\alpha}{2(1-\alpha)}+\alpha\\&=-\frac{2\alpha^2-\alpha+1}{2(1-\alpha)}.\end{aligned}$$

This contradicts hypothesis (2.11) and therefore, we have

$$|w(z)| < 1 \quad z \in \mathbf{D}.$$

Hence, from (2.13), we have

$$\Re\mathrm{e}\{p(z)\} = \Re\mathrm{e}\left\{\frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)}\right\} > \frac{1+\alpha}{2}, \quad (z \in \mathbf{D}).$$

It completes the proof of Theorem 2.5. □

For $p = 2$, $\alpha = 0$, Theorem 2.5 becomes Corollary 2.2, therefore Theorem 2.5 is in some sense a continuation of Theorem 2.1. Furthermore, Theorem 2.5 is an improvement of Theorem 2.4, it describes the following corollary.

COROLLARY 2.6. *Let* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $2 \le p$ *be analytic in* $\mathbf{D}$ *and suppose*

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > -1, \quad (z \in \mathbf{D}).$$

Then we have

$$\Re\mathrm{e}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > 0, \quad (z \in \mathbf{D}).$$

COROLLARY 2.7. *Let* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $2 \le p$ *be analytic in* $\mathbf{D}$ *and suppose*

$$\Re\mathrm{e}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > -1, \quad (z \in \mathbf{D}). \tag{2.15}$$

Then we have

$$\Re\mathrm{e}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0, \quad (z \in \mathbf{D}), \tag{2.16}$$

for each k, $k = 1, 2, 3, \ldots, p-1$.

PROOF. Putting $\alpha = 1$ in Theorem 2.5, we have

$$\Re\mathrm{e}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > 0, \quad (z \in \mathbf{D})$$

and applying Theorem (2.4), we have

$$\mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0, \quad (z \in \mathbf{D}), \tag{2.17}$$

for each k, $k = 1, 2, 3, \dots, p-1$. It completes the proof of Corollary 2.7. □

Corollary 2.8. *Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, be analytic in $\mathbf{D}$ and suppose that*

$$\left|\mathfrak{Im}\left\{\frac{f^{(p+1)}(z)}{f^{(p)}(z)}\right\}\right| < \frac{3\pi|z|}{2}, \quad (z \in \mathbf{D}\backslash\{0\}). \tag{2.18}$$

Then $f(z)$ is at most p-valent in $\mathbf{D}$.

Proof. From the hypothesis, we have

$$\left|\mathfrak{Im}\int_0^z \frac{f^{(p+1)}(t)}{f^{(p)}(t)}\,\mathrm{d}t\right| \le \int_0^z \left|\mathfrak{Im}\left\{\frac{f^{(p+1)}(t)}{f^{(p)}(t)}\right\}\right| |\mathrm{d}t| < \int_0^z \frac{3}{2}\pi|t|\,|\mathrm{d}t| \le \frac{3}{4}\pi|z|^2$$
$$< \frac{3}{4}\pi,$$

where $|z| < 1$, $\arg\{z\} = \theta$, $t = \rho e^{i\theta}$ and $0 \le \rho \le |z|$. Applying Nunokawa's result (1.5), we immediately obtain that $f(z)$ is at most p-valent in $\mathbf{D}$. □

Recall here some related results.

Lemma 2.9 ([7, Th. 2, p. 93]). *Let $f(z) \in \mathscr{A}(p)$, $f^{(k)}(z) \ne 0$ in $0 < |z| < 1$ for $k = 1, 2, \dots, p$ and suppose that*

$$|\arg\{f^{(p)}(z)\}| < \frac{\pi}{2}\left(1 + \frac{1}{\pi}\log p\right) \quad z \in \mathbf{D}. \tag{2.19}$$

Then $f(z)$ is p-valent in $\mathbf{D}$.

Lemma 2.10 ([8]). *Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in $\mathbf{D}$, $f^{(k)}(z) \ne 0$ in $0 < |z| < 1$ for $k = 1, 2, 3, \dots, p$ and suppose that*

$$\left|\mathfrak{Im}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}\right| \le \frac{\pi}{2}\left\{1 + \frac{2}{\pi}\log p\right\}\alpha|z|^{\alpha} \quad z \in \mathbf{D}, \tag{2.20}$$

for some $\alpha > 0$. Then $f(z)$ is p-valent in $\mathbf{D}$.

THEOREM 2.11. *Let* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, *be analytic in* $\mathbf{D}$ *and suppose that*

$$\left|\mathfrak{Im}\left\{\frac{1}{z} + \frac{f^{(p+1)}(z)}{f^{(p)}(z)} - \frac{f^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \pi|z|, \quad (z \in \mathbf{D}\backslash\{0\}). \tag{2.21}$$

Then $f(z)$ *is at most* p*-valent in* $\mathbf{D}$.

PROOF. Applying the same method as the above proof, we have

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \frac{\pi}{2} \quad (z \in \mathbf{D}).$$

Applying [4, Theorem 5] shows that $f(z)$ is p-valently starlike in $\mathbf{D}$ and so, $f(z)$ at most p-valent in $\mathbf{D}$. □

THEOREM 2.12. *Let* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $p \geq 2$ *be analytic in* $\mathbf{D}$ *and suppose that*

$$\mathfrak{Re}\{f^{(p)}(z)\} > -\frac{p!}{2}, \quad (z \in \mathbf{D}). \tag{2.22}$$

Then $f(z)$ *is at most* p*-valent in* $\mathbf{D}$.

PROOF. Let us put

$$p(z) = \frac{f^{(p-1)}(z)}{p!z}, \quad p(0) = 1, \quad (z \in \mathbf{D}).$$

Then it follows that

$$f^{(p)}(z) = p!(p(z) + zp'(z)).$$

If there exists a point z_0, $|z_0| < 1$, such that

$$\mathfrak{Re}\{p(z)\} > 0 \quad \text{for } |z| < |z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\} = 0$$

then from Lemma 1.2 we have

$$z_0 p'(z_0) = \mathfrak{Re}\{z_0 p'(z_0)\} \leq -\frac{1}{2(1 + |p(z_0)|^2)}.$$

Therefore, we have

$$\Re e\{f^{(p)}(z_0)\} = \Re e\{p!(p(z_0) + z_0 p'(z_0))\} = \Re e\{p! z_0 p'(z_0)\}$$
$$\leq -\Re e\left\{\frac{p!}{2}(1 + |p(z_0)|^2)\right\} \leq -\frac{p!}{2}.$$

This contradicts hypothesis and so, it shows that

$$\Re e\{p(z)\} = \frac{1}{p!}\Re e\left\{\frac{f^{(p-1)}}{z}\right\} > 0, \quad (z \in \mathbf{D}).$$

Applying Nunokawa's result [4, Th. 8], we obtain that $f(z)$ is at most p-valent in $\mathbf{D}$. □

THEOREM 2.13. *Let* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $p \geq 2$ *be analytic in* $\mathbf{D}$ *and suppose that*

$$\Re e\left\{\frac{zf^{(k)}(z)}{z^{p-k+1}}\right\} > -\frac{p(p-1)(p-2)\cdots(p-k+2)}{2}, \quad (z \in \mathbf{D}). \tag{2.23}$$

Then we have

$$\Re e\left\{\frac{zf^{(k-1)}(z)}{z^{p-k}}\right\} > 0, \quad (z \in \mathbf{D})$$

or $f(z)$ *is at most* p*-valent in* $\mathbf{D}$.

PROOF. Let us put

$$p(z) = \frac{1}{p(p-1)(p-2)\cdots(p-k+2)} \frac{f^{(k-1)}(z)}{z^{p-k}}, \quad p(0) = 1, \quad (z \in \mathbf{D}).$$

Then it follows that

$$f^{(k-1)}(z) = \alpha \frac{p(z) z^{p-k}}{z},$$

where $\alpha = p(p-1)(p-2)\cdots(p-k+2)$ and

$$f^{(k)}(z) = \frac{\alpha\{(p'(z) z^{p-k} + p(z)(p-k) z^{p-k-1}) z - p(z) z^{p-k}\}}{z^2}$$

and so,

$$\frac{z^2 f^{(k)}(z)}{z^{p-k+2}} = \frac{zf^{(k)}}{z^{p-k+1}}$$
$$= \alpha(zp'(z) + p(z)(p-k) - p(z))$$
$$= \alpha(zp'(z) + (p-k-1)p(z)).$$

If there exists a point z_0, $|z_0| < 1$, such that

$$\Re\mathrm{e}\{p(z)\} > 0 \quad \text{for } |z| < |z_0|$$

and

$$\Re\mathrm{e}\{p(z_0)\} = 0$$

then from Lemma 1.2 we have

$$z_0 p'(z_0) = \Re\mathrm{e}\{z_0 p'(z_0)\} \leq -\frac{1 + |p(z_0)|^2}{2}.$$

Therefore, we have

$$\Re\mathrm{e}\left\{\frac{f^{(k)}(z_0)}{z_0^{p-k+1}}\right\} \leq -\frac{\alpha}{2}(1 + |p(z_0)|^2)$$
$$\leq -\frac{\alpha}{2}.$$

This contradicts hypothesis and so, it shows that

$$\Re\mathrm{e}\left\{\frac{zf^{(k-1)}(z)}{z^{p-k}}\right\} > 0, \quad (z \in \mathbf{D})$$

Applying Nunokawa's result [4, Th. 8], we obtain that $f(z)$ is at most p-valent in $\mathbf{D}$. □

In [9] the authors proved that if $f \in \mathcal{A}(p)$, $p \geq 2$ and there exists a positive integer k, $1 \leq k \leq p-1$ for which $(f^{(k)}(z)/z^{p-k})'$ is real for real z and

$$\Re\mathrm{e}\left\{\left(\frac{f^{(k)}(z)}{z^{p-k}}\right)'\right\} > 0 \quad z \in \mathbf{D},$$

then, we have

$$\Re\mathrm{e}\left\{\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\right\} \geq p-k \geq 0 \quad z \in \mathbf{D}.$$

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