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KÖHLER THEORY FOR COUNTABLE QUADRUPLE SYSTEMS

By

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Abstract. From the late 1970s to the early 1980s, Köhler developed a theory for constructing finite quadruple systems with point-transitive Dihedral automorphism groups by introducing a certain algebraic graph, now widely known as the (first) Köhler graph in finite combinatorics. In this paper, we define the countable Köhler graph and discuss countable extensions of a series of Köhler's works, with emphasis on various gaps between the finite and countable cases. We show that there is a simple 2-fold quadruple system over $\mathbf{Z}$ with a point-transitive Dihedral automorphism group if the countable Köhler graph has a so-called $[1,2]$-factor originally introduced by Kano (1986) in the study of finite graphs. We prove that a simple Dihedral ℓ-fold quadruple system over $\mathbf{Z}$ exists if and only if $\ell = 2$. The paper also covers some related remarks about Hrushovski's constructions of countable projective planes.

1. Introduction

A *t-design* is an incidence structure consisting of a set V of points with a collection $\mathscr{B}$ of k-element subsets of V, called *blocks*, such that the number λ of blocks containing a t-element subset T of V is independent of the choice of T; for example, see [18, 21]. This is often denoted by t-(v, k, λ) if V is a finite set of v elements, and by t-$(\aleph_0, k, \lambda)$ if V is a countable set. More generally, Cameron and

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Webb [4] defined 'infinite t-designs' allowing uncountable sets V and removing the assumption of finiteness of k and λ, but in this paper, we mainly focus on countable t-designs with k, λ finite. A 3-design with $k = 4$ is a *λ-fold quadruple system* (QS) and in particular, a one-fold QS is called a *Steiner quadruple system* (SQS). A t-design is *simple* if $\mathcal{B}$ contains no repeated blocks. Some of the standard terminology in 'Design Theory', not explained but used in this paper, can be found in [2].

There are some necessary 'divisibility conditions' for the existence of a finite t-(v, k, λ) design, namely, that

$$\lambda\binom{v-i}{t-i} \equiv 0 \pmod{\binom{k-i}{t-i}} \quad \text{for every } i = 0, 1, \dots, t-1. \tag{1.1}$$

A famous conjecture (cf. [15]) is that the divisibility conditions are also sufficient, with some possible exceptions. This conjecture was a long-standing open problem, until Keevash [14] recently announced the proof of the conjecture. After the work of Keevash, constructions of t-designs have received special attention in combinatorics and incidence geometry, since Keevash's theorem is non-constructive.

On the other hand, countable t-designs have received attention in model theory and related areas. Köhler [18] first proved an existence theorem for countable t-designs with $\lambda = 1$. Under some mild assumptions, a result by Cameron and Webb [4, Proposition 7.2] also implies that there exists a simple countable t-design for every t, k, λ with $t < k$. Thus, as in the finite case, to establish constructions of simple countable t-designs is a challenging problem. There are many achivements for t-$(\aleph_0, t+1, 1)$ designs and 2-designs in general [7, 8, 9, 24]. However, very little is known on constructions of countable t-designs for $t \geq 3$ and $\lambda \geq 2$, as remarked in the textbook *Design Theory* by Beth, Jungnickel, and Lenz [2]; see also [4, p. 80].

In the infinite case, a series of influential works on quadruple systems was conducted by Köhler from the late 1970s to the early 1980s. For example, Köhler [19] developed a method for constructing finite SQS with a point-transitive Dihedral automorphism group, by introducing the concept of the (*first*) *Köhler graph* and proving the equivalence between one-factors and Dihedral SQS. In [17], Köhler also introduced the notion of difference cycles and thereby constructed simple 3-fold QS. To discuss countable extensions of a series of Köhler's works is a main aim of this paper.

This paper is organized as follows. In Section 2 we briefly review the original Köhler theory. Section 3 is the body of this paper. In Subsection 3.1, we define the *countable Köhler graph* $\mathscr{G}$ and study the structure of $\mathscr{G}$, with emphasis on various gaps between $\mathscr{G}$ and finite Köhler graphs. In Subsection 3.2, we prove that if a $[1,2]$-factor of $\mathscr{G}$ is explicitly constructed, then so is a Dihedral simple 2-fold QS. We also prove that a Dihedral simple λ-fold QS over $\mathbf{Z}$ exists if and only if $\lambda = 2$. This is a remarkable gap between the finite and countable cases because finite Dihedral simple λ-fold QS actually exist for $\lambda = 1, 3$. Section 4 is the Conclusion where discussions and further remarks about Hrushovski's construction ([11]) will be made; for instance, see Propositions 4.7 and 4.9.

As far as the authors know, this is the first paper relating to constructions of countable t-designs with point-transitive (non-cyclic) automorphism groups with $t \geq 3$, though there are some publications on countable t-designs with point-intransitive automorphism groups; for example, see [5, 7]. 'Infinite Design Theory' is originally a part of mathematical logic, but is of combinatorial nature here and there. To share such feelings among people in combinatorics and model theory is an important aim of this paper.

2. The Classical Köhler Theory

It is easy to see (cf. [2]) that if a finite SQS with v points exists, then $v \equiv 2, 4 \pmod 6$. Let $\mathbf{Z}_v$ be the residue ring modulo v acting regularly on the point set. Let σ_v be the automorphism of $\mathbf{Z}_v$ defined by $x^{\sigma_v} = -x$, and $\tilde{\mathbf{Z}}_v$ be the semidirect product of $\mathbf{Z}_v$ and $\langle \sigma \rangle$ which is isomorphic to the Dihedral group D_v of order $2v$. Let

$$\binom{\mathbf{Z}_v}{k} = \{B \in \mathbf{Z}_v \mid |B| = k\}.$$

For a subset X of $\mathbf{Z}_v$, let $\mathrm{Orb}_{\tilde{\mathbf{Z}}_v}(X)$ be the $\tilde{\mathbf{Z}}_v$-orbit of X, namely,

$$\mathrm{Orb}_{\tilde{\mathbf{Z}}_v}(X) = \{\pm X + z \mid z \in \mathbf{Z}_v\}. \tag{2.1}$$

We use the notation $[z_1, \ldots, z_k]$ for $\mathrm{Orb}_{\tilde{\mathbf{Z}}_v}(\{0, z_1, \ldots, z_k\})$. For example, for a triple $\{0, x, y\} \in \binom{\mathbf{Z}_v}{3}$, we have

$$\begin{aligned} &\{X \mid \{0\} \sqcup X \in [x, y]\} \\ &\qquad = \{\{x, y\}, \{-x, y - x\}, \{-y, x - y\}, \{-x, -y\}, \{x, x - y\}, \{y, y - x\}\}. \end{aligned}$$

Let $\mathscr{T}_v = \left\{[x, y] \mid \{0, x, y\} \in \binom{\mathbf{Z}_v}{3}\right\}$. $\mathscr{T}_v$ contains two types of special orbits given by

$$\mathscr{T}_v{}' = \left\{ [x, -x] \mid \{0, x, -x\} \in \binom{\mathbf{Z}_v}{3} \right\}, \tag{2.2}$$

$$\mathscr{T}_v{}'' = \left\{ \left[x, \frac{v}{2}\right] \,\middle|\, \left\{0, x, \frac{v}{2}\right\} \in \binom{\mathbf{Z}_v}{3} \right\}. \tag{2.3}$$

With the standard isomorphism f_v between the additive group $\mathbf{Z}_v$ and the multiplicative group of the vth roots of unity, each element of $\mathscr{T}_v{}'$ (resp. $\mathscr{T}_v{}''$) can be regarded as an isosceles triangle (resp. right triangle) in the complex plane. We also set

$$\mathscr{Q}_v = \left\{ [x, y, x+y] \mid \{0, x, y, x+y\} \in \binom{\mathbf{Z}_v}{4} \right\}, \tag{2.4}$$

$$\mathscr{Q}_v' = \left\{ \left[x, -x, \frac{v}{2}\right] \,\middle|\, \left\{0, x, -x, \frac{v}{2}\right\} \in \binom{\mathbf{Z}_v}{4},\ x \neq \frac{v}{2} \right\}. \tag{2.5}$$

With f_v defined above, each element of $\mathscr{Q}_v'$ can be realized as a "kite" in the complex plane $\mathbf{C}$.

Proposition 2.1 ([17, 19]). *Let $B \in \binom{\mathbf{Z}_v}{4}$. Then $\mathrm{Orb}_{\mathbf{Z}_v}(B) \in \mathscr{Q}_v \cup \mathscr{Q}_v'$ if and only if*

$$\mathrm{Orb}_{\tilde{\mathbf{Z}}_v}(B) = \mathrm{Orb}_{\mathbf{Z}_v}(B). \tag{2.6}$$

Definition 2.2 (Dihedral quadruple system). A quadruple system $(\mathbf{Z}_v, \mathscr{B})$ is *Dihedral* if $\mathrm{Orb}_{\tilde{\mathbf{Z}}_v}(B) \subseteq \mathscr{B}$ and (2.6) holds for every $B \in \mathscr{B}$.

By the definition, a Dihedral QS over $\mathbf{Z}_v$ admits D_v as a point-transitive automorphism group.

The following notion was introduced by Köhler in [19]:

Definition 2.3 (Köhler graph). The (*first*) *Köhler graph of order v* is a finite incidence structure $\mathscr{G}_v = (\mathscr{V}_v, \mathscr{E}_v)$ such that $\mathrm{Orb}_{\tilde{\mathbf{Z}}_v}(T) \in \mathscr{T}_v$ is incident with $\mathrm{Orb}_{\tilde{\mathbf{Z}}_v}(B) \in \mathscr{E}$ if $T' \subset B$ for some $T' \in \mathrm{Orb}_{\tilde{\mathbf{Z}}_v}(T)$, where

$$\mathscr{V}_v = \{[x, y] \mid x, y \in \mathbf{Z}_v,\ x \neq \pm y,\ 2x \notin \{0, y, 2y\},\ 2y \notin \{0, x, 2x\}\}, \tag{2.7}$$

$$\mathscr{E}_v = \{[x, y, x+y] \mid x, y \in \mathbf{Z}_v,\ 0 \notin \{2x, 2y\},\ \{\pm x, \pm 2x\} \cap \{\pm y, \pm 2y\} = \varnothing\}. \tag{2.8}$$

The graph $\mathscr{G}_v$ is indeed a graph in the usual sense, meaning any element of $\mathscr{E}_v$ is incident with exactly two members of $\mathscr{V}_v$. We remark that $\mathscr{T}_v = \mathscr{V}_v \sqcup$

$(\mathscr{T}_v{}' \cup \mathscr{T}_v{}'')$. A one-factor of a graph $G = (V, E)$ is a subset $\mathscr{F}$ of E such that for any vertex $x \in V$ there uniquely exists $e \in \mathscr{F}$ for which $x \in e$. A one-factor of $\mathscr{G}_v$ covers all elements of $\mathscr{V}_v$ exactly once and $\mathscr{Q}'_v$ covers all elements of $\mathscr{T}_v{}' \cup \mathscr{T}_v{}''$ exactly once.

The following theorem is due to Köhler:

Theorem 2.4 ([19]). *Let $\mathscr{F}$ be a one-factor of $\mathscr{G}_v$. Then*

$$\left\{ B \in \binom{\mathbf{Z}_v}{4} \,\middle|\, \mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) \in \mathscr{F} \sqcup \mathscr{Q}'_v \right\} \tag{2.9}$$

forms the set of quadruples of a Dihedral SQS over $\mathbf{Z}_v$.

Theorem 2.5 ([17, 19]). *Then the following hold*:

(i) *Let $v \equiv 2 \pmod 4$. Then $(\mathbf{Z}_v, \mathscr{Q}_v \cup \mathscr{Q}'_v)$ forms a simple Dihedral* 3*-fold QS.*

(ii) *Let $v \equiv 2, 10 \pmod{24}$. Then a Dihedral SQS over $\mathbf{Z}_v$ exists if and only if a Dihedral simple* 2*-fold QS over $\mathbf{Z}_v$ exists.*

Kleemann [16] proved a similar result for $v \equiv 0 \pmod 4$, where $(\mathbf{Z}_v, \mathscr{Q}_v \cup \mathscr{Q}'_v)$ forms a non-simple 3-fold QS [16]. For instance, we refer the reader to [2, Chapter 10] for a brief summary of Köhler's works; see also [20] for abelian-group-extensions of Köhler's works.

In the next sections we develop countable extensions of a series of Köhler's works, with emphasis on various gaps between the originals and our countable analogues.

3. Countable Köhler Theory

3.1. Countable Köhler Graph. Let σ be an automorphism of $\mathbf{Z}$ defined by $x^\sigma = -x$. Let $\tilde{\mathbf{Z}}$ be the semidirect product of $\mathbf{Z}$ and $\langle \sigma \rangle$. As in the finite case, we use the notation $[z_1, \dots, z_k]$ for $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(\{0, z_1, \dots, z_k\})$. It is obvious that if $\{0, x, y\} \in \binom{\mathbf{Z}}{3}$, then

$$\begin{aligned} &\{X \mid \{0\} \sqcup X \in [x, y]\} \\ &\quad = \{\{x, y\}, \{-x, y - x\}, \{-y, x - y\}, \{-x, -y\}, \{x, x - y\}, \{y, y - x\}\}. \end{aligned} \tag{3.1}$$

Similarly, if $\{0, x, y, x + y\} \in \binom{\mathbf{Z}}{4}$, then

$$\begin{aligned} &\{X \mid \{0\} \sqcup X \in [x, y, x + y]\} \\ &\quad = \{\{x, y, x + y\}, \{-x, y, -x + y\}, \{x, -y, x - y\}, \{-x, -y, -x - y\}\}. \end{aligned} \tag{3.2}$$

Let $\mathscr{T}$ denote the set of all $\tilde{\mathbf{Z}}$-orbits of triples in $\mathbf{Z}$, and

$$\mathscr{T}' = \{[x, -x] \mid x \in \mathbf{Z} \backslash \{0\}\}. \tag{3.3}$$

Since $\mathbf{Z}$ has no nontrivial involutions, the following will be natural candidates for countable analogues of $\mathscr{T}_v$ and $\mathscr{E}_v$:

$$\mathscr{V} = \{[x, y] \mid x, y \in \mathbf{Z},\ x \neq \pm y,\ 2x \notin \{0, y\},\ 2y \notin \{0, x\}\}, \tag{3.4}$$

$$\mathscr{E} = \{[x, y, x+y] \mid x, y \in \mathbf{Z},\ 0 \notin \{x, y\},\ x \neq \pm y,\ x \neq \pm 2y,\ y \neq \pm 2x\}. \tag{3.5}$$

It is shown by (3.1) that

$$\mathscr{T} = \mathscr{V} \sqcup \mathscr{T}'. \tag{3.6}$$

Remark 3.1. In the countable case, we cannot consider triples corresponding to "right triangles" nor quadruples corresponding to "kite quadruples" in the ℓ_1-space $(\mathbf{Z}, \|\cdot\|_1)$. We can understand $\mathscr{T}'$ in an intuitive geometric language, that is, the elements of $\mathscr{T}'$ correspond to the "isosceles triangles" in $(\mathbf{Z}, \|\cdot\|_1)$.

Now we shall define the countable Köhler graph:

Definition 3.2 (Countable Köhler graph). The *countable Köhler graph for* $\mathbf{Z}$ is an incidence structure $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ where $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \in \mathscr{T}$ is incident with $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(B)$ if $T' \subset B$ for some $T' \in \mathrm{Orb}_{\tilde{\mathbf{Z}}}(T)$.

From now on, we show $\mathscr{G}$ is a graph in the usual sense and then extensively study the structure of $\mathscr{G}$.

Lemma 3.3. *The following hold*:

(i) *Let* $\{0, x, y\} \in \binom{\mathbf{Z}}{3}$ *such that* $[x, y] \in \mathscr{V}$. *Then*

$$x \neq \pm y, \quad 2x \notin \{0, y\}, \quad 2y \notin \{0, x\}. \tag{3.7}$$

(ii) *Let* $\{0, x, y, x+y\} \in \binom{\mathbf{Z}}{4}$ *such that* $[x, y, x+y] \in \mathscr{E}$. *Then*

$$0 \notin \{x, y\}, \quad x \neq \pm y, \quad x \neq \pm 2y, \quad y \neq \pm 2x. \tag{3.8}$$

Proof of Lemma 3.3. Straightforward from (3.4) and (3.5). □

Lemma 3.4. *Let* $B \in \binom{\mathbf{Z}}{4}$ *such that* $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) \in \mathscr{E}$, *and let* $T \in \binom{B}{3}$. *Then* B *is the only member of* $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(B)$ *containing* T.

Proof of Lemma 3.4. Without loss of generality we may assume $0 \in B$, that is, $B = \{0, x, y, x + y\}$ satisfying (3.8). Assume $T = \{0, x, y\}$. Then, by (3.2) and (3.8),

$$
\begin{aligned}
&|\{C \in \mathrm{Orb}_{\hat{\mathbf{Z}}}(B) \mid T \subset C\}| \\
&\quad = |\{C' \in \{\{x, y, x + y\}, \{-x, y - x, y\}, \\
&\qquad \{-y, x - y, x\}, \{-y, -x, -x - y\}\} \mid \{x, y\} \subset C'\}| \\
&\quad = |\{\{x, y, x + y\}\}| = 1.
\end{aligned}
$$

The same thing holds for $T = \{x, y, x + y\}$, since

$$\{x, y, x + y\} - (x + y) = \{0, -x, -y\} \subset \{0, -x, -y, -x - y\} = B - (x + y).$$

Similarly for $T = \{0, x, x + y\}$, we get the desired result since

$$\{0, x, x + y\} - x = \{0, -x, y\} \subset \{0, -x, y, -x + y\} = B - x.$$

Switching the role of x and y also leads to the desired result for $T = \{0, y, x + y\}$. $\square$

Lemma 3.5. *If* $[x, y] \in \mathscr{V}$, *then* $[x, y] \notin \{[x, x + y], [x, y - x], [y, x - y]\}$.

Proof of Lemma 3.5. Assume $[x, y] = [x, x + y]$ or $[x, y - x]$. Then by (3.1),

$$
\begin{aligned}
&\{\{x, y\}, \{-x, y - x\}, \{-y, x - y\}, \{-x, -y\}, \{x, x - y\}, \{y, y - x\}\} \\
&\quad \cap \{\{x, x + y\}, \{x, y - x\}\} = \varnothing.
\end{aligned}
$$

This is however impossible by Lemma 3.3 (i). Switching the role of x and y, we get $[x, y] \neq [y, x - y]$. $\square$

Proposition 3.6. *Each edge* $[x, y, x + y] \in \mathscr{E}$ *is incident with exactly two vertices* $[x, y], [x, x + y] \in \mathscr{V}$. *Moreover, the possible edges incident with a given vertex* $[x, y] \in \mathscr{V}$ *are given by*

$$[x, y, x + y], \quad [x, y - x, y], \quad [y, x - y, x], \tag{3.9}$$

with the adjacent vertices

$$[x, x + y], \quad [x, y - x], \quad [y, x - y], \tag{3.10}$$

respectively.

PROOF OF PROPOSITION 3.6. We note that

$$[x, y] = \mathrm{Orb}_{\tilde{\mathbf{Z}}}(\{x, y, x+y\}), \quad [x, x+y] = [y, x+y]$$

since

$$\{0, x, y\} = -\{x, y, x+y\} + (x+y),$$
$$\{0, x, x+y\} = -\{0, y, x+y\} + (x+y).$$

So it suffices to show $[x, x+y] \in \mathscr{V}$, because $[x, y] \in \mathscr{V}$ by Lemmas 3.3 (i) and (ii). Since

$$0 \notin \{x, x+y\}, \quad x \neq \pm(x+y), \quad x \neq 2(x+y), \quad x+y \neq 2x$$

by Lemma 3.3 (ii), we have $[x, x+y] \in \mathscr{V}$ by Lemma 3.3 (i). Moreover the possible edges incident with a given vertex $[x, y] \in \mathscr{V}$ are given by (3.9), since

$$\{x, y\} \cup \{x, x+y\} = \{x, y, x+y\},$$
$$\{x, y\} \cup \{x, y-x\} = \{x, y-x, y\},$$
$$\{x, y\} \cup \{y, x-y\} = \{y, x-y, x\}. \qquad \square$$

The above results also hold for finite Köhler graphs $\mathscr{G}_v$ (see e.g. [19] and [2, Chapter 10]), whereas the following are not always true for $\mathscr{G}_v$.

LEMMA 3.7. *If* $[x, y] \in \mathscr{V}$, *then* $[x, x+y]$, $[x, y-x]$, $[y, x-y]$ *are distinct each other.*

PROOF OF LEMMA 3.7. We first claim that $[x, x+y] \notin \{[x, y-x], [y, x-y]\}$. Suppose $[x, x+y] = [x, y-x]$. By (3.1), this is equivalently,

$$\{x, x+y\} \in \{\{x, y-x\}, \{-x, y-2x\}, \{x-y, 2x-y\},$$
$$\{-x, x-y\}, \{x, 2x-y\}, \{y-x, y-2x\}\},$$

which is however impossible by Lemma 3.3 (i). Thus we also get

$$[x, x+y] = \mathrm{Orb}_{\tilde{\mathbf{Z}}}(-\{0, x, x+y\} + (x+y)) = [y, x+y] \neq [y, x-y].$$

Similar arguments also yield that $[x, y-x] \neq [y, x-y]$. $\square$

REMARK 3.8. Lemma 3.7 is not true for finite Köhler graphs $\mathscr{G}_v$. In fact, if $\mathbf{Z}_v$ contains elements x, y with $5x = 2x + y = 0$ (in $\mathbf{Z}_v$), then $[x, x+y] = [x, y-x]$.

Let us determine the degree sequence of $\mathscr{G}$. Proposition 3.6 implies that $\deg([x,y])$, the degree of $[x,y]$ in $\mathscr{G}$, is at most 3. To determine $\deg([x,y])$, we only have to check whether $[x,x+y]$, $[x,y-x]$, $[y,x-y]$ belong to $\mathscr{V}$ by Lemma 3.5, Lemma 3.7 and Proposition 3.6.

Proposition 3.9. *Assume* $[x,y]\in\mathscr{V}$. *Then the following hold*:
(i) $\deg([x,y])=3$ *if and only if*

$$0\notin\{2x+y,x+2y,3x-y,3x-2y,3y-2x,3y-x\}. \tag{3.11}$$

In particular, $\deg([x,y])=2$ *or* 3.
(ii) $\deg([x,y])=2$ *if and only if*

$$[x,y]\in\{[x,3x],[y,3y]\}. \tag{3.12}$$

Proof of Proposition 3.9 (i). By Lemma 3.3 (i), $[x,x+y]\in\mathscr{V}$ is equivalently

$$x\neq\pm(x+y),\quad 2x\notin\{0,x+y\},\quad 2(x+y)\notin\{0,x\},$$

meaning $0\notin\{2x+y,2y+x\}$. Similarly, by Lemma 3.3 (i), $[x,y-x]\in\mathscr{V}$ is equivalently

$$x\neq\pm(y-x),\quad 2x\notin\{0,y-x\},\quad 2(y-x)\notin\{0,x\},$$

i.e. $0\notin\{3x-y,3x-2y\}$. Switching the role of x and y, we see that $[y,x-y]\in\mathscr{V}$ if and only if $0\notin\{3y-x,3y-2x\}$. Clearly, if any two distinct elements of (3.11) equal zero, we have $x=y=0$. This means that $\deg([x,y])=2$ or 3. □

Proof of Proposition 3.9 (ii). It suffices to consider the case where exactly one of the elements of (3.11) is zero. Suppose $2x+y=0$. Then, by Lemma 3.3 (i), we have

$$[x,x+y]\notin\mathscr{V}\quad\text{and}\quad\{[x,y-x],[y,x-y]\}\subset\mathscr{V}.$$

Proposition 3.6 thus implies $\deg([x,y])=2$ and in this case,

$$[x,y]=[x,-2x]=\mathrm{Orb}_{\tilde{\mathbf{Z}}}(-\{0,x,3x\}+x)=[x,3x].$$

Switching the role of x and y, we get the desired result for $x+2y=0$. Similarly, if $0\in\{3x-y,3x-2y\}$, then

$$\{[x,x+y],[y,x-y]\}\subset\mathscr{V}\quad\text{and}\quad[x,y-x]\notin\mathscr{V}.$$

Switching the role of x and y, we get the desired result for $0 \in \{3y - x, 3y - 2x\}$. $\square$

Proposition 3.9 (ii) says that $\mathcal{G}$ does not have vertices of degree 1 or 2. This is a gap between the structure of $\mathcal{G}$ and that of finite Köhler graphs $\mathcal{G}_v$.

Next we discuss the connectedness of the countable graph $\mathcal{G}$. We begin by introducing a certain algebraic object introduced by Köhler in [18].

DEFINITION 3.10 (Difference sequence). Let $Z = \{z_1, \dots, z_t\} \in \binom{\mathbf{Z}}{t}$; without loss of generality we may assume $z_1 < \cdots < z_t$. The map $\Delta : \binom{\mathbf{Z}}{t} \to \mathbf{N}^{t-1}$ defined by

$$\Delta Z = (z_2 - z_1, z_3 - z_2, \dots, z_t - z_{t-1})$$

or the image ΔZ is called the *difference sequence of* Z.

FACT 3.11 (cf. [2, 18]). *Let* $X = \{x_1, \dots, x_t\}$, $Y = \{y_1, \dots, y_t\} \in \binom{\mathbf{Z}}{t}$ *with* $x_1 < \cdots < x_t$, $y_1 < \cdots < y_t$. *Then the following are equivalent*:

(i) $\Delta X = \Delta Y$;

(ii) $\mathrm{Orb}_{\mathbf{Z}}(X) = \mathrm{Orb}_{\mathbf{Z}}(Y)$.

In particular there is a bijection between $\mathbf{N}^2$ *and the set of* $\mathbf{Z}$*-orbits of triples in* $\mathbf{Z}$.

The following is a refinement of Fact 3.11:

PROPOSITION 3.12. *Let* $X = \{x_1, \dots, x_t\}$, $Y = \{y_1, \dots, y_t\} \in \binom{\mathbf{Z}}{t}$ *with* $x_1 < \cdots < x_t$, $y_1 < \cdots < y_t$. *Then the following are equivalent*:

(i) $\Delta X = \Delta Y$ *or* $\Delta X = \Delta(-Y)$;

(ii) $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(X) = \mathrm{Orb}_{\tilde{\mathbf{Z}}}(Y)$.

In particular there is a bijection between $\{(x, y) \in \mathbf{N}^2 \mid y \geq x\}$ *and* $\mathcal{T}$.

REMARK 3.13. Let

$$\Gamma = \{(x, y) \in \mathbf{N}^2 \mid y > x\}. \tag{3.13}$$

There is a bijective map $\eta : \Gamma \to \mathcal{V}$ defined by

$$\eta((x, y)) = [x, x + y]. \tag{3.14}$$

Let us briefly check that η is surjective. Let $[z, w] \in \mathcal{V}$. Recall (3.1), namely

$$\begin{aligned}\{X \mid \{0\} \sqcup X \in [z, w]\} \\ = \{\{z, w\}, \{-z, w - z\}, \{-w, z - w\}, \{-z, -w\}, \{z, z - w\}, \{w, w - z\}\}.\end{aligned}$$

We thus need to consider only the case when $0 < z < w$, and then take $(x, y) = (z, w - z) \in \Gamma$ if $w > 2z$, and $(x, y) = (w - z, z) \in \Gamma$ if $w < 2z$. To show the injectivity, we may use (3.1) again.

Lemma 3.14. *For a positive integer x, let $\mathcal{V}_x = \{[x, nx] \mid n \geq 3\}$. Then the subgraph $\mathcal{G}_x$ induced from $\mathcal{V}_x$ is connected.*

Proof of Lemma 3.14. It follows by Proposition 3.6 that

$$\{[x, nx, (n+1)x] \mid n \in \mathbf{Z},\ n \geq 3\} \in \mathcal{E}.$$

Since two vertices $[x, nx]$ and $[x, (n+1)x]$ are joined by $[x, nx, (n+1)x]$ for $n \geq 3$, we get the desired result. □

For each x, we denote the connected component covering $\mathcal{V}_x$ by $\mathcal{C}_x$.

Proposition 3.15. *The following hold*:
(i) $\mathcal{G} = \bigcup_{x=1}^{\infty} \mathcal{C}_x$.
(ii) *Every $\mathcal{C}_x$ is isomorphic to $\mathcal{C}_1$.*

Proof of Proposition 3.15 (i). Let $P_1 := (x_1, x_0) \in \Gamma$. It follows by the Euclidean algorithm that

$$\begin{aligned} x_0 &= q_1 x_1 + x_2, \quad && 0 \leq x_2 < x_1 \\ x_1 &= q_2 x_2 + x_3, \quad && 0 \leq x_3 < x_2 \\ &\ \vdots \\ x_{n-1} &= q_n x_n, && 0 \leq x_n < x_{n-1} \end{aligned}$$

We claim that $\eta(P_1)$ lies in the component $C_{\gcd(x_0, x_1)}$. If $x_2 = 0$, we are already done. Assume $x_2 \neq 0$. We consider a sequence of latttice points given by

$$P_1, (x_1, x_0 - x_1), (x_1, x_0 - 2x_1), \ldots, (x_1, x_0 - (q_1 - 1)x_1).$$

This is equivalent to choosing a sequence of vertices

$$\eta(P_1), \eta((x_1, x_0 - x_1)), \ldots, \eta((x_1, x_0 - (q_1 - 1)x_1))$$

where each two consecutive vertices

$$\eta((x_1, x_0 - mx_1)) = [x_1, x_0 - (m-1)x_1],$$

$$\eta((x_1, x_0 - (m+1)x_1)) = [x_1, x_0 - mx_1]$$

are joined by $[x_1, x_0 - mx_1, x_0 - (m-1)x_1] \in \mathscr{E}$ by Proposition 3.6. Since $[x_1, x_1 + x_2] = [x_2, x_1 + x_2]$ by (3.1), the vertex $[x_1, x_0 - (q_1 - 1)x_1] = [x_1, x_1 + x_2]$ corresponds to the lattice point $P_2 := (x_2, x_1) \in \Gamma$, which is adjacent to $\eta((x_1, x_0 - (q_1 - 1)x_1)) = [x_1, 2x_1 + x_2]$. In summary, we get a simple path from $\eta(P_1)$ to $\eta(P_2)$, say ℓ_1. Now, by repeating the same arguments, we obtain n simple paths $\ell_1, \ell_2, \dots, \ell_n$, each starting from $P_i := (x_i, x_{i-1})$ and ending with $P_{i+1} := (x_{i+1}, x_i)$. The result thus follows by Lemma 3.14. $\square$

Proof of Proposition 3.15 (ii). Let us fix $x \geq 2$. By (3.1) and Lemma 3.3, there is a bijective map $\phi : V(\mathscr{C}_1) \to V(\mathscr{C}_x)$ with $\phi([a, b]) = [ax, bx]$. By Proposition 3.6, the neighbour of $[a, b]$ lie in

$$\{[a, b-a], [b, a-b], [a, a+b]\} \cap V(\mathscr{C}_1).$$

This is equivalently,

$$\{\phi([a, b-a]), \phi([b, a-b]), \phi([a, a+b])\} \cap V(\mathscr{C}_x)$$

since x is a nonzero element in $\mathbf{Z}$. Proposition 3.9 implies that $\{[a, b], [a, b-a]\} \in E(\mathscr{C}_1)$ if and only if $\{\phi([a, b]), \phi([a, b-a])\} \in E(\mathscr{C}_x)$, and similarly for $\{[a, b], [b, a-b]\}$ and $\{[a, b], [a, a+b]\}$. Hence ϕ induces an isomorphism between $\mathscr{C}_1$ and $\mathscr{C}_x$, which completes the proof. $\square$

In general a finite Köhler graph $\mathscr{G}_n$ is a union of non-isomorphic connected components. This is again a gap between the finite and countable cases.

3.2. Dihedral Simple Quadruple Systems. Let

$$\mathscr{Q} = \left\{ [x, y, x+y] \mid \{0, x, y, x+y\} \in \binom{\mathbf{Z}}{4} \right\}. \tag{3.15}$$

There are no "kite quadruples" in the space $(\mathbf{Z}, \|\cdot\|_1)$ since $\mathbf{Z}$ has no non-trivial involutions, as mentioned in Remark 3.1.

We begin with a countable analogue of Proposition 2.1:

Proposition 3.16. *Let* $B \in \binom{\mathbf{Z}}{4}$. *Then* $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) \in \mathscr{Q}$ *if and only if*

$$\mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) = \mathrm{Orb}_{\mathbf{Z}}(B). \tag{3.16}$$

Proof of Proposition 3.16. Suppose $B \in \mathscr{Q}$. Without loss of generality we may assume $0 \in B$, that is, $B = \{0, x, y, x+y\}$ for some $x, y \in \mathbf{Z}$. The "only if" part can be easily shown by noting that $B^{\sigma} = B - (x+y) \in \mathrm{Orb}_{\mathbf{Z}}(B)$. Conversely, let $B = \{0, x, y, z\} \in \binom{\mathbf{Z}}{4}$ satisfy (3.16). Since $-B = B + a$ for some $a \in \mathbf{Z}$, we have

$$\begin{aligned}\{-x, -y, -z\} \in \{&\{x, y, z\}, \{-x, y-x, z-x\}, \{-y, x-y, z-y\},\\ &\{-z, x-z, y-z\}\}\end{aligned}$$

If $\{-x, -y, -z\} = \{x, y, z\}$, we must have $0 \in \{x, y, z\}$ by a simple descent, which is clearly impossible. Suppose that $\{-x, -y, -z\} = \{-x, y-x, z-x\}$. Then $\{-y, -z\} = \{y-x, z-x\}$ and so $x = y+z$; in this case $B = \{0, y, z, y+z\}$. In the remaining two cases, by switching the role of x and z (resp. y), we get $B = \{0, x, y, x+y\}$ (resp. $B = \{0, x, z, x+z\}$). □

Definition 3.17 (Dihedral (countable) quadruple system). We say that a countable quadruple system $\mathscr{D} = (\mathbf{Z}, \mathscr{B})$ is *Dihedral* if $\mathrm{Orb}_{\mathbf{Z}}(B) \subseteq \mathscr{B}$ and (3.16) holds for every $B \in \mathscr{B}$.

In this section we give a necessary and sufficient condition for the existence of simple Dihedral quadruple systems over $\mathbf{Z}$. Before doing so, we give some preliminary lemmas.

Lemma 3.18. *$\mathscr{Q} = \mathscr{E} \sqcup \mathscr{Q}_0$, where*

$$\mathscr{Q}_0 = \{[x, 2x, 3x] \mid x \in \mathbf{Z} \backslash \{0\}\}. \tag{3.17}$$

Proof of Lemma 3.18. Let $B = \{0, x, y, x+y\} \in \binom{\mathbf{Z}}{4}$, and suppose $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) \notin \mathscr{E}$. By Lemma 3.3 (ii), we see that $x = \pm 2y$ or $y = \pm 2x$. In the former case, if $x = -2y$, then

$$B = \{0, -2y, y, -y\} = \{0, y, 2y, 3y\} + (-2y) \in [y, 2y, 3y].$$

Switching the role of x and y, we have $B \in [x, 2x, 3x]$ if $y = -2x$. There is nothing to prove when $x = 2y$ or $y = 2x$. □

Lemma 3.19. *For $x \in \mathbf{Z} \backslash \{0\}$, let*

$$T^x = \{0, x, 3x\} \in \binom{\mathbf{Z}}{3}, \quad T_x = \{0, x, -x\} \in \binom{\mathbf{Z}}{3}, \tag{3.18}$$

$$Q_x = \{-2x, -x, 0, x\} \in \binom{\mathbf{Z}}{4}.$$

The following hold:

(i) Q_x *and* Q_{-x} *are the only members in* $\bigcup_{\mathcal{O} \in \mathcal{Q}} \mathcal{O}$ *containing* T_x.

(ii) $Q_x + 2x$ *is the unique member in* $\bigcup_{\mathcal{O} \in \mathcal{Q}_0} \mathcal{O}$ *containing* T^x.

Proof of Lemma 3.19 (i). Clearly, Q_x and Q_{-x} satisfy (3.16) and contain T_x. Conversely, let $B = \{0, x, -x, y\} \in \bigcup_{\mathcal{O} \in \mathcal{Q}} \mathcal{O}$. Then by Proposition 3.16, there exists $z \in \mathbf{Z}$ such that

$$B = -B + z = \{z, z - x, z + x, z - y\}.$$

If $z \in \{0, y\}$, we must have $y = 0$, which is clearly impossible. So $z = \pm x$. If $z = x$, then we have $\{2x, x - y\} = \{-x, y\}$ and so $2x = y$, which implies that $B = Q_{-x}$. Switching the role of x and $-x$, we get $B = Q_x$ if $z = -x$. □

Proof of Lemma 3.19 (ii). Clearly, $T^x \subset \{0, x, 2x, 3x\} \in \bigcup_{\mathcal{O} \in \mathcal{Q}_0} \mathcal{O}$. Conversely, let $B = \{0, x, 3x, y\} \in \bigcup_{\mathcal{O} \in \mathcal{Q}_0} \mathcal{O}$. Then by Proposition 3.16,

$$B = -B + z = \{z, z - x, z - 3x, z - y\} \quad \text{for some } z \in \mathbf{Z}.$$

Clearly, $z \neq 0$. If $0 \in \{z - x, z - y\}$, then $y \in \{-2x, 4x\}$ and $B \notin \bigcup_{\mathcal{O} \in \mathcal{Q}_0} \mathcal{O}$. If $z = 3x$, then we have $y = 2x$, namely $T^x \subset Q_x + 2x$. □

Proposition 3.20. *If there is a Dihedral simple* λ*-fold QS over* $\mathbf{Z}$, *then* $\lambda = 2$.

Proof of Proposition 3.20. The result follows by Lemma 3.19. □

To show the converse direction, we begin by introducing a countable analogue of the concept of $[k-1, k]$-*factors* originally introduced by Kano [13] for finite graphs:

Definition 3.21 ([13]). Let k be a positive integer and G be a (possibly finite) graph. A subset $\mathcal{F}$ of $E(G)$ is called a $[k-1, k]$-*factor of* G if every vertex has degree $k-1$ or k with respect to the subgraph G with edge set $\mathcal{F}$; we say that a vertex of degree $k-1$ (resp. degree k) is *covered* $k-1$ times (resp. k times) by $\mathcal{F}$.

Theorem 3.22. *Let* $\mathcal{F}$ *be a* $[1, 2]$*-factor of* $\mathcal{G}$ *where the elements of* $\{[x, 3x] \mid x \in \mathbf{Z} \backslash \{0\}\}$ *are the only vertices of degree* 1. *Then*

$$\mathcal{B} := \left\{ B \in \binom{\mathbf{Z}}{4} \,\middle|\, \mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) \in \mathcal{F} \sqcup \mathcal{Q}_0 \right\} \tag{3.19}$$

forms the set of quadruples of a Dihedral simple 2*-fold QS over* $\mathbf{Z}$.

Proof of Theorem 3.22. Let

$$\mathscr{T}'' = \{[x, 3x] \mid x \in \mathbf{Z}\backslash\{0\}\}.$$

$\mathscr{T}''$ is clearly a subset of $\mathscr{V}$ and so $\mathscr{T}' \cap \mathscr{T}'' = \varnothing$ by (3.6). We count the number of quadruples in $\mathscr{B}$ which contains a given triple $T \in \binom{\mathbf{Z}}{3}$. First, assume that $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \in \mathscr{T}' \sqcup \mathscr{T}''$. By Lemma 3.19 we get

$$\left|\left\{B \in \binom{\mathbf{Z}}{4} \,\middle|\, \mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) \in \mathscr{Q}_0,\ T \subset B\right\}\right| = \begin{cases} 2 & \text{if } \mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \in \mathscr{T}', \\ 1 & \text{if } \mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \in \mathscr{T}''. \end{cases}$$

Next, let $T \in \binom{\mathbf{Z}}{3}$ with $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \in \mathscr{V}$. Then we see that

$$\begin{aligned} &\left|\left\{B \in \binom{\mathbf{Z}}{4} \,\middle|\, \mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) \in \mathscr{F},\ T \subset B\right\}\right| \\ &\quad = \left|\left\{B \in \binom{\mathbf{Z}}{4} \,\middle|\, \mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) \in \mathscr{F},\ \mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \text{ is incident with } \mathrm{Orb}_{\tilde{\mathbf{Z}}}(B)\right\}\right| \\ &\quad = |\{\mathrm{Orb}_{\tilde{\mathbf{Z}}}(B) \in \mathscr{F} \mid \mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \text{ is incident with } \mathrm{Orb}_{\tilde{\mathbf{Z}}}(B)\}| \\ &\quad = \begin{cases} 2 & \text{if } \mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \in \mathscr{V}\backslash\mathscr{T}'', \\ 1 & \text{if } \mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \in \mathscr{T}'' \end{cases} \end{aligned}$$

where the second equality follows by Lemma 3.4 and the last equality follows by the assumption on $\mathscr{F}$. By (3.2), if $B \in \bigcup_{\mathscr{O} \in \mathscr{Q}_0} \mathscr{O}$ and $T \in \binom{B}{3}$, then $\mathrm{Orb}_{\tilde{\mathbf{Z}}}(T) \in \mathscr{T}' \sqcup \mathscr{T}''$. Therefore, $\mathscr{F} \sqcup \mathscr{Q}_0$ forms a simple 3-$(\aleph_0, 4, 2)$ design, because $\mathscr{F} \cap \mathscr{Q}_0 = \varnothing$ by Lemma 3.18. □

Proposition 3.23. *There exists a* $[0,1]$*-factor of* $\mathscr{G}$ *in which* $\{[x, 3x] \mid x \in \mathbf{Z}\backslash\{0\}\}$ *is the set of isolated vertices.*

Proof of Proposition 3.23. For an integer $x > 0$, let

$$\mathscr{U}_x = \{[x, x + y] \mid x < y\}.$$

Then $\mathscr{V} = \bigsqcup_{x>0} \mathscr{U}_x$; recall Remark 3.13. The set $\mathscr{U}_x\backslash\{[x, 3x]\}$ can be partitioned into pairs of vertices as follows:

$$\begin{aligned} &\{\{[x, 2x + y], [x, 3x + y]\} \mid 1 \le y \le x - 1\} \\ &\quad \sqcup \bigsqcup_{z=0}^{\infty} \{\{[x, 4x + 2zx + y], [x, 5x + 2zx + y]\} \mid 0 \le y \le x - 1\}. \end{aligned}$$

By Lemma 3.3 (i) and Proposition 3.6, this pairing gives a one-factor $\mathscr{F}^{(x)}$ of the subgraph induced from $\mathscr{U}_x \backslash \{[x, 3x]\}$, namely,

$$\mathscr{F}^{(x)} = \{[x, 2x + y, 3x + y] \mid 1 \le y \le x - 1\} \tag{3.20}$$

$$\sqcup \bigsqcup_{z=0}^{\infty} \{[x, 4x + 2zx + y, 5x + 2zx + y] \mid 0 \le y \le x - 1\}.$$

In summary,

$$\mathscr{F}_1 := \bigsqcup_{x=1}^{\infty} \mathscr{F}^{(x)} \tag{3.21}$$

is a $[0, 1]$-factor covering all vertices of $\mathscr{G}$ exactly once, except for the vertices $[x, 3x]$. $\square$

Proposition 3.24. *There exists a one-factor $\mathscr{F}_2$ of $\mathscr{G}$ which is disjoint to $\mathscr{F}_1$ given in* (3.21).

Proof of Proposition 3.24. Let

$$\Gamma_1 = \Gamma \cap \{(x, y) \in \mathbf{N}^2 \mid 2x > y > x\}, \quad \Gamma_2 = \Gamma \backslash \Gamma_1.$$

With η defined in (3.14), we inductively partition $\eta(\Gamma_1)$ into pairs of vertices as follows. First, consider the half lines

$$l_1^a : f_1^{(a)}(x) = x + 1, \quad l_1^b : f_1^{(b)}(x) = 2x - 1.$$

Let $a_{\min}^{[1]} = 2$ and

$$A_1 = \{(x_1^{(a)}, y_1^{(a)}) \in \mathbf{N}^2 \mid y_1^{(a)} = f_1^{(a)}(x_1^{(a)}),\ x_1^{(a)} \ge a_{\min}^{[1]}\},$$

$$B_1 = \{(x_1^{(b)}, y_1^{(b)}) \in \mathbf{N}^2 \mid y_1^{(b)} = f_1^{(b)}(x_1^{(b)}),\ x_1^{(b)} \ge f_1^{(a)}(a_{\min}^{[1]})\}.$$

Then take the pairing

$$\mathscr{P}_1 = \{\{(x, x + 1), (x + 1, 2x + 1)\} \mid x \ge a_{\min}^{[1]},\ x \in \mathbf{N}\};$$

see Figure 1. Next, consider the half lines

$$l_2^a : f_2^{(a)}(x) = x + 2, \quad l_2^b : f_2^{(b)}(x) = 2x - 2.$$

Let $a_{\min}^{[2]}$ $(= 4)$ be the smallest integer $x_2^{(a)}$ such that

$$(x_2^{(a)}, y_2^{(a)}) \notin B_1 \quad \text{and} \quad (x_2^{(a)}, y_2^{(a)}) \in l_2^a \cap \Gamma_1,$$

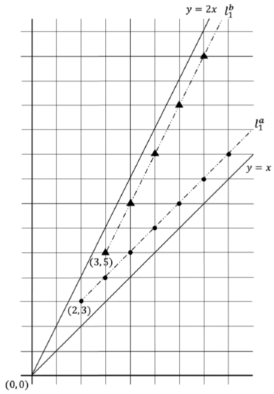

Figure 1: Parings points with rational slopes, I

and

$$A_2 = \{(x_2^{(a)}, y_2^{(a)}) \in \mathbf{N}^2 \mid y_2^{(a)} = f_2^{(a)}(x_1^{(a)}),\ x_2^{(a)} \geq a_{\min}^{[2]}\},$$

$$B_2 = \{(x_2^{(b)}, y_2^{(b)}) \in \mathbf{N}^2 \mid y_2^{(b)} = f_2^{(b)}(x_1^{(b)}),\ x_2^{(b)} \geq f_1^{(a)}(a_{\min}^{[2]})\}.$$

Then take the pairing

$$\mathscr{P}_2 = \{\{(x, x+2), (x+2, 2x+2)\} \mid x \geq a_{\min}^{[2]},\ x \in \mathbf{N}\};$$

see Figure 2. In general, given half lines $l_1^a, \dots, l_n^a$, $l_1^b, \dots, l_n^b$, subsets $A_1, \dots, A_n$, $B_1, \dots, B_n$ of Γ_1, and integers $a_{\min}^{[1]}, \dots, a_{\min}^{[n]}$, we consider the half lines

$$l_{n+1}^a : f_{n+1}^{(a)}(x) = x + n + 1, \quad l_{n+1}^b : f_{n+1}^{(b)}(x) = 2x - n - 1.$$

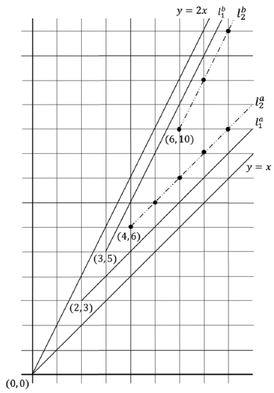

Figure 2: Parings points with rational slopes, II

Let $a_{\min}^{[n+1]}$ be the smallest integer $x_{n+1}^{(a)}$ such that

$$(x_{n+1}^{(a)}, y_{n+1}^{(a)}) \notin \bigcup_{k=1}^{n} B_k \quad \text{and} \quad (x_{n+1}^{(a)}, y_{n+1}^{(a)}) \in l_{n+1}^{a} \cap \Gamma_1,$$

and

$$A_{n+1} = \{(x_{n+1}^{(a)}, y_{n+1}^{(a)}) \in \mathbf{N}^2 \mid y_{n+1}^{(a)} = f_{n+1}^{(a)}(x_{n+1}^{(a)}),\ x_{n+1}^{(a)} \ge a_{\min}^{[n+1]}\},$$

$$B_{n+1} = \{(x_{n+1}^{(b)}, y_{n+1}^{(b)}) \in \mathbf{N}^2 \mid y_{n+1}^{(b)} = f_{n+1}^{(b)}(x_{n+1}^{(b)}),\ x_{n+1}^{(b)} \ge f_{n+1}^{(a)}(a_{\min}^{[n+1]})\}.$$

Then take the pairing

$$\mathscr{P}_{n+1} = \{\{(x, x+n+1), (x+n+1, 2x+n+1)\} \mid x \ge a_{\min}^{[n+1]},\ x \in \mathbf{N}\}.$$

By the construction, $\{\mathscr{P}_n\}_{n\ge 1}$ is a partition of Γ_1 which produces a one-factor $\mathscr{F}_2'$ of the subgraph induced from $\eta(\Gamma_1)$ as follows:

$$\mathscr{F}_2' = \bigsqcup_{n=1}^{\infty}\{[x, 2x+n, 3x+n] \mid x \ge a_{\min}^{[n]},\ x \in \mathbf{N}\}. \tag{3.22}$$

Next, we partition $\eta(\Gamma_2)$ into pairs as follows:

$$\bigsqcup_{x=1}^{\infty}\{\{[x, 3x+2zx+y], [x, 4x+2zx+y]\} \mid 0 \le y \le x-1,\ z \in \mathbf{Z}_{\ge 0}\}.$$

By Proposition 3.6 and Lemma 3.3 (i), this pairing produces a one-factor $\mathscr{F}_2''$ of the subgraph induced from $\eta(\Gamma_2)$, namely,

$$\mathscr{F}_2'' = \{[x, 3x+2zx+y, 4x+2zx+y] \mid 0 \le y \le x-1,\ z \in \mathbf{Z}_{\ge 0}\}.$$

In summary, by noting that $\mathscr{F}_2$ and $\mathscr{F}_1$ are disjoint, we see that

$$\mathscr{F}_2 := \mathscr{F}_2' \sqcup \mathscr{F}_2'' \tag{3.23}$$

is the desired one-factor covering all vertices of $\mathscr{G}$. □

Finally we get the following theorem:

Theorem 3.25. *There exists a Dihedral simple λ-fold QS over $\mathbf{Z}$ if and only if $\lambda = 2$.*

Proof of Theorem 3.25. The "only if part" is shown in Proposition 3.20. To prove the "if part", we use $\mathscr{F}_1$, $\mathscr{F}_2$ given in Propositions 3.23 and 3.24, in order to get a $[1,2]$-factor of $\mathscr{G}$. The theorem thus follows by Theorem 3.22. $\square$

REMARK 3.26. With $\mathscr{F}_1$ given in (3.21), we moreover take the quadruples

$$\mathscr{F}_3 = \bigsqcup_{x=1}^{\infty} \bigsqcup_{y=-\infty}^{\infty} (\{0, x, 2x, 3x\} + 2y).$$

Then $\mathscr{F}_1 \cup \mathscr{F}_3$ forms a partial 3-$(\aleph_0, 4, 1)$ design, $(\mathbf{Z}, \mathscr{B})$ say, where the triples of the form $\{0, x, 3x\} + (2y+1)$ or $\{0, 2x, 3x\} + (2y+1)$ are missing. This is maximal, meaning that there is no way to extend $\mathscr{B}$, in order to cover the "odd translates" of the triples $\{0, x, 3x\}$ or $\{0, 2x, 3x\}$. Of course, the one-factor $\mathscr{F}_2$ of (3.23) provides another maximal partial 3-$(\aleph_0, 4, 1)$ design, which is however based on an inductive argument and may not be 'non-iteratively' constructive for researchers in finite combinatorics. We can obtain a 'completely' non-iterative construction of 3-$(\aleph_0, 4, 2)$ designs if the monotone-increasing sequence $a_{\min}^{[1]}, a_{\min}^{[2]}, \ldots$ can be written explicitly.

4. Conclusion, Further Remarks, and Discussions

In this paper we develop a Köhler theory for Dihedral countable quadruple systems and discuss the gaps between the finite and countable cases. The following are the main results:

(i) To define the countable Köhler graph $\mathscr{G}$ and study the graph structure such as degree sequence, connectedness and so on.

(ii) To show there is a simple 3-$(\aleph_0, 4, 2)$ design with a Dihedral point-transitive automorphism group if and only if there is a $[1,2]$-factor of $\mathscr{G}$ in which the triples in $\bigsqcup_{x=1}^{\infty} [x, 3x]$ appear only once.

(iii) A necessary and sufficient condition for the existence of a simple Dihedral 3-$(\aleph_0, 4, \lambda)$ design is $\lambda = 2$.

Finite projective planes $PG(2,q)$ for prime powers q are 2-$(q^2+q+1, q+1, 1)$ designs, which are one of the most important objects in Finite Design Theory. No iterative constructions have been found for such finite designs. But, the situation is quite different in Infinite Design Theory and countable projective planes can be constructed by Hrushovski's amalgamation. This is a remarkable gap between finite and infinite designs, which is however not fully recognized in finite combinatorics at least. The observations we shall make in this section have an aim to inform Hrushovski's amalgamation and related ideas in model theory to researchers in finite combinatorics. An interesting question asks whether there exist a generic graph M and a family of finite designs $\mathscr{D}_n = (V_n, \mathscr{B}_n)$ such that every $\mathscr{D}_n$ is embedded in M as a bipartite graph with vertex set $V_n \sqcup \mathscr{B}_n$. If this is the case, one may regard M as 'limit' of finite designs $\mathscr{D}_n$. Motivated by this, we tried to find a generic graph in which all finite $\mathrm{PG}(2, q)$ are embedded. Though our trials are not necessarily going well at present, we believe that it still provides an insight to the finite-combinatorics society.

In the Zermelo-Fraenkel set theory with the Well-Ordering principle, Cameron and Webb [4] tightened the definition of 'infinite t-designs' by allowing uncountable sets V and removing the assumption of finiteness of k and λ; see also Beutelspacher and Cameron [3].

Definition 4.1 ([4]). A t-$(v, k, \{\lambda_{i,j}\}_{i,j})$ design is a pair consisting of a v-element set V and a collection $\mathscr{B}$ of k-element subsets of V, called blocks, such that

(i) no block is a proper subset of any other block;
(ii) there is a non-zero cardinal κ such that for any block $B \in \mathscr{B}$, $V \backslash B$ has cardinality κ.
(iii) for any non-negative integers i, j with $i + j \leq t$, the cardinality $\lambda_{i,j}$ of the set of blocks containing a given i-subset X but not intersecting a given j-subset Y, where X and Y are disjoint, is independent of the choice of X and Y.

A *simple design* is defined by replacing the condition (i) with "no block contains any other block".

Remark 4.2. 3-$(\aleph_0, 4, 2)$ designs given in Theorem 3.25 naturally satisfy the conditions (i) through (iii). More generally, countable t-designs with t, k finite (in our definition) always satisfy the conditions (i) through (iii); see [21, Section 2] or [4, Theorem 3.1, Proposition 4.1] for more details.

Cameron and Webb [4] gave some examples of infinite t-designs with k, $\lambda_{i,j}$ infinite. An example is a 2-$(\aleph_0, \aleph_0, \{\lambda_{i,j}\}_{i,j})$ design whose point set V and block set $\mathscr{B}$ are the vertex set and the set of the maximal cliques/independent vertex sets of the Rado graph, respectively. Another example is a t-$(\aleph_0, \aleph_0, \{\lambda_{i,j}\}_{i,j})$ that involves the affine plane over the algebraic closure $\overline{\mathbf{F}}_p$ (cf. [25]).

We now describe one more example of 2-$(\aleph_0, \aleph_0, \{\lambda_{i,j}\}_{i,j})$ designs with $\lambda_{2,0}$ constant. Some of the standard terminology used below can be found in [1, 12].

Let $\mathbf{K}$ be the set of all finite simple graphs. We define a *Hrushovski's predimension function* by

$$\delta(G) = 2|V(G)| - |E(G)| \quad \text{for every } G \in \mathbf{K}. \tag{4.1}$$

Let $G, H \in \mathbf{K}$ with $H \subseteq G$, where "$\subseteq$" means H is an induced subgraph of G. We say that H is a *closed submodel* of G if $\delta(H) \leq \delta(K)$ for every $H \subseteq K \subseteq G$, and write $H \leq G$. Let $\mathbf{K}_{1/2}$ be a subclass of $\mathbf{K}$ consisting of finite graphs G with $\emptyset \leq G$.

Definition 4.3 (Generic graph). Let $\mathbf{K}' \subset \mathbf{K}_{1/2}$. We say that a countable graph M is $\mathbf{K}'$*-generic* if it satisfies the following conditions:

(i) for $H \subset_{\text{fin}} M$, there exists some finite graph G such that $H \subseteq G \leq M$;
(ii) if $G \subset_{\text{fin}} M$, then $G \in \mathbf{K}'$;
(iii) for $H, G \in \mathbf{K}'$ with $H \leq M$ and $H \leq G$, there is an isomorphic copy G' of G over H such that $G' \leq M$.

Definition 4.4 (Amalgamation class). A subset $\mathbf{K}'$ of $\mathbf{K}_{1/2}$ is called an *amalgamation class* if, $\emptyset \in \mathbf{K}'$, $\mathbf{K}'$ is closed under induced subgraphs, and for $H, K, L \in \mathbf{K}'$, whenever embeddings $f_0 : H \to K$ and $g_0 : H \to L$ satisfy $f_0(H) \leq K$ and $g_0(H) \leq L$ then there exist $G \in \mathbf{K}'$ and embeddings $f_1 : K \to G$ and $g_1 : L \to G$ such that $f_1(K) \leq G$, $g_1(L) \leq G$, and $f_1 \circ f_0 = g_1 \circ g_0$.

The following fact is well known in model theory:

Fact 4.5 ([11]). *Let $\mathbf{K}'$ be an amalgamation class of $\mathbf{K}_{1/2}$. Then there uniquely exists a $\mathbf{K}'$-generic graph.*

Baldwin [1] constructed a countable non-Desarguesian projective plane using Hrushovski's construction with a $\mathbf{K}'$-generic graph for some subclass $\mathbf{K}'$ of $\mathbf{K}_{1/2}$. He also has shown that there are continuously many such non-isomorphic countable non-Desarguesian projective planes. We shall simplify Baldwin's arguments and thereby give a family of infinite 2-designs.

Lemma 4.6. *The subclass of $\mathbf{K}_{1/2}$ defined by*

$$\tilde{\mathbf{K}} := \{G \in \mathbf{K}_{1/2} \mid G : C_4\text{-free}\} \tag{4.2}$$

is an amalgamation class.

PROOF OF LEMMA 4.6. Clearly, $\varnothing \in \tilde{\mathbf{K}}$ and $\tilde{\mathbf{K}}$ is closed under induced subgraphs. Let $H, K, K' \in \tilde{\mathbf{K}}$ be such that there exist embeddings f, f' of H into K, K' with $f(H) \leq K$, $f'(H) \leq K'$, respectively. Identifying $f(H)$ with $f'(H)$, we may assume that $H \leq K$ and $H \leq K'$. Let $\tilde{K}$, $\tilde{K}'$ be maximal subgraphs of K, K' for which

(i) $H \subseteq \tilde{K} \leq K$ and $H \subseteq \tilde{K}' \leq K'$, and

(ii) there is an isomorphism between $\tilde{K}$ and $\tilde{K}'$ preserving H.

We define a finite graph G by identifying $\tilde{K}$ with $\tilde{K}'$ and then amalgamating K, K' over $\tilde{K}$. Since

$$\delta(G) = \delta(K) + \delta(K') - \delta(\tilde{K}) \geq \delta(K) \geq 0,$$

we have $G \in \mathbf{K}$. It remains to prove the C_4-freeness of G. Suppose the contrary. By inspection, we can choose $x, z \in V(\tilde{K})$, $y \in V(K) \backslash V(\tilde{K})$, $y' \in V(K') \backslash V(\tilde{K})$ such that the subgraph of G induced from $\{x, y, y', z\}$ is a 4-cycle. Let L, L' be the subgraphs induced from $V(\tilde{K}) \cup \{y\}$, $V(\tilde{K}) \cup \{y'\}$, respectively. We claim that there is no vertex $w \in V(\tilde{K}) \backslash \{x, z\}$ for which $\{y, w\} \in E(L)$ or $\{y', w\} \in E(L')$. In fact, if $\{y, w\} \in E(L)$ for some $w \in V(\tilde{K}) \backslash \{x, z\}$, then there is an integer $\ell \geq 3$ such that

$$\begin{aligned}\delta(L) &= 2(|V(\tilde{K})| + 1) - (|E(\tilde{K})| + \ell)| = 2|V(\tilde{K})| - |E(\tilde{K})| - (\ell - 2) \\ &< 2|V(\tilde{K})| - |E(\tilde{K})| = \delta(\tilde{K}).\end{aligned}$$

This is a contradiction since $\tilde{K} \leq K$ and $\tilde{K} \subseteq L \subseteq K$. We thus conclude that L and L' are isomorphic. Also, we have

$$\begin{aligned}\delta(L) &= 2(|V(\tilde{K})| + 1) - (|E(\tilde{K})| + 2)| = 2|V(\tilde{K})| - |E(\tilde{K})| \\ &= \delta(\tilde{K}),\end{aligned}$$

and $\delta(L') = \delta(\tilde{K}')$ similarly. Hence, $L \leq K$ and $L' \leq K'$. This is again a contradiction to the maximality of the choices of $\tilde{K}$ and $\tilde{K}'$. $\square$

By combining Fact 4.5 with Lemma 4.6, we get a $\tilde{\mathbf{K}}$-generic graph M.

PROPOSITION 4.7. *With the above generic graph M, let $\mathscr{V} = V(M)$ and $\mathscr{B} = \{B_x \mid x \in \mathscr{V}\}$, where*

$$B_x = \{y \in V(M) \mid \{x, y\} \in E(M)\} \quad \textit{for every } x \in V(M).$$

Then the incidence structure $(\mathscr{V}, \mathscr{B}, \in)$ is a projective plane, which produces a 2-$(\aleph_0, \aleph_0, \{\lambda_{i,j}\}_{i,j})$ design with $\lambda_{2,0} = 1$.

Proof of Proposition 4.7. Clearly, by the definition of $\mathscr{B}$, B_x is the unique block incident with a given point $x \in \mathscr{V}$. Moreover, there are four points where no block is incident with more than two points because any set of four independent vertices in $V(M)$ belongs to $\tilde{\mathbf{K}}$. We finally show that $|B_x \cap B_{x'}| = 1$ for any distinct points $x, x' \in \mathscr{V}$. Let $x, x' \in \mathscr{V}$ be two distinct points. Let X be a finite induced subgraph of M such that $X \leq M$. Such a X exists since M is a generic graph. If X contains a path of length two from x to x', there exists some $y \in V(X) \subseteq V(M)$ such that $\{x, y\}, \{x', y\} \in E(M)$. If X contains no path of length two from x to x', then let X' be a graph such that $V(X') = V(X) \cup \{y'\}$ and $E(X') = E(X) \cup \{\{x, y'\}, \{x', y'\}\}$. In this case, $X \leq X'$ and $X' \in \tilde{\mathbf{K}}$. Since M is a generic graph for $\tilde{\mathbf{K}}$, we can embed X' into M over X. Therefore, there exists some $y \in V(M)$ such that $\{x, y\}, \{x', y\} \in E(M)$. The uniqueness of y follows by the absence of 4-cycles in $\tilde{\mathbf{K}}$. □

A generic graph for a subclass of $\mathbf{K}_{1/2}$ is known to have a CM-trivial theory [23]. Also, it is known that any structure with a CM-trivial theory cannot interpret a field structure [22]. Since any Desarguesian projective plane interpets a field structure in model theoretic sense above, the projective plane we have constructed is non-Desarguesian.

Consider a projective plane of order n as a bipartite graph. It has $2(n^2 + n + 1)$ vertices and $(n^2 + n + 1)(n + 1)$ edges (incidence relations). The value of δ is $(3 - n)(n^2 + n + 1)$. Therefore, it does not belong to $\tilde{\mathbf{K}}$ if $n > 3$.

By inspection, we can see that projective planes of order 2 and 3 belong to $\tilde{\mathbf{K}}$. A projective plane of order 3 has a δ-rank 0 and any proper induced subgraph has a positive δ-rank. Moreover, we have the following:

Lemma 4.8. *Suppose G is a projective plane of order 2 or 3 as a bipartite graph. Then for any induced subgraphs A, B, C of G, if $A \leq B$, $A \leq C$, $A \neq B$, $A \neq C$, then G is not a free amalgam of B and C over A.*

Proof. In the case that G is a projective plane of order 2, the lemma can easily be checked by a computer. In the case that G is a projective plane of order 3, it can easily be checked by a computer that any proper subgraph of G has a positive δ-rank. Hence, a free amalgam of B and C over A has a positive δ-rank. But, $\delta(G) = 0$. □

With this lemma, we have the following:

Proposition 4.9. *Let $\tilde{\mathbf{K}}'$ be a class of graphs $G \in \tilde{\mathbf{K}}$ such that no projective plane of order 2 or 3 is an induced subgraph of G. Then $\tilde{\mathbf{K}}'$ is an amalgamation class. No finite projective plane is an induced subgraph of a generic graph for $\tilde{\mathbf{K}}'$.*

The 2-$(\aleph_0, \aleph_0, 1)$ design $\mathscr{D}$ given in Proposition 4.7 has existential closure number 0. The *existential closure number* $\Xi(\mathscr{D})$, which is a finite analogue of the axiom of the countable random graph, is defined by the minimum non-negative integer n such that the block-intersection graph of $\mathscr{D}$ is n-existentially closed. Here a graph $G = (V, E)$ is *n-existentially closed* if for every $S \in \binom{V}{n}$ and $T \subset S$, there exists a vertex $x \notin S$ which is adjacent to every vertex in T, and is not adjacent to any vertex in $S \setminus T$; see e.g. [21]. In Example 2.5 and Example 2.6 of [10], Horsley et al. constructed a 2-$(\aleph_0, \aleph_0, 1)$ design with existential closure number 1 or 2, which is different from our 2-design. But, using a technique by Horsley et al. [10, pp. 323–324] similar to that of constructing affine planes from projective planes, we can reconstruct a 2-$(\aleph_0, \aleph_0, 1)$ design with $\Xi(\mathscr{D}) = 0$ from Horsley's 2-design.

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