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SELECTIONS AND DELETED SYMMETRIC PRODUCTS

By

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Abstract. We give a very simple example of a connected second countable space X whose hyperspace $[X]^{n+1}$ of unordered $(n+1)$-tuples of points has a continuous selection, but $[X]^n$ has none. This settles an open question posed by Michael Hrušák and Ivan Martínez-Ruiz. The substantial part of the paper sheds some light on this phenomenon by showing that in the presence of connectedness this is essentially the only possible example of such spaces.

1. Introduction

All spaces in this paper are Hausdorff topological spaces. Let $\mathscr{F}(X)$ be the collection of all nonempty closed subsets of a space X. Each subcollection $\mathscr{D} \subset \mathscr{F}(X)$ will carry the (relative) *Vietoris topology* τ_V, and will be simply called a *hyperspace*. The basic τ_V-neighbourhoods for this topology on $\mathscr{D}$ are the sets

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{D} : S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \varnothing, \text{ whenever } V \in \mathscr{V} \right\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X. A map $f : \mathscr{D} \to X$ is a *selection* for $\mathscr{D}$ if $f(S) \in S$ for every $S \in \mathscr{D}$; and f is called *continuous* if it is continuous with respect to the Vietoris topology on $\mathscr{D}$.

Let $n \geq 1$ be an integer. The hyperspace $\mathscr{F}_n(X) = \{S \in \mathscr{F}(X) : |S| \leq n\}$ is commonly called the *n-fold symmetric product* of X, and was studied by many authors relative to the hyperspace selection problem. In this paper, we are interested in the hyperspace $[X]^n = \{S \in \mathscr{F}(X) : |S| = n\}$, which is known as the *n-fold deleted symmetric product*, or the *n-fold configuration space*. The Vietoris topology on $[X]^n$ has a very simple description emulating the product

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topology. Namely, a subset $\Omega \subset [X]^n$ is τ_V-open if and only if for every $S \in \Omega$ there exists a pairwise disjoint family $\mathscr{V} = \{V_x : x \in S\}$ of open subsets of X such that $x \in V_x$, $x \in S$, and $\langle \mathscr{V} \rangle \subset \Omega$. In particular, we have the following alternative way to express continuity of selections for $[X]^n$.

Proposition 1.1. *A selection $\sigma : [X]^n \to X$ is continuous if and only if for every $S \in [X]^n$, there exists a pairwise disjoint family $\mathscr{V} = \{V_x : x \in S\}$ of open subsets of X such that $x \in V_x$, $x \in S$, and $\sigma(\langle \mathscr{V} \rangle) \subset V_{\sigma(S)}$.*

A selection $\sigma : \mathscr{F}_2(X) \to X$ is usually called a *weak selection* for X. Such selections offer a natural interface to order-like relations on X by letting $x \preceq_\sigma y$ if $\sigma(\{x, y\}) = x$ [13, Definition 7.1]. The resulting relation $\preceq_\sigma$ is both total and antisymmetric, but not necessarily transitive. The corresponding strict relation $x \prec_\sigma y$ defined by $x \preceq_\sigma y$ and $x \neq y$, plays an important role in describing continuity of weak selections. Namely, $\sigma : \mathscr{F}_2(X) \to X$ is continuous iff for every $x, y \in X$ with $x \prec_\sigma y$, there are open sets $U, V \subset X$ such that $x \in U$, $y \in V$ and $s \prec_\sigma t$ for every $s \in U$ and $t \in V$ [7, Theorem 3.1]. Accordingly, continuity of weak selections is expressed only in terms of the elements of $[X]^2$. Moreover, each selection for $[X]^2$ has a unique extension to a selection for $\mathscr{F}_2(X)$. In contrast to weak selections, it was shown in [9, Proposition 3.10] that there exists a separable space X which admits a continuous selection for $[X]^3$ and yet has no continuous weak selection. Thus, the following question was posed in [9].

Question 1 ([9, Question 4.4]). Does there exist a second countable space X that admits a continuous selection for $[X]^n$ for some $n > 2$, but does not admit a continuous weak selection?

Here is a very simple example. Let $T = \left\{\left(t, \sin\frac{1}{t}\right) \in \mathbf{R}^2 : 0 < t \leq 1\right\}$ be the topological sine curve, and $X = T \cup \{(0, \pm 1)\}$. Then X is a connected second countable space which has no continuous weak selection because it has three noncut points $(0, -1)$, $(0, 1)$ and $(1, \sin 1)$, see Section 2. However, each triple $S \in [X]^3$ has a unique point $\sigma(S) \in S \cap T$ with a maximal t-coordinate. It is easy to see that the so defined selection $\sigma : [X]^3 \to X$ is continuous. Also, one can easily generalise this example by adding more noncut points to T, see Example 2.5.

The aim of this paper is to show that in the realm of connected spaces this is essentially the only possible example. Briefly, in Section 2 we show that a connected space X with a continuous selection for $[X]^n$ for some $n \geq 2$, has at most n noncut points, whereas the cut points of X form a connected set, Theorem 2.4. In the same section, we also obtain that all cut points of X are strong, Theorem 2.7. In Section 3, we relate these properties to a class of spaces, called *almost weakly orderable*, which are defined by the property that among any three points of X with two of them being cut, there is one that separates the other two. The paper culminates in Section 4, where we obtain that a connected space X has a continuous selection for $[X]^n$ for some $n \geq 2$ if and only if it is almost weakly orderable, see Theorem 4.1. In the presence of local compactness or local connectedness, this implies the orderability of X, Corollary 4.6. In Section 5, Theorem 4.1 is applied to obtain several other interesting applications. For instance, we show that a connected space X is weakly orderable if and only if $[X]^n$ has precisely n continuous selections for some (every) $n \geq 2$, Corollary 5.2; also that $[X]^n$ has a continuous selection if and only if $[X]^{n+1}$ has at least two continuous selections, Corollaries 5.4 and 5.5.

2. Cut and Noncut Points

For each pair of sets $A, Z \subset X$ and $n \in \mathbf{N}$, we are going to associate the subset

$$[A, Z]^n = \{S \in [A \cup Z]^n : A \subset S\} \subset [X]^n. \tag{2.1}$$

If $A = \varnothing$, then clearly $[A, Z]^n = [Z]^n$; similarly, $[A, Z]^n = \varnothing$ whenever $|A| > n$.

It is well known that the hyperspace $\mathscr{F}_n(X)$ is connected if and only if so is X [13, Theorem 4.10]. Regarding deleted symmetric products, it was shown by Kurilić [12, Theorems 5.1 and 5.2] that $[X]^n$ is connected, whenever so is X. For the reader's convenience, we give a simple proof of the latter fact (see Theorem 6.1 in the Appendix); the fact itself is crucial to establish the following property of selections.

Proposition 2.1. *Let $Z \subset X$ be a connected subset, and $A \subset X$ be disjoint from Z. If $\sigma : [X]^n \to X$ is a continuous selection for some $n \geq 2$, then either $\sigma([A, Z]^n) \subset Z$ or $\sigma([A, Z]^n) = \{a\}$ for some $a \in A$.*

Proof. The nontrivial case is when $0 < |A| < n \leq |A \cup Z|$. In this case, the collection $\Omega = \{\sigma^{-1}(a) \cap [A, Z]^n : a \in A\}$ is pairwise disjoint and closed because σ is continuous. In fact, each member of Ω is clopen in $[A, Z]^n$, which follows from Proposition 1.1 because $A \cap Z = \varnothing$. However, by Theorem 6.1 and Proposition 6.3, $[A, Z]^n$ is connected. Thus, either $\sigma^{-1}(a) \cap [A, Z]^n = \varnothing$ for every $a \in A$, or $[A, Z]^n \subset \sigma^{-1}(a)$ for some $a \in A$. □

DEFINITION 2.2. A point $p \in X$ of a connected space X is *cut* if $X \setminus \{p\}$ is not connected, and p is *noncut* if $X \setminus \{p\}$ is connected. We set

$$\mathrm{ct}(X) = \{p \in X : p \text{ is a cut point of } X\}, \tag{2.2}$$

$$\mathrm{nct}(X) = \{p \in X : p \text{ is a noncut point of } X\}. \tag{2.3}$$

If $p \in \mathrm{ct}(X)$, then $X \setminus \{p\}$ is not connected, therefore $X \setminus \{p\} = U \cup V$ for some nonempty disjoint open sets. In this case, it will convenient to say that (U, V) is a *p-cut* of X. Evidently, $\overline{U}$ and $\overline{V}$ are connected subsets with $\overline{U} \cap \overline{V} = \{p\}$.

In what follows, for a singleton $A = \{p\}$ and a subset $Z \subset X$, we will simply write $[p, Z]^n$ instead of $[\{p\}, Z]^n$, see (2.1). If a connected space X has a continuous weak selection $\sigma : [X]^2 \to X$, then $p = \sigma(S) \in \mathrm{nct}(X)$ for some $S \in [X]^2$ if and only if $\sigma(T) = p$ for every $T \in [X]^2$ with $p \in T$ [11] (see also [5, Corollary 2.7]). In other terms, we have that $p \in \sigma([X]^2) \cap \mathrm{nct}(X)$ if and only if $\sigma([p, X]^2) = \{p\}$. The property remains valid for continuous selections for $[X]^n$ as well.

THEOREM 2.3. *Let X be a connected space and $\sigma : [X]^n \to X$ be a continuous selection for some $n \geq 2$. Then $p \in \sigma([X]^n) \cap \mathrm{nct}(X)$ if and only if $\sigma([p, X]^n) = \{p\}$.*

PROOF. Let $p \in \sigma([X]^n) \cap \mathrm{nct}(X)$. Then $\sigma^{-1}(p) \cap [p, X]^n \neq \varnothing$ and $X \setminus \{p\}$ is connected. Hence, it follows from Proposition 2.1 that $[p, X]^n \subset \sigma^{-1}(p)$ because $[p, X]^n = [p, X \setminus \{p\}]^n$, see (2.1). Conversely, assume to the contrary that $\sigma([p, X]^n) = \{p\}$ and $p \in \mathrm{ct}(X)$. Next, set $Y = \overline{U}$ and $Z = \overline{V}$ for some p-cut (U, V) of X, and take nonempty sets $A \subset U$ and $B \subset V$ with $S = A \cup B \in [X]^n$. Since $p \in Z$, there exists $T \in [A, Z]^n$ with $p \in T$ and, by assumption, $\sigma(T) = p$. Hence, by Proposition 2.1, $\sigma([A, Z]^n) \subset Z$ because Z is connected. In particular, $\sigma(S) = \sigma(A \cup B) \in B$. The same is true for $[B, Y]^n$ in place of $[A, Z]^n$; therefore, we also have $\sigma(S) = \sigma(A \cup B) \in A$. Since A and B are disjoint, this is impossible. Thus, $p \in \mathrm{nct}(X)$ provided that $\sigma([p, X]^n) = \{p\}$. □

If a connected space X has a continuous weak selection $\sigma : [X]^2 \to X$, then $|\mathrm{nct}(X)| \leq 2$ and $\mathrm{ct}(X)$ is open and connected [11] (see also [5, Corollary 2.7]). The theorem below extends this property for all $n \geq 2$.

Theorem 2.4. *Let X be a connected space, and $\sigma : [X]^n \to X$ be a continuous selection for some $n \geq 2$. Then*

(i) $|X \backslash \sigma([X]^n)| < n$ *and* $X \backslash \sigma([X]^n) \subset \mathrm{nct}(X)$.

(ii) $|\mathrm{nct}(X) \cap \sigma([X]^n)| \leq 1$.

(iii) $\mathrm{ct}(X)$ *is open and connected.*

Proof. If $Q \subset X$ and $|Q| \geq n$, then Q contains an element $S \in [X]^n$, consequently $\sigma(S) \in S \subset Q$. Thus, $|X \backslash \sigma([X]^n)| < n$. Since $[X]^n$ is connected (by Theorem 6.1), so is $\sigma([X]^n)$ because σ is continuous. Hence, $X \backslash \sigma([X]^n) \subset \mathrm{nct}(X)$ because $X \backslash \sigma([X]^n)$ is finite, which is (i). Since (ii) follows from Theorem 2.3, it remains to show (iii). By (i) and (ii), $\mathrm{ct}(X)$ is open in X. Let $Y = \sigma([X]^n)$ and $\eta = \sigma \restriction [Y]^n$. If $p \in Y \cap \mathrm{nct}(X)$, then Theorem 2.3 implies that $\eta([p, Y]^n) \subset \sigma([p, X]^n) = \{p\}$. Hence, by the same theorem, p is a noncut point of Y and $\mathrm{ct}(X) = Y \backslash \{p\}$ is connected. This is (iii). □

Now, we also have the following more general example related to Question 1.

Example 2.5. For every $n \geq 2$ there exists a connected second countable space X such that $[X]^{n+1}$ has a continuous selection, but $[X]^n$ has none.

Proof. Let $n \geq 2$, and $Z_n \subset \{0\} \times [-1, 1]$ be a subset consisting of n elements. Then $X = T \cup Z_n$ is as required, where $T = \{(t, \sin\frac{1}{t}) \in \mathbf{R}^2 : 0 < t \leq 1\}$ is the topological sine curve. Indeed, by Theorem 2.4, $[X]^n$ has no continuous selection because X has $n + 1$ noncut points. However, each $S \in [X]^{n+1}$ contains a unique point $\sigma(S)$ with a maximal t-coordinate. This $\sigma : [X]^{n+1} \to X$ is a continuous selection, see Proposition 1.1. □

Definition 2.6. A point $p \in X$ of a connected space X is a *strong cut point* [4] if $X \backslash \{p\}$ has exactly two components; equivalently, if X has a p-cut consisting of connected sets.

A space X is *weakly orderable* (*KOTS* in the terminology of [14]; and sometimes called also "Eilenberg orderable") if it has a coarser open interval topology generated by a linear ordering $\preceq$ on X, called a *compatible* order for X. If X is connected and has a continuous weak selection $\sigma : [X]^2 \to X$, then it is weakly orderable. In fact, the order-like relation $\preceq_\sigma$ induced by σ (see the Introduction) is a compatible linear order on X [13, Lemma 7.2]. It is well known and easy to prove that all cut points in a connected weakly orderable space are strong, see for instance Kok [10]. We conclude this section by showing that this still holds if we only assume that $[X]^n$ has a continuous selection for some $n \geq 2$.

Theorem 2.7. *Let X be a connected space with a continuous selection for $[X]^n$ for some $n \geq 2$. Then each cut point of X is strong.*

The proof of Theorem 2.7 is based on the following simple observation.

Proposition 2.8. *Let X be a connected space, $\sigma : [X]^n \to X$ be a continuous selection for some $n \geq 2$, and (U, V) be a p-cut for some $p \in X$. If $Y = \overline{U}$ and $\sigma^{-1}(p) \cap [Y]^n \neq \emptyset$, then $\sigma([p, Y]^n) = \{p\}$.*

Proof. Let $S \in \sigma^{-1}(p) \cap [Y]^n$ and $A = S \setminus \{p\}$. Since $Z = \overline{V}$ is connected and $\sigma(A \cup \{p\}) = p$, it follows from Proposition 2.1 that $\sigma([A, Z]^n) \subset Z$ and, therefore, $\sigma(A \cup \{x\}) = x$ for every $x \in Z$. Since $Y = \overline{U}$ is also connected, the same reasoning implies that $\sigma([x, Y]^n) = \{x\}$ for every $x \in V$. Accordingly, $\sigma([p, Y]^n) = \{p\}$ because $p \in \overline{V}$ and σ is continuous. □

Proof of Theorem 2.7. Suppose that $\sigma : [X]^n \to X$ is a continuous selection for some $n \geq 2$, and (U, V) is a p-cut for some $p \in X$. The proof consists of showing that p is a noncut point of both $Y = \overline{U}$ and $Z = \overline{V}$. According to Proposition 2.8, either $\sigma([p, Y]^n) = \{p\}$ or $p \notin \sigma([Y]^n)$. In either case, by Theorems 2.3 and 2.4, p is a noncut point of Y. Precisely the same reasoning applies to show that p is also a noncut point of Z. □

3. Almost Weak Orderability

Definition 3.1. For a connected space X, a point $p \in X$ is said to *separate* $x, y \in X$ if $x \in U$ and $y \in V$ for some p-cut (U, V) of X.

If p separates x and y, then p is a cut point of X (see Definition 2.2), and neither x nor y separates the other two points (see [10, Lemma 2.1]). A connected space X is weakly orderable if and only if among every three points of X there is one which separates the other two, see [10, Theorem 4.1] (in a footnote of [2], the result is credited to D. Zaremba-Szczepkowicz). Evidently, this property incorporates the fact that $|\mathrm{nct}(X)| \leq 2$ for each weakly orderable connected space X.

In the present section, we use a slight modification of this property to deal with the selection problem for deleted symmetric products on connected spaces.

Definition 3.2. We shall say that a connected space X is *almost weakly orderable* if it has finitely many noncut points and among every three points of X with two of them being cut, there is one which separates the other two.

We proceed with some properties showing a natural relationship with weak orderability.

Proposition 3.3. *Let X be a connected almost weakly orderable space, and $Y \subset X$ be a connected subset. Then* $\mathrm{ct}(Y) \subset \mathrm{ct}(X)$ *and* $|\mathrm{nct}(Y) \setminus \mathrm{nct}(X)| \leq 2$. *In particular, Y is also almost weakly orderable.*

Proof. Take $y \in \mathrm{ct}(Y)$, and let (E, D) be a y-cut of Y. Since $E, D \subset Y$ are nonempty open sets of Y, they are infinite. Since $\mathrm{nct}(X)$ is finite, there are cut points $p, q \in \mathrm{ct}(X)$ such that $p \in E$ and $q \in D$. Accordingly, y separates p and q in Y. Since X is almost weakly orderable and $p, q \in \mathrm{ct}(X)$, one of the points p, q or y must separate the other two in X, hence in Y as well. That point is clearly y because it is the only point of this triple that separates the other two in Y. Thus, $y \in \mathrm{ct}(X)$ and we have that $\mathrm{ct}(Y) \subset \mathrm{ct}(X)$. The second part follows by a very similar argument. Namely, take three points of $Y \setminus \mathrm{nct}(X)$. Then one of these points separates the other two in X, hence also in Y. Accordingly, one of these points is a cut point of Y. □

A subset E of a connected space X is an *endset* if $X \setminus E$ is connected. It is evident that p is a noncut point of X iff the singleton $\{p\}$ is an endset for X. Thus, noncut points are often called *endpoints*. However, a set of endpoints is not necessarily an endset. Here is a simple example. Let $X = S \cup \{(0, \pm 1)\}$, where $S = \left\{\left(\pm t, \sin \frac{1}{t}\right) \in \mathbf{R}^2 : 0 < t < 1\right\}$. Then X is a connected space having two endpoints $(0, -1)$ and $(0, 1)$, but the two-point set $\{(0, \pm 1)\}$ is not an endset. In contrast, the endpoints of almost weakly orderable spaces form an endset.

Corollary 3.4. *Let X be a connected almost weakly orderable space. Then* $\mathrm{ct}(X)$ *is connected and weakly orderable. In particular, each cut point of X is strong.*

Proof. Take a noncut point $p \in X$, and set $Y = X \setminus \{p\}$. Then Y is a connected subset of X, so Proposition 3.3 implies that $\mathrm{ct}(Y) \subset \mathrm{ct}(X)$ and Y is itself almost weakly orderable. However, $\mathrm{ct}(X) \subset \mathrm{ct}(Y)$ because Y is dense and $\mathrm{ct}(X) \subset Y$. Thus, $\mathrm{ct}(X) = \mathrm{ct}(Y)$. Since X has finitely many noncut points, this implies that $\mathrm{ct}(X)$ is a connected almost weakly orderable space having only cut points. Hence, among every three distinct points of $\mathrm{ct}(X)$ there is one which separates the other two. Accordingly, $\mathrm{ct}(X)$ is also weakly orderable [10, Theorem 4.1]. □

Let $(Y, \preceq)$ be a (partially) ordered set. For subsets $A, B \subset Y$, we will write $A \prec B$ to express that $y \prec z$ for every $y \in A$ and $z \in B$. In case $A = \{y\}$, we will simply write $y \prec B$ instead of $\{y\} \prec B$; similarly, $A \prec z$ for $B = \{z\}$. Finally, we will use the standard notation for the intervals of $(Y, \preceq)$; for instance, $(\leftarrow, y)_{\preceq}$ will stand for the $\preceq$-open interval of all $z \in Y$ with $z \prec y$; $(y, \rightarrow)_{\preceq}$ for that of all $z \in Y$ with $y \prec z$; $(y, z)_{\preceq} = (y, \rightarrow)_{\preceq} \cap (\leftarrow, z)_{\preceq}$; etc.

According to Corollary 3.4, the cut points $\mathrm{ct}(X)$ of a connected almost weakly orderable space X form a connected weakly orderable space. Moreover, by Proposition 3.3, the cut points of X remain cut points of $\mathrm{ct}(X)$, and all cut points of both spaces are strong (see Definition 2.6). This implies the following immediate consequence.

Corollary 3.5. *Let X be a connected almost weakly orderable space, and $\preceq$ be a compatible linear order on* $\mathrm{ct}(X)$. *Then each cut point $p \in X$ has a unique p-cut (U, V) such that $U \cap \mathrm{ct}(X) = (\leftarrow, p)_{\preceq}$ and $V \cap \mathrm{ct}(X) = (p, \rightarrow)_{\preceq}$.*

For a connected space X and $p, q \in X$, let (see Definition 3.1)

$$\mathbf{S}(p, q) = \{x \in X : x \text{ separates } p \text{ and } q\}. \tag{3.1}$$

An important property of this set is that $\mathbf{S}(p, q) \subset Y$ for every connected subset $Y \subset X$ with $p, q \in \overline{Y}$. That is, $\mathbf{S}(p, q)$ behaves as the "segment" between the points p and q. In case of almost weakly orderable spaces, this is essentially true and is based on the following considerations.

Proposition 3.6. *Let X be a connected almost weakly orderable space, $\preceq$ be a compatible linear order on* $\mathrm{ct}(X)$, *and $p, q \in X$. Then $\mathbf{S}(p, q) \neq \varnothing$ if and only if $U \cap \mathrm{ct}(X) \prec V \cap \mathrm{ct}(X)$ or $V \cap \mathrm{ct}(X) \prec U \cap \mathrm{ct}(X)$ for some open sets $U, V \subset X$ with $p \in U$ and $q \in V$.*

Proof. If $y \in \mathbf{S}(p, q)$, then y is a cut point of X and, by Corollary 3.5, X has a y-cut (U, V) with $U \cap \mathrm{ct}(X) = (\leftarrow, y)_{\preceq}$ and $V \cap \mathrm{ct}(X) = (y, \rightarrow)_{\preceq}$.

Accordingly, $U \cap \mathrm{ct}(X) \prec V \cap \mathrm{ct}(X)$. To show the converse, let $U, V \subset X$ be open sets such that $p \in U$, $q \in V$ and $U \cap \mathrm{ct}(X) \prec V \cap \mathrm{ct}(X) = B$. Then $A = \bigcup_{z \prec B} (\leftarrow, z)_{\preceq}$ is an open subset of $\mathrm{ct}(X)$ with $A \cap V = \varnothing$, in fact $A \prec B$. Moreover, A is not closed in $\mathrm{ct}(X)$ because $\mathrm{ct}(X)$ is connected and $\varnothing \neq U \cap \mathrm{ct}(X) \subset A$. If $y \in \bar{A} \cap \mathrm{ct}(X)$ with $y \notin A$, then $A = (\leftarrow, y)_{\preceq}$ and $V \cap \mathrm{ct}(X) \subset (y, \rightarrow)_{\preceq}$. Let (E, D) be a y-cut in X such that $E \cap \mathrm{ct}(X) = (\leftarrow, y)_{\preceq}$ and $D \cap \mathrm{ct}(X) = (y, \rightarrow)_{\preceq}$. We are left to show that $p \in E$ and $q \in D$. However, this is evident because $p \in D$ will imply that $\varnothing \neq U \cap D \cap \mathrm{ct}(X) \subset U \cap (y, \rightarrow)_{\preceq} = \varnothing$. Similarly, $q \in E$ is impossible. □

Let X be a connected almost weakly orderable space, and $\preceq$ be a compatible linear order on $\mathrm{ct}(X)$. We can now extend $\preceq$ to a partial order $\preceq_{\mathbf{ct}}$ on X by writing for points $p, q \in X$ that $p \prec_{\mathbf{ct}} q$ if $U \cap \mathrm{ct}(X) \prec V \cap \mathrm{ct}(X)$ for some open sets $U, V \subset X$ with $p \in U$ and $q \in V$. According to Proposition 3.6, $\preceq_{\mathbf{ct}}$ is the maximal extension of $\preceq$ which is still compatible with the topology of X. Namely, we have the following immediate consequence.

Corollary 3.7. *Let X be a connected almost weakly orderable space, $\preceq$ be a compatible linear order on* $\mathrm{ct}(X)$ *and $\preceq_{\mathbf{ct}}$ be defined as above. Then points $p, q \in X$ are $\prec_{\mathbf{ct}}$-comparable if and only if* $\mathbf{S}(p, q) \neq \varnothing$. *In fact, $p \prec_{\mathbf{ct}} q$ if and only if $U \prec_{\mathbf{ct}} V$ for some open sets $U, V \subset X$ with $p \in U$ and $q \in V$.*

Motivated by Corollary 3.7, we will refer to $\preceq_{\mathbf{ct}}$ as the *separation partial ordering* on X induced by $\preceq$, and will use the same notation for both relations. Let us explicitly remark that the idea of a separation order induced by cut points goes back to Whyburn [16]; the interested reader is also referred to [8, 17], and the more recent monograph [15].

Proposition 3.8. *If X is a connected almost weakly orderable space, then every two separation partial orderings on X are either identical or inverse to each other.*

Proof. Suppose that $\preceq$ and $\preceq_*$ are separation partial orderings on X. Since $\mathrm{ct}(X)$ is connected and weakly orderable with respect to both $\preceq$ and $\preceq_*$, it follows from [3, Theorem II] that on the points of $\mathrm{ct}(X)$, these orders are either identical or inverse to each other. According to the definition of $\preceq$ and $\preceq_*$, they are themselves either identical or inverse to each other. □

Proposition 3.9. *Whenever $\preceq$ is a separation partial ordering on a connected almost weakly orderable space X and $p \in X$ is a noncut point, we have that either $p \prec \mathrm{ct}(X)$ or $\mathrm{ct}(X) \prec p$. In particular, for each cut point $q \in X$, the $\preceq$-open intervals $(\leftarrow, q)_{\preceq}$ and $(q, \rightarrow)_{\preceq}$ form a q-cut of X.*

Proof. Let $p \in X$ be a noncut point, and $q \in X$ be a cut one. Take a q-cut (E, D) of X with $p \in E$. By Corollary 3.4, q is a strong cut point of X, hence q is not separating any pair of points of E. Since X is almost weakly orderable and p is a noncut point, any point of $E \cap \mathrm{ct}(X)$ is separating p and q. Thus, $\mathbf{S}(p,q) \neq \varnothing$ because $E \cap \mathrm{ct}(X) \neq \varnothing$. According to Corollary 3.7, p and q are $\prec$-comparable, so p is $\prec$-comparable with each cut point of X. By the same corollary, the sets $A_p = \{x \in \mathrm{ct}(X) : x \prec p\}$ and $B_p = \{x \in \mathrm{ct}(X) : p \prec x\}$ are open in $\mathrm{ct}(X)$. Since $\mathrm{ct}(X)$ is connected (by Corollary 3.4), it follows that $\mathrm{ct}(X) = A_p$ or $\mathrm{ct}(X) = B_p$, i.e. $\mathrm{ct}(X) \prec p$ or $p \prec \mathrm{ct}(X)$. The second part now follows from Corollary 3.5 and the definition of $\preceq$, which completes the proof. □

Let X and $\preceq$ be as in Proposition 3.9, and $p, q \in X$ with $p \prec q$. It follows from this proposition that $\mathbf{S}(p,q) = (p,q)_{\preceq}$ is an open connected subset of $\mathrm{ct}(X)$. However, the $\preceq$-closed interval $[p,q]_{\preceq} = \{x \in X : p \preceq x \preceq q\}$ is not necessarily closed in X. In fact, one can easily prove that X is weakly orderable provided all $\preceq$-closed intervals are closed in X, but this fact will play no role in this paper.

We conclude this section with the following two special cases when extra conditions on the noncut points of a connected almost weakly orderable space imply weak orderability.

Proposition 3.10. *Let X be a connected almost weakly orderable space which is locally connected at each of its noncut points. Then X is weakly orderable.*

Proof. Let $\preceq$ be a separation partial ordering on X. According to Proposition 3.9, $\mathrm{nct}(X) = A \cup B$ for some sets A and B with $A \prec \mathrm{ct}(X) \prec B$. It now suffices to show that $|A| \leq 1$ and $|B| \leq 1$. To this end, take $p \in A$, and contrary to the claim, assume that A contains another point $q \in A$. Then, by condition, there are open connected sets $U, V \subset X$ such that $p \in U$, $q \in V$ and $U \cap V = \varnothing$. Since $\mathrm{ct}(X)$ is dense in X, there are cut points $x, y \in X$ with $x \in U$ and $y \in V$. Accordingly, x and y are $\prec$-comparable, say $x \prec y$, and we get that $q \prec x \prec y$. It now follows that $(q, y)_{\preceq} = \mathbf{S}(q, y) \subset V$ because V is connected and $q, y \in V$.

However, this is impossible because $x \in (q, y)_{\preceq}$ and $x \in U$, but $U \cap V = \emptyset$. Accordingly, $|A| \leq 1$. Similarly, $|B| \leq 1$. □

Lemma 3.11. *Let X be a connected almost weakly orderable space which is locally compact at each of its noncut points. Then X is weakly orderable.*

Proof. By Proposition 3.10, it suffices to show that X is locally connected at each of its noncut points. This can be shown following the idea of the proof of [1, Proposition 1.2]. Namely, let $\preceq$ be a separation partial ordering on X and $\mathrm{nct}(X) = A \cup B$ with $A \prec \mathrm{ct}(X) \prec B$. Next, contrary to the claim, assume that X is not locally connected at some point $p \in \mathrm{nct}(X)$, say $p \in A$. Hence, p is contained in an open set U such that $K = \overline{U}$ is compact, $K \cap A = \{p\}$, but K does not contain any interval $(p, y)_{\preceq}$ for $y \in \mathrm{ct}(X)$. Therefore, the set $H = \bigcap_{x \in \mathrm{ct}(X) \setminus K} K \cap (\leftarrow, x)_{\preceq}$ is nonempty, in fact $H = \{p\}$. To get a contradiction, for every $x \in \mathrm{ct}(X) \setminus K$, set $S_x = (K \setminus U) \cap (\leftarrow, x)_{\preceq}$ which is a clopen set in $K \setminus U$ because $S_x = (K \setminus U) \cap (\leftarrow, x]_{\preceq} = K \setminus (U \cup (x, \rightarrow)_{\preceq})$, see Proposition 3.9. The set S_x is also nonempty because $U \cap (\leftarrow, x)_{\preceq} \neq \emptyset$ and $(\leftarrow, x)_{\preceq}$ is connected. Finally, $S_x \subset S_y$ whenever $x, y \in \mathrm{ct}(X) \setminus K$ with $x \prec y$. Since K is compact, we must have that $\bigcap_{x \in \mathrm{ct}(X) \setminus K} S_x \neq \emptyset$. However, this is impossible because $p \in U$ and, therefore, $\bigcap_{x \in \mathrm{ct}(X) \setminus K} S_x \subset H \setminus U = \{p\} \setminus U = \emptyset$. The proof is complete. □

4. $[X]^n$-Selections Versus Weak Selections

A connected space X is weakly orderable if and only if it has a continuous weak selection, equivalently a continuous selection for $[X]^2$. In this section, we will prove the following natural generalisation.

Theorem 4.1. *A connected space X has a continuous selection for $[X]^n$ for some $n \geq 2$ if and only if it is almost weakly orderable.*

In one direction, the proof of Theorem 4.1 is based on the following properties of the set $\mathbf{S}(p, q)$, see (3.1).

Proposition 4.2. *Let X be a connected space, and $\sigma : [X]^n \to X$ be a continuous selection for some $n \geq 2$. If $p \in \mathrm{nct}(X) \cap \sigma([X]^n)$, then $\mathbf{S}(p, q) = \mathrm{ct}(X)$ for any other noncut point $q \in X$.*

Proof. Let $y \in X$ be any cut point, and (U, V) be a y-cut of X with $p \in U$. By Theorem 2.4, it suffices to show that $U \subset \sigma([X]^n)$. To this end, take an $A \in [V]^{n-1}$ and observe that, by Theorem 2.3, $\sigma(A \cup \{p\}) = p$. Hence, by Proposition 2.1, we have that $\sigma(A \cup \{x\}) = x$, for every $x \in \overline{U}$. The proof is complete. $\square$

Lemma 4.3. *Let X be a connected space, $\sigma : [X]^n \to X$ be a continuous selection for some $n \geq 2$. If $q \in \sigma([X]^n)$, $p \in \mathrm{nct}(X)$ and U is the component of $X \setminus \{q\}$ with $p \in U$, then $\mathbf{S}(p,q) = \mathrm{ct}(U)$.*

Proof. Since q is either a strong cut point of X (by Theorem 2.7), or a noncut one, the set U is open and q is a noncut point of $Y = \overline{U}$. Moreover, $\mathbf{S}(p,q) \subset U$ because U is connected and $p, q \in Y = \overline{U}$. Since σ is also a continuous selection for $[Y]^n$, all cut points of Y are strong cut points of Y, therefore

$$\mathbf{S}_Y(p,q) = \{y \in Y : y \text{ separates } p \text{ and } q \text{ in } Y\} \subset \mathbf{S}(p,q).$$

Thus, it is now sufficient to show that $\mathbf{S}_Y(p,q) = \mathrm{ct}(Y)$ which, by Proposition 4.2, is reduced to showing that $\{p,q\} \cap \mathrm{nct}(Y) \cap \eta([Y]^m) \neq \varnothing$ for some $m \geq 2$ and a continuous selection $\eta : [Y]^m \to Y$. To this end, take an $S \in [X]^n$ with $\sigma(S) = q$. Evidently, $q \in \mathrm{nct}(Y) \cap \sigma([Y]^n)$ provided that $S \in [Y]^n$. The other two cases are considered below.

(i) If $S \cap U = \varnothing$, set $A = S \setminus \{q\}$. Then by Proposition 2.1, $\sigma([A, Y]^n) \subset Y$ because $S \in [A, Y]^n$ and $\sigma(S) = q \in Y$. Accordingly, $\sigma(A \cup \{x\}) = x$ for every $x \in U$, so $\sigma(A \cup \{p\}) = p$. Since $p \in \mathrm{nct}(X)$, it follows from Theorem 2.3 that $\sigma([p, Y]^n) \subset \sigma([p, X]^n) = \{p\}$. Hence, for the same reason, $p \in \mathrm{nct}(Y) \cap \sigma([Y]^n)$.

(ii) If $S \setminus Y \neq \varnothing \neq S \cap U$, set $B = S \setminus U$ and $C = S \setminus Y$. Then $k = |C| < |B| < n$ and $S \in [B, U]^n$. Since $q = \sigma(S) \in B \setminus C$, by Proposition 2.1, we now have that $\sigma([B, U]^n) = \{q\}$ and $\sigma([C, Y]^n) \subset Y$. So, one can define a continuous selection $\eta : [Y]^{n-k} \to Y$ by $\eta(T) = \sigma(C \cup T)$, $T \in [Y]^{n-k}$, see Proposition 6.3. Then $q \in T \in [Y]^{n-k}$ implies that $C \cup T = B \cup (T \setminus \{q\}) \in [B, U]^n$, therefore $\eta(T) = \sigma(C \cup T) = q$. Thus, $q \in \mathrm{nct}(Y) \cap \eta([Y]^{n-k})$. $\square$

The other direction of Theorem 4.1 is based on the following considerations of order-determined selections on partially ordered sets. Let $(X, \preceq)$ be a partially ordered set, and $\sigma : [X]^n \to X$ be a selection for some $n \geq 2$.

Definition 4.4. We shall say that σ is *$\preceq$-determined* if for every $S \in [X]^n$, each point of S is $\preceq$-comparable with $\sigma(S)$. A $\preceq$-determined selection $\sigma : [X]^n \to X$ will be called *$\preceq$-balanced* if

(4.1) $\quad |\{x \in S : x \preceq \sigma(S)\}| = |\{x \in T : x \preceq \sigma(T)\}| \quad$ for every $S, T \in [X]^n$.

We finalise the preparation for the proof of Theorem 4.1 with the following characterisation of continuity of selections.

Lemma 4.5. *Let X be a connected almost weakly orderable space, $\preceq$ be a separation partial ordering on X, and $\sigma : [X]^n \to X$ be a selection for some $n \geq 2$. Then σ is continuous if and only if it is $\preceq$-balanced.*

Proof. If σ is continuous, then it is also $\preceq$-determined. Indeed, take an $S \in [X]^n$. If $q = \sigma(S)$ is a cut point of X, then it is $\preceq$-comparable with any other point of X, by Proposition 3.9. Otherwise, if q is a noncut point of X, it follows from Proposition 4.2 that any cut point of X separates q from any other noncut point of X. In other words, q is $\preceq$-comparable with any other noncut point of X, hence q is also $\preceq$-comparable with any element of S. Thus, in either case, σ is $\preceq$-determined. For such a selection, consider the function $k_\sigma : [X]^n \to \mathbf{N}$ defined by $k_\sigma(S) = |\{x \in S : x \preceq \sigma(S)\}|$, for every $S \in [X]^n$. Since $\sigma(S)$ is $\preceq$-comparable with each $x \in S$, using Corollary 3.7, there exists a pairwise disjoint collection $\mathscr{U} = \{U_x : x \in S\}$ of open subsets of X such that $x \in U_x$, for every $x \in S$, and

$$U_x \prec U_{\sigma(S)} \quad \text{or} \quad U_{\sigma(S)} \prec U_x, \quad \text{whenever } x \in S \backslash \{\sigma(S)\}. \tag{4.2}$$

Consider the τ_V-neighbourhood $\Omega = \langle \mathscr{U} \rangle$ of S in $[X]^n$. Whenever $T \in \Omega$, it follows from (4.2) that $\sigma(T) \in U_{\sigma(S)}$ if and only if $k_\sigma(T) = k_\sigma(S)$. Hence, by Proposition 1.1, σ is continuous at S if and only if k_σ is continuous at S (equivalently, constant in a neighbourhood of S). Since $[X]^n$ is connected (by Theorem 6.1) and $\mathbf{N}$ is discrete, σ is continuous if and only if k_σ is constant. The latter is clearly equivalent to σ being $\preceq$-balanced, see (4.1). The proof is complete. □

Proof of Theorem 4.1. By Theorem 2.4, X has finitely many noncut points. To show that it is almost weakly orderable, take distinct points $p, q, y \in X$ with $q, y \in \mathrm{ct}(X)$, and assume that p doesn't separate q and y; also, that q doesn't separate p and y. Thus, we are left to show that $y \in \mathbf{S}(p, q)$. Since p is either a strong cut point of X (by Theorem 2.7) or a noncut one, $X \backslash \{p\}$ has an open component W with $q, y \in W$. It is evident that q and y remain cut points of W because W is open in X, whereas p is a noncut point of $Z = \overline{W}$. In fact, q and y are strong cut points of Z because $[Z]^n$ also has a continuous selection being a subset of $[X]^n$. Moreover, $\mathbf{S}(p, q) \subset W$ because W is connected and

$p, q \in Z = \overline{W}$. Thus, we are left to show that y separates p and q in Z. Let U be the component of $Z \backslash \{q\}$ with $p \in U$, hence with $y \in U$ as well. Since $[Z]^n$ has a continuous selection, Lemma 4.3 implies that y separates p and q in Z because $y \in \mathrm{ct}(U)$.

Conversely, suppose that X is almost weakly orderable, and $\preceq$ is a separation partial ordering on X. Then $\mathrm{nct}(X) = A \cup B$ with $A \prec \mathrm{ct}(X) \prec B$, see Proposition 3.9. Let $n = |A| + |B| + 1$ so that each $S \in [X]^n$ contains a cut point of X, and set $k = |A|$. We can now define a $\preceq$-balanced selection $\sigma : [X]^n \to X$ with $|\{x \in S : x \preceq \sigma(S)\}| = k + 1$, for every $S \in [X]^n$. Namely, $S \in [X]^n$ implies that $S \cap \mathrm{ct}(X) \neq \emptyset$. If $A \subset S$, let $\sigma(S)$ be the $\preceq$-minimal element of $S \cap \mathrm{ct}(X)$. If $A \backslash S \neq \emptyset$, then $S \cap \mathrm{ct}(X)$ contains at least $|A \backslash S| + 1$ points, so we can take $\sigma(S) \in S \cap \mathrm{ct}(X)$ such that $|\{x \in S \cap \mathrm{ct}(X) : x \preceq \sigma(S)\}| = |A \backslash S| + 1$. Accordingly, $|\{x \in S : x \preceq \sigma(S)\}| = k + 1$ and Lemma 4.5 completes the proof. $\square$

A space X is *orderable* (or *linearly ordered*) if it has the open interval topology generated by a linear ordering on X. It is well known that a connected weakly orderable space is orderable if and only if it is locally connected, or locally compact. For a discussion on this, the interested reader is referred to [6]. In view of this equivalence, the following is an immediate consequence of Proposition 3.10, Lemma 3.11 and Theorem 4.1.

COROLLARY 4.6. *For a connected space X with a continuous selection for $[X]^n$ for some $n \geq 2$, the following are equivalent*:

(a) *X is orderable.*

(b) *X is locally connected.*

(c) *X is locally compact.*

5. Selections as Order-Determined Choice

If X is a connected space and $\sigma : [X]^2 \to X$ is a continuous selection, then X is weakly orderable with respect to the relation $\preceq_\sigma$ generated by σ [13, Lemma 7.2], see the Introduction. In fact, in this case, $\sigma(S) = \min_{\preceq_\sigma} S$ is the $\preceq_\sigma$-minimal element of S, for every $S \in [X]^2$. If $\eta : [X]^2 \to X$ is any other continuous selection, then the linear order $\preceq_\eta$ is inverse to $\preceq_\sigma$ [3, Theorem II], hence $\eta(S) = \max_{\preceq_\sigma} S$, for every $S \in [X]^2$. This also follows easily from Theorem 6.1 because the set $\Omega = \{S \in [X]^2 : \sigma(S) = \eta(S)\}$ is clopen in $[X]^2$. Based on the same idea, we extend this result to continuous selections for $[X]^n$ for $n \geq 2$. To this end, for a partially ordered set $(X, \preceq)$ and a $\preceq$-balanced selection

$\sigma : [X]^n \to X$, we are going to associate the unique integer $|\sigma|_{\preceq} \in \mathbf{N}$ with the property that

(5.1) $$|\sigma|_{\preceq} = |\{x \in S : x \preceq \sigma(S)\}|, \quad \text{for some (every) } S \in [X]^n.$$

It is evident that a partially ordered set $(X, \preceq)$ has at most n $\preceq$-balanced selections for $[X]^n$. According to Theorem 4.1 and Proposition 3.8, this implies the following immediate consequence.

Corollary 5.1. *Let X be a connected space with a continuous selection for $[X]^n$ for some $n \geq 2$. Then $[X]^n$ has at most n continuous selections.*

We now have also the following characterisation of weak orderability of almost weakly orderable spaces.

Corollary 5.2. *For a connected space X and $n \geq 1$, the following are equivalent*:

(a) *X is weakly orderable.*
(b) *$[X]^{n+1}$ has precisely $n+1$ continuous selections.*
(c) *$[X]^{n+1}$ has at least n continuous selections.*

Proof. Suppose that X is weakly orderable with respect to a linear order $\preceq$ on it. According to Lemma 4.5, $[X]^{n+1}$ has precisely $n+1$ continuous selections $\sigma_1, \dots, \sigma_{n+1}$; each with the property that $|\sigma_k|_{\preceq} = k$, $1 \leq k \leq n+1$. They can be defined inductively by letting for $S \in [X]^{n+1}$ that $\sigma_1(S) = \min_{\preceq} S$ and $\sigma_{k+1}(S) = \min_{\preceq}(S \setminus \{\sigma_1(S), \dots, \sigma_k(S)\})$, $k \leq n$.

Suppose that $[X]^{n+1}$ has at least n selections. By Theorem 4.1, X is almost weakly orderable. Let $\preceq$ be a separation partial ordering on X. By Lemma 4.5, each continuous selection for $[X]^{n+1}$ is $\preceq$-balanced. Since $[X]^{n+1}$ has at least n such selections, $\preceq$ is a linear order on each element of $[X]^{n+1}$. Thus, $\preceq$ is a linear order on X, and X is weakly orderable with respect to $\preceq$, see Corollary 3.7. $\square$

Corollary 5.3. *Let X be a connected space which has two different continuous selections $\sigma_1, \sigma_2 : [X]^n \to X$ for some $n \geq 2$, such that*

$$\sigma_1([X]^n) \cap \mathrm{nct}(X) \neq \varnothing \neq \mathrm{nct}(X) \cap \sigma_2([X]^n).$$

Then X is weakly orderable.

PROOF. Let $p \in \sigma_1([X]^n) \cap \mathrm{nct}(X)$ and $q \in \sigma_2([X]^n) \cap \mathrm{nct}(X)$. By Theorem 4.1, X is almost weakly orderable. Take a separation partial ordering $\preceq$ on X with $p \prec \mathrm{ct}(X)$, see Proposition 3.9. By Lemma 4.5, σ_1 and σ_2 are $\preceq$-balanced. Hence, $|\sigma_1|_{\preceq} \neq |\sigma_2|_{\preceq}$ because $\sigma_1 \neq \sigma_2$, while $|\sigma_1|_{\preceq} = 1$ because $\sigma_1([p, X]^n) = \{p\}$, by Theorem 2.3. Since we also have that $\sigma_2([q, X]^n) = \{q\}$, the points p and q are different and being contained in some member of $[X]^n$, they are $\preceq$-comparable. By Proposition 3.9, this implies that $p \prec \mathrm{ct}(X) \prec q$ and $|\sigma_2|_{\preceq} = n$. If $y \in X \setminus \{p, q\}$, then $y \in S \cap T$ for some $S, T \in [X]^n$ with $p \in S$ and $q \in T$. Therefore, $p \prec y \prec q$ because $|\sigma_1|_{\preceq} = 1 < n = |\sigma_2|_{\preceq}$. That is, $\preceq$ is a linear order on X. □

COROLLARY 5.4. *Let X be a connected space which has at least two continuous selections for $[X]^{n+1}$ for some $n \geq 2$. Then $[X]^n$ also has a continuous selection.*

PROOF. Let $\sigma_1, \sigma_2 : [X]^{n+1} \to X$ be continuous selections with $\sigma_1 \neq \sigma_2$. If

$$\sigma_1([X]^n) \cap \mathrm{nct}(X) \neq \varnothing \neq \mathrm{nct}(X) \cap \sigma_2([X]^n),$$

then X is weakly orderable (by Corollary 5.3), and $[X]^n$ has a continuous selection (by Corollary 5.2). Suppose that $\sigma_2([X]^n) \subset \mathrm{ct}(X)$, and take a separation partial ordering $\preceq$ on X with $|\sigma_1|_{\preceq} < |\sigma_2|_{\preceq}$. We can now define a $\preceq$-balanced selection $\eta : [X]^n \to X$ with $|\eta|_{\preceq} = |\sigma_1|_{\preceq}$. Namely, take $T \in [X]^n$ and a cut point $q \in X$ such that $x \prec q$ for every $x \in T \cap \mathrm{ct}(X)$. Then $S = T \cup \{q\} \in [X]^{n+1}$ and $\sigma_1(S) \prec \sigma_2(S) \preceq q$. Setting $\eta(T) = \sigma_1(S)$ and using Lemma 4.5, the proof is complete. □

In fact, we also have the converse of Corollary 5.4.

COROLLARY 5.5. *Let X be a connected space which has a continuous selection for $[X]^n$ for some $n \geq 2$. Then $[X]^{n+1}$ has at least two continuous selection.*

PROOF. Let $\sigma : [X]^n \to X$ be a continuous selection, and $\preceq$ be a separation partial ordering $\preceq$ on X such that if $z \in \mathrm{nct}(X) \cap \sigma([X]^n)$, then $z \preceq x$ for every $x \in X$, see Proposition 4.2. Since σ is continuous, by Lemma 4.5, it is $\preceq$-balanced, and we have that $|\sigma|_{\preceq} = k$ for some $k \leq n$. By the same lemma, it suffices to define $\preceq$-balanced selections $\eta_1, \eta_2 : [X]^{n+1} \to X$ with $|\eta_1|_{\preceq} = k$ and $|\eta_2|_{\preceq} = k + 1$. This can be done as follows. Let $T \in [X]^{n+1}$, and q be the $\preceq$-maximal cut point of X contained in T, i.e. $q = \max_{\preceq} T \cap \mathrm{ct}(X)$. Such a point does exist because

$T \cap \mathrm{ct}(X) \neq \varnothing$, see Theorem 2.4. Consider the set $S = T \setminus \{q\}$ and the point $p = \sigma(S)$. If p is a noncut point of X, then by the properties of $\preceq$, we have that $p \prec q$. If p is a cut point of X, then by the properties of q we have again that $p \prec q$. Thus, p is $\preceq$-comparable with each point of T, and $|\{x \in T : x \preceq p\}| = |\{x \in S : x \preceq p\}| = k$. Hence, we can define $\eta_1(T) = p$. As for $\eta_2(T)$, take in mind that $\mathbf{S}(p, q) \subset \mathrm{ct}(X)$, see Proposition 3.9. If $T \cap \mathbf{S}(p, q) = \varnothing$, then $|\{x \in T : x \preceq q\}| = k + 1$, and we can take $\eta_2(T) = q$. Otherwise, if $T \cap \mathbf{S}(p, q) \neq \varnothing$, take $\eta_2(T) = \min_{\preceq} T \cap \mathbf{S}(p, q)$. It is evident that $|\{x \in T : x \preceq \eta_2(T)\}| = k + 1$, and $\eta_2(T) \in \mathrm{ct}(X)$. Hence, $\eta_2(T)$ is also $\preceq$-comparable with each point of T, which completes the proof. □

We conclude with the following consequence about the distribution of continuous selections for deleted symmetric products on connected spaces.

Corollary 5.6. *Let X be a connected almost weakly orderable space, $\preceq$ be a separation partial ordering on X, and $n \geq 2$. If σ_1 and σ_2 are continuous selections for $[X]^{n+1}$ and $|\sigma_1|_{\preceq} < k < |\sigma_2|_{\preceq}$, then $[X]^{n+1}$ also has a continuous selection η with $|\eta|_{\preceq} = k$.*

Proof. Suppose that $k = |\sigma_1|_{\preceq} + 1 < |\sigma_2|_{\preceq}$, and let us show that $[X]^{n+1}$ has a continuous selection η with $|\eta|_{\preceq} = k$. So, take $T \in [X]^{n+1}$ and let $p = \sigma_1(T)$ and $q = \sigma_2(T)$. Then $|\{x \in T : x \preceq p\}| = k - 1 < k < |\{x \in T : x \preceq q\}|$ and, therefore, $p \prec x \prec q$ for some $x \in T$. That is, $\varnothing \neq T \cap \mathbf{S}(p, q) \subset \mathrm{ct}(X)$, and we can now take $\eta(T) = \min_{\preceq}\{x \in T : p \prec x\}$ which is a well defined cut point of X. Hence, $\eta(T)$ is $\preceq$-comparable with each $x \in T$, and $|\{x \in T : x \preceq \eta(T)\}| = k$. □

Appendix

Here, we give a short proof of the following result of Kurilić about connectedness of n-fold deleted symmetric products [12, Theorems 5.1 and 5.2].

Theorem 6.1. *If X is a connected space and $n \geq 1$, then $[X]^{n+1}$ is also connected.*

Our proof of Theorem 6.1 is based on the following considerations. A family $\mathscr{P}$ of subsets of a given set is *connected* if for every $E, D \in \mathscr{P}$ there exists a finite sequence $P_1, P_2, \dots, P_k$ of elements of $\mathscr{P}$ with $E = P_1$, $D = P_k$ and $P_i \cap P_{i+1} \neq \varnothing$ for every $i = 1, \dots, k - 1$. The proof of the following property of connected families is easy and is left to the reader.

PROPOSITION 6.2. *Let $\mathscr{P}$ be a connected family in a space X.*

(i) *If each $P \in \mathscr{P}$ is covered by a connected family $\mathscr{Q}_P$, then $\bigcup_{P \in \mathscr{P}} \mathscr{Q}_P$ is itself a connected family.*

(ii) *If each element of $\mathscr{P}$ is a connected subset of X, then $\bigcup \mathscr{P}$ is also a connected subset of X.*

Now, we proceed by pointing out a particular connected cover of $[X]^{n+1}$ consisting of connected subsets of $[X]^{n+1}$. In this, we are going to use the following simple observation, see (2.1).

PROPOSITION 6.3. *Let $A, Z \subset X$ be disjoint sets in a space X with $|A| \leq n$. Then the map $\varphi : [Z]^{n+1-|A|} \to [A, Z]^{n+1}$ defined by $\varphi(T) = A \cup T$ for every $T \in [Z]^{n+1-|A|}$, is a homeomorphism.*

Applying induction on n by assuming that $[Z]^k$ is connected for every connected space Z and $k \leq n$, it follows from Proposition 6.3 that the elements of the collection

$$\mathscr{P}[X]^{n+1} = \{[A, Z]^{n+1} : A \neq \varnothing,\ A \cap Z = \varnothing \text{ and } Z \text{ is connected}\}$$

are connected subsets of $[X]^{n+1}$. Thus, by Proposition 6.2, the proof of Theorem 6.1 is reduced to showing that $\mathscr{P}[X]^{n+1}$ is a connected cover of $[X]^{n+1}$.

LEMMA 6.4. *Let X be a connected space, $p \in X$ and $Q \in [X]^{n+1}$ with $p \notin Q$. Then $[p, Q]^{n+1}$ is covered by a connected subcollection $\mathscr{Q} \subset \mathscr{P}[X]^{n+1}$.*

PROOF. If $Z = X \setminus \{p\}$ is connected, take $\mathscr{Q} = \{[p, Z]^{n+1}\} \subset \mathscr{P}[X]^{n+1}$. If $X \setminus \{p\}$ is not connected, take a p-cut (U, V) of X. Then both $Y = \overline{U}$ and $Z = \overline{V}$ are connected. If Q is contained in one of the sets U or V, say $Q \subset U$, then $\mathscr{R} = \{[A, Z]^{n+1} : A \in [Q]^n\} \subset \mathscr{P}[X]^{n+1}$ is a cover of $[p, Q]^{n+1}$. Take a point $q \in V \subset Z$, and observe that $[q, Y]^{n+1} \cap [A, Z]^{n+1} \neq \varnothing$ for every $A \in [Q]^n$. Hence, $\mathscr{Q} = \mathscr{R} \cup \{[q, Y]^{n+1}\}$ is a connected subcollection of $\mathscr{P}[X]^{n+1}$ covering $[p, Q]^{n+1}$. Suppose finally that $A = U \cap Q \neq \varnothing \neq Q \cap V = B$, in which case $|A| \leq n$ and $|B| \leq n$. Then $\mathscr{Q} = \{[A, Z]^{n+1}, [B, Y]^{n+1}\} \subset \mathscr{P}[X]^{n+1}$ is as required. Indeed, $\mathscr{Q}$ is connected because $Q = A \cup B \in [A, Z]^{n+1} \cap [B, Y]^{n+1}$. It is also a cover of $[p, Q]^{n+1}$ because $S \in [p, Q]^{n+1}$ implies that $Q \setminus S$ is a singleton, hence $A \subset S$ if $Q \setminus S \subset V$ and $B \subset S$ if $Q \setminus S \subset U$. The proof is complete. □

The required property of $\mathscr{P}[X]^{n+1}$ is now precisely the assertion of the proposition below, which also completes the proof of Theorem 6.1.

Proposition 6.5. *If X is a connected space, then $\mathscr{P}[X]^{n+1}$ is a connected cover of $[X]^{n+1}$.*

Proof. Let $S, T \in [X]^{n+1}$ and $P = S \cup T$. It now suffices to show that $[P]^{n+1}$ is covered by a connected subcollection of $\mathscr{P}[X]^{n+1}$. Whenever $x \in P$, set $Q_x = P \setminus \{x\}$. Then $[x, Q_x]^{n+1} \cap [y, Q_y]^{n+1} \neq \varnothing$ for every $x, y \in P$, so $\{[x, Q_x]^{n+1} : x \in P\}$ is a connected cover of $[P]^{n+1}$. Thus, $|P| = n+2$ implies that $|Q_x| = n+1$ for every $x \in P$, and the property follows from Proposition 6.2 and Lemma 6.4. If $|P| > n+2$, this follows by induction using Proposition 6.2 and the fact that $\{[Q_x]^{n+1} : x \in P\}$ is a connected cover of $[P]^{n+1}$ with $|Q_x| = |P| - 1$, for every $x \in P$. □

References

[1] G. Artico, U. Marconi, J. Pelant, L. Rotter, and M. Tkachenko, Selections and suborderability, Fund. Math. **175** (2002), 1–33.

[2] R. Duda, On ordered topological spaces, Fund. Math. **63** (1968), 295–309.

[3] S. Eilenberg, Ordered topological spaces, Amer. J. Math. **63** (1941), 39–45.

[4] S. P. Franklin and G. V. Krishnarao, "On the topological characterization of the real line": An addendum., J. London Math. Soc. (2) **3** (1971), 392.

[5] V. Gutev, Weak orderability of second countable spaces, Fund. Math. **196** (2007), no. 3, 275–287.

[6] V. Gutev, Selections and hyperspaces, Recent progress in general topology, III (K. P. Hart, J. van Mill, and P. Simon, eds.), Atlantis Press, Springer, 2014, pp. 535–579.

[7] V. Gutev and T. Nogura, Selections and order-like relations, Appl. Gen. Topol. **2** (2001), 205–218.

[8] J. G. Hocking and G. S. Young, Topology, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1961.

[9] M. Hrušák and I. Martínez-Ruiz, Selections and weak orderability, Fund. Math. **203** (2009), 1–20.

[10] H. Kok, Connected orderable spaces, Mathematisch Centrum, Amsterdam, 1973, Mathematical Centre Tracts, No. 49.

[11] K. Kuratowski, S. B. Nadler, Jr., and G. S. Young, Continuous selections on locally compact separable metric spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **18** (1970), 5–11.

[12] M. S. Kurilić, Properties of topological n-partitions, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **25** (1995), no. 2, 31–47.

[13] E. Michael, Topologies on spaces of subsets, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[14] J. v. Mill and E. Wattel, Selections and orderability, Proc. Amer. Math. Soc. **83** (1981), no. 3, 601–605.

[15] S. B. Nadler, Jr., Continuum theory, Monographs and Textbooks in Pure and Applied Mathematics, vol. 158, Marcel Dekker, Inc., New York, 1992, An introduction.

[16] G. T. Whyburn, Concerning the cut points of continua, Trans. Amer. Math. Soc. **30** (1928), no. 3, 597–609.

[17] S. Willard, General topology, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

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