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STABILITY OF CERTAIN REFLECTIVE SUBMANIFOLDS IN COMPACT SYMMETRIC SPACES

By

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Abstract. In [1], J. Berndt and H. Tamaru classified all the cohomogeneity one actions on Riemannian symmetric spaces of noncompact type with a totally geodesic singular orbit. Also they provided that there is a one-to-one correspondence between the totally geodesic singular orbits of cohomogeneity one actions on a Riemannian symmetric space of noncompact type and those on its dual simply connected compact Riemannian symmetric space. In this paper, we determine stability of the totally geodesic singular orbits in simply connected compact symmetric spaces which obtained by the duality stated above.

Introduction

Totally geodesic submanifolds in symmetric spaces are also symmetric spaces and they are the so-called subspaces in the category of symmetric spaces. In [4], we classified all the maximal totally geodesic submanifolds in compact symmetric spaces of rank two. If the ambient symmetric space is not of type G_2, then the maximal totally geodesic submanifolds are reflective submanifolds. When we turns from rank two to three, the cases which we should take in account much increase.

The motivation for this parer is the determination of stability for all the totally geodesic submanifolds as minimal submanifolds in compact Riemannian symmetric spaces. Research on their stability often fails to grasp geometric structure. But there are some results in which we can find relation between stability and geometric structure. In [8], Mashimo proved that if a Cartan

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embedding of a compact symmetric space has its orthogonal complement with non-trivial center, then it is unstable. Also, in [13] Tanaka proved that if a monomorphism between compact symmetric spaces has a smooth section of the normal bundle with a trivial line bundle, then it is unstable. Also there is a known result about the stability of symmetric R-spaces in Hermitian symmetric spaces ([12]): If a symmetric R-space is simply connected, then it is stable. If it is not simply connected, then it is unstable. As we know on, all the geometric structure of stable totally geodesic submanifolds in compact Riemannian symmetric spaces have not yet been solved. In [5], we determined the stability of maximal totally geodesic submanifolds in compact symmetric spaces of rank two. This paper is a part of the author's doctoral thesis, Tokyo University of Science ([4], [5]).

We consider the cohomogeneity one actions on compact Riemannian symmetric spaces. Let M be a totally geodesic submanifold of a compact Riemannian symmetric space N. Then M arises as a singular orbit of cohomogeneity one action on N if and only if the isotropy representation of M acts transitively on the unit sphere in the normal space of M.

Thus, what we wish to show in this paper is the determination of stability of totally geodesic singular orbits which are obtained by the cohomogeneity one actions on compact symmetric spaces.

This paper is organized as follows. In Section 1, we will explain the theory of totally geodesic submanifolds in compact Riemannian symmetric spaces. In Section 2, we will refer to the results of cohomogeneity one actions on Riemannian symmetric spaces of noncompact type which were provided by Berndt and Tamaru ([1]). In Section 3, we recall the stability of totally geodesic submanifolds in compact Riemannian symmetric spaces. In Section 4, we determine the stability of totally geodesic submanifolds in compact irreducible simply connected Riemannian symmetric spaces which are obtained in Section 2.

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1 Totally Geodesic Submanifolds in Compact Symmetric Spaces

We introduce a "polar" and the "meridian" in a compact symmetric space which were introduced by Chen-Nagano.

Definition 1.1 ([3]). Let o be a point in a symmetric space N. We call a connected component of the fixed-point set of s_o, the symmetry at o, in N a *polar* of o and we denote it by N^+ or $N^+(p)$ for a point p in N^+. We call the connected component of the fixed-point set of $s_p \circ s_o$ in N through p the *meridian* of $N^+(p)$ in N and denote it by $N^-(p)$ or simply by N^-. When a polar consists of a single point, which differs from o, we call it a *pole*.

Remark 1.2. Polars and meridians are totally geodesic submanifolds in N; they are thus symmetric spaces. Every polar and the corresponding meridian are known for each compact connected Riemannian symmetric space ([3], [9], [10]). One of the most important properties of them is that every compact connected symmetric space N is determined by one pair of $(N^+(p), N^-(p))$ completely ([10]). N^+ is an isotropy orbit and N^- has the same rank as N has.

Definition 1.3. Let M be a totally geodesic submanifold of N and let p be a point in M. We denote by $T_p^{\perp}M$ the orthogonal complement of T_pM in T_pN. If there is a totally geodesic submanifold $M^{\perp}$ of N through p whose tangent space at p coincides with $T_p^{\perp}M$, then $M^{\perp}$ is called the *orthogonal complement* to M in N at p.

Remark 1.4. A polar $N^+(p)$ and the meridian $N^-(p)$ are the orthogonal complements to each other in N at p.

We introduce a reflective submanifold in a Riemannian manifold which was first introduced by Leung.

Definition 1.5 ([6]). Let N be a Riemannian manifold and let M be a submanifold in N. M is a *reflective submanifold* if M is a connected component of the fixed-point set of some involutive isometry of N.

Remark 1.6. Any reflective submanifold is a totally geodesic submanifold. Hence any reflective submanifold in a Riemannian symmetric space is a Riemannian symmetric space.

Proposition 1.7 ([6]). *Let M be a submanifold of a Riemannian symmetric space N, then M is a reflective submanifold if and only if M and $M^{\perp}$ are totally geodesic submanifolds.*

Next we refer to a Hermann action. There is close relation between Hermann actions and reflective submanifolds.

Definition 1.8. An isometric action of a compact Lie group H on a compact Riemannian symmetric space $N = U/L$ is called a *Hermann action* if the pair (U, H) is a symmetric pair.

The following proposition is very useful for the determination of stability of a reflective submanifold in compact Riemannian symmetric spaces.

Proposition 1.9 ([5]). *Let $N = U/L$ be a compact Riemannian symmetric space and let M be a reflective submanifold of N. Then M is a totally geodesic orbit of a Hermann action.*

Our object is that we determine the stability of a reflective submanifold M in a compact Riemannian symmetric space which has the orthogonal complement $M^{\perp}$ of rank one. In order to reach this object, we need the following propositions.

Proposition 1.10. *Let U be a compact connected Lie group and let σ and τ be different commuting involutive automorphisms of U. We put $L := U_o^{\tau}$, $H := U_o^{\sigma}$ and $H' := U_o^{\tau\sigma}$, where U^{τ}, U^{σ} and $U^{\tau\sigma}$ denote the fixed-point set of τ, σ and $\tau\sigma$ in U, respectively. Also we denote their identity components by U_o^{τ}, U_o^{σ} and $U_o^{\tau\sigma}$, respectively. Then we have the following*:

(1) $L \cap H' = H \cap H' = L \cap H$.

(2) *The pair $(H, L \cap H)$ is a compact symmetric pair with the involutive automorphism $\tau|_H$.*

(3) *The pair $(H', L \cap H)$ is a compact symmetric pair with the involutive automorphism $\tau|_{H'} = \sigma|_{H'}$.*

(4) *The pair $(L, L \cap H)$ is a compact symmetric pair with the involutive automorphism $\sigma|_L$.*

Proof. First we note that σ and τ are commuting involutive automorphisms, thus we prove (1). An automorphism $\tau|_H$ leaves H invariant and is an involutive automorphism of H. Thus we prove (2). Similarly, $\tau|_{H'} = \sigma|_{H'}$ and $\sigma|_L$ leave H' and L invariant, respectively. Also these are involutive automorphisms of H' and L, respectively. Thus we prove (3) and (4). □

The next proposition is an immediate consequence of Proposition 1.10.

Proposition 1.11. *With the above notation, a compact symmetric space* $H/L \cap H$ *is a reflective submanifold in* U/L *and the orthogonal complement to* $H/L \cap H$ *is* $H'/L \cap H$.

Proof. Let $\mathfrak{u} = \mathfrak{l} \oplus \mathfrak{p}$ be the canonical decomposition of U/L. By Proposition 1.10, we have the canonical decomposition $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{l}) \oplus (\mathfrak{h} \cap \mathfrak{l}^{\perp})$ of $H/L \cap H$, where $\mathfrak{l}^{\perp}$ is the orthogonal complement to $\mathfrak{l}$ in $\mathfrak{u}$. Also we have the canonical decomposition $\mathfrak{h}' = (\mathfrak{h} \cap \mathfrak{l}) \oplus (\mathfrak{h}^{\perp} \cap \mathfrak{l}^{\perp})$ of $H'/L \cap H$. Now $\mathfrak{h} \cap \mathfrak{l}^{\perp}$ and $\mathfrak{h}^{\perp} \cap \mathfrak{l}^{\perp}$ are the orthogonal complements to each other and both $\mathfrak{h} \cap \mathfrak{l}^{\perp}$ and $\mathfrak{h}^{\perp} \cap \mathfrak{l}^{\perp}$ are Lie triple systems. Thus $H/L \cap H$ is a reflective submanifold in U/L by Proposition 1.7. □

Remark 1.12. By Proposition 1.10 and Proposition 1.11 we can concretely determine any reflective submanifold and its orthogonal complement as symmetric pairs.

Lemma 1.13. *We use the same notation in Proposition* 1.10. *If a compact symmetric pair* $(H, H \cap L)$ *is not an effective compact symmetric pair, then there exists an almost effective compact symmetric pair* (G, K) *such that* $H/H \cap L$ *is isomorphic to* G/K.

Proof. By the assumption, there is a suitable normal subgroup H^N of H such that H^N is a subgroup of $H \cap L$. We put $G := H/H^N$ and $K := (H \cap L)/H^N$. Then the pair (G, K) is an almost effective compact symmetric pair. Thus $H/H \cap L$ is isomorphic to G/K as a symmetric space. □

2 Cohomogeneity One Actions on Riemannian Manifolds

In this section, we recall basic facts about cohomogeneity one actions on Riemannian manifolds and introduce some results concerning the classification of cohomogeneity one actions on Riemannian symmetric spaces of noncompact type ([1]).

Let M be a Riemannian manifold and let G be a Lie group acting smoothly on M by isometries. An orbit $G \cdot p$ is a *principal orbit* at $p \in M$ if for each $q \in M$, G_p is conjugate with a subgroup of G_q, where G_p is the isotropy group at p.

Each principal orbit is an orbit of maximal dimension. A non-principal orbit of maximal dimension is called an *exceptional orbit*. An orbit whose dimension is less than the dimension of a principal orbit is called a *singular orbit*. The *cohomogeneity* of the action is the codimension of a principal orbit. We denote this cohomogeneity by $\mathrm{coh}(G, M)$.

DEFINITION 2.1. An isometric action of a connected Lie group G on a Riemannian manifold M is a *cohomogeneity one action* if the $\mathrm{coh}(G, M)$ is equal to one.

Let $\mathfrak{g}^* = \mathfrak{k} \oplus \mathfrak{m}^*$ be the Cartan decomposition of Riemannian symmetric spaces $M^* = G^*/K$ of noncompact type. We identify $\mathfrak{m}^*$ with the tangent space $T_o M^*$ of M^* at some point $o^* \in M^*$. Let $(\mathfrak{g}^*)^{\mathbf{C}}$ be the complexification of $\mathfrak{g}^*$ and put $\mathfrak{g} := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}^*$. Then $\mathfrak{g}$ is a compact real form of $(\mathfrak{g}^*)^{\mathbf{C}}$. The simply connected Riemannian symmetric space $M = G/K$ associated with the pair $(\mathfrak{g}, \mathfrak{k})$ is called the *compact dual* of M^*, where G is the simply connected Lie group with the Lie algebra $\mathfrak{g}$.

This dual relation gives the following correspondence. There is a correspondence between the totally geodesic submanifolds of M and the totally geodesic submanifolds of M^*. Thus, the relation give rise to the following proposition.

PROPOSITION 2.2 ([1]). *Let N^* be a Riemannian symmetric space of non-compact type and let N be a its dual simply connected compact Riemannian symmetric space. Then, there is a one-to-one correspondence between the set of totally geodesic singular orbits of cohomogeneity one actions on N^* and the set of those on N.*

Also, they provided the following proposition.

PROPOSITION 2.3 ([1]). *Let M be a reflective submanifold of a connected Riemannian symmetric space N of noncompact type. Then M is a singular orbit of a cohomogeneity one action on N if and only if the rank of $M^{\perp}$ is one.*

By Proposition 2.2 and Proposition 2.3, we can see that M is a singular orbit of a cohomogeneity one action on N if and only if $M^{\perp}$ is a compact symmetric space of rank one. Then we obtain Table 1.

Table 1: Totally geodesic singular orbits of cohomogeneity one actions on simply connected irreducible compact symmetric spaces

N	Totally geodesic singular orbits with $\mathrm{coh}(G,N)=1$	Remark
$G_k^o(\mathbf{R}^n)$	$G_{k-1}^o(\mathbf{R}^{n-1}),\ G_k^o(\mathbf{R}^{n-1})$	1^*
$G_k^o(\mathbf{R}^{2k})$	$G_{k-1}^o(\mathbf{R}^{2k-1})=G_k^o(\mathbf{R}^{2k-1})$	$k\geq 4$
$G_2^o(\mathbf{R}^{2n})$	$S^{2n-2},\ G_2^o(\mathbf{R}^{2n-1}),\ \mathbf{C}P^{n-1}$	$n\geq 3$
$G_3^o(\mathbf{R}^6)=AI(4)$	$G_2^o(\mathbf{R}^5)=G_3^o(\mathbf{R}^5),\ S^1\cdot AI(3)$	
$G_k(\mathbf{C}^n)$	$G_{k-1}(\mathbf{C}^{n-1}),\ G_k(\mathbf{C}^{n-1})$	2^*
$G_k(\mathbf{C}^{2k})$	$G_{k-1}(\mathbf{C}^{2k-1})=G_k(\mathbf{C}^{2k-1})$	$k\geq 3$
$G_2(\mathbf{C}^{2n})$	$G_2(\mathbf{C}^{2n-1}),\ \mathbf{C}P^{2n-2},\ \mathbf{H}P^{n-1}$	$n\geq 3$
$G_k(\mathbf{H}^n)$	$G_{k-1}(\mathbf{H}^{n-1}),\ G_k(\mathbf{H}^{n-1})$	3^*
$G_k(\mathbf{H}^{2k})$	$G_{k-1}(\mathbf{H}^{2k-1})=G_k(\mathbf{H}^{2k-1})$	$k\geq 2$
$AI(n)$	$S^1\cdot AI(n-1)$	4^*
$AII(n)$	$S^1\cdot AII(n-1)$	$n\geq 4$
$AII(3)$	$S^1\cdot S^5,\ SU(3)$	
$DIII(n)$	$DIII(n-1)$	$n\geq 5$
$CI(n)$	$S^2\times CI(n-1)$	$n\geq 3$
$SU(n)$	$S(U(1)\times U(n-1))$	$n\geq 5$
$SU(4)$	$S(U(1)\times U(3)),\ Sp(2)$	
$SU(3)$	$S^1\cdot S^3,\ AI(3)$	
$Spin(n)$	$Spin(n-1)$	5^*
$Sp(n)$	$Sp(n-1)\times Sp(1)$	$n\geq 3$
EII	FI	
$EIII$	$\mathbf{O}P^2$	
EIV	$S^1\cdot S^9,\ AII(3)$	
FI	$G_4^o(\mathbf{R}^9)$	
F_4	$Spin(9)$	

1^*: $1<k<n-k$, $(k,n)\neq(2,2m)$, $m>2$, 2^*: $1<k<n-k$, $(k,n)\neq(2,2m)$, $m>2$, 3^*: $1<k<n-k$, 4^*: $n=3$ or $n\geq 5$ and 5^*: $n=5$ or $n\geq 7$.

3 On Stability of Totally Geodesic Submanifolds

In this section, we give a review of stability of totally geodesic submanifolds in compact symmetric spaces.

Definition 3.1. Let M be a compact totally geodesic submanifold immersed in a compact irreducible Riemannian symmetric space (N,h) and we denote the immersion by $f: M \to N$. Then f is stable if the second derivative of the volume $\mathrm{Vol}(M, f_t^*h)$ at $t = 0$ is non-negative for every smooth variation $\{f_t\}$ of f with $f_0 = f$.

The second variation formula of $\mathrm{Vol}(M, f_t^*h)$ is given as follows:

$$\frac{d^2}{dt^2}\,\mathrm{Vol}(M, f_t^*h)\bigg|_{t=0} = \int_M \langle J(V), V\rangle\, dv,$$

where dv denotes the Riemannian measure of (M, f^*h) and V is an element of $\Gamma(N(M))$, the space of smooth sections of the normal bundle of M. Here J is defined as

$$J = -\Delta^{\perp} - A_f + R_f,$$

where $\Delta^{\perp}$ is the rough Laplacian of $N(M)$, A_f and R_f are smooth sections of $\mathrm{End}(N(M))$ defined by $\langle A_f(u), v\rangle = \mathrm{Tr}_{f^*h}(A_u A_v)$ and $\langle R_f(u), v\rangle = \sum_{i=1}^{\dim M} \langle R^N(e_i, u)e_i, v\rangle$ for $u, v \in \Gamma(N(M))$, where we denote by $\{e_i\}$, A and R^N an orthonormal frame of tangent bundle $T(M)$, the shape operator of f and the curvature tensor of (N,h), respectively. J is a self-adjoint strongly elliptic linear differential operator and has discrete eigenvalues $\mu_1 < \mu_2 < \cdots < \infty$. We put $E_\mu = \{V \in \Gamma(N(M)) \mid J(V) = \mu V\}$, then $\dim E_\mu < \infty$.

Definition 3.2. The index of f is a number $\sum_{\mu<0} \dim E_\mu$, denoted by $\mathrm{index}(f)$. Clearly, f is stable if and only if $\mathrm{index}(f) = 0$.

We assume that $f: M = G/K \to N = U/L$ is a totally geodesic imbedding. We choose U so that G is a Lie subgroup of U. We denote the Lie algebra of G and U by $\mathfrak{g}$ and $\mathfrak{u}$ respectively. And let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathfrak{u} = \mathfrak{l} \oplus \mathfrak{p}$ be the canonical decompositions. We have the decomposition $\mathfrak{u} = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$ as a G-module as well as K-module decompositions $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{k}^{\perp}$ and $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{m}^{\perp}$, where $\mathfrak{m}$ (resp. $\mathfrak{m}^{\perp}$) is isomorphic with $T_o M$ (resp. $T_o^{\perp} M$) as a K-module. We decompose $\mathfrak{g}^{\perp}$ into the sum of simple G-modules $\mathfrak{g}_i^{\perp}$ and denote by μ and μ_i the corresponding representations of G $(1 \le i \le k)$. We have the decompositions $\mathfrak{g}_i^\perp = \mathfrak{k}_i^\perp \oplus \mathfrak{m}_i^\perp$ as K-modules where $\mathfrak{k}_i^\perp = \mathfrak{k}^\perp \cap \mathfrak{g}_i^\perp$ and $\mathfrak{m}_i^\perp = \mathfrak{m}^\perp \cap \mathfrak{g}_i^\perp$ for each i $(1 \le i \le k)$.

THEOREM 3.3 ([11]). *With the above notation, the index of f is given as follows*:

$$\text{(1)} \qquad \operatorname{index}(f) = \sum_{i=1}^{k} \sum_{\substack{\lambda \in D(G) \\ a_\lambda > a_i}} \dim \operatorname{Hom}_K(V_\lambda, (\mathfrak{m}_i^\perp)^{\mathbf{C}}) \dim V_\lambda,$$

where $D(G)$ denotes all the equivalence classes of complex irreducible representations of G and V_λ denotes the representation space of an element λ in $D(G)$ and a_λ denotes the eigenvalue of the Casimir operator of λ. Here a_i denotes the eigenvalue of the Casimir operator of μ_i. $\operatorname{Hom}_K(V_\lambda, (\mathfrak{m}_i^\perp)^{\mathbf{C}})$ *denotes the K-module homomorphisms from V_λ into the complexification $(\mathfrak{m}_i^\perp)^{\mathbf{C}}$ of $\mathfrak{m}_i^\perp$.*

Also we consider the case that N is a compact connected semisimple Lie group U with a bi-invariant Riemannian metric and M is a connected semisimple subgroup G. Applying Theorem 3.3 to this case, we have the following:

LEMMA 3.4. *Index(f) is given as follows*:

$$\text{(2)} \qquad \operatorname{index}(f) = \sum_{i=1}^{k} \sum_{\substack{\lambda, \mu \in D(G) \\ a_\lambda + a_\mu > a_i}} \dim \operatorname{Hom}_G(V_\lambda \otimes V_\mu, (\mathfrak{g}_i^\perp)^{\mathbf{C}}) \dim(V_\lambda \otimes V_\mu),$$

where we follow the notation in Theorem 3.3.

Now we apply (1) to inclusion maps $\iota : M \to N$ and $\iota^\perp : M^\perp \to N$ of a reflective submanifold $M = H/H \cap L$ and the orthogonal complement $M^\perp = H'/H \cap L$ in $N = U/L$. Here we can take $L = U^\tau$, $H = U^\sigma$ and $H' = U^{\tau\sigma}$ by Proposition 1.10 and Proposition 1.11. We fix a point o with $L(o) = o$ and assume that $o \in M$. Let $\mathfrak{h}$, $\mathfrak{h}'$ and $\mathfrak{u}$ be the Lie algebra of H, H' and U respectively. And let $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{l}) \oplus \mathfrak{h} \cap \mathfrak{l}^\perp$, $\mathfrak{h}' = (\mathfrak{h} \cap \mathfrak{l}) \oplus \mathfrak{h}^\perp \cap \mathfrak{l}^\perp$ and $\mathfrak{u} = \mathfrak{l} \oplus \mathfrak{p}$ be the canonical decompositions, where $\mathfrak{h} \cap \mathfrak{l}^\perp$ (resp. $\mathfrak{h}^\perp \cap \mathfrak{l}^\perp$) is isomorphic to T_oM (resp. $T_oM^\perp$) as a $(H \cap L)$-module. Since $\mathfrak{p} = (\mathfrak{h} \cap \mathfrak{l}^\perp) \oplus (\mathfrak{h}^\perp \cap \mathfrak{l}^\perp)$, we have $(\mathfrak{h} \cap \mathfrak{l}^\perp)^\perp = \mathfrak{h}^\perp \cap \mathfrak{l}^\perp$ and $(\mathfrak{h}^\perp \cap \mathfrak{l}^\perp)^\perp = \mathfrak{h} \cap \mathfrak{l}^\perp$. Here we put $\mathfrak{h} \cap \mathfrak{l}^\perp := \mathfrak{m}$ and $\mathfrak{h}^\perp \cap \mathfrak{l}^\perp := \mathfrak{m}^\perp$. Also we have the following decompositions: $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ and $\mathfrak{m}^\perp = \mathfrak{m}_1^\perp \oplus \cdots \oplus \mathfrak{m}_t^\perp$ as $(H \cap L)$-modules. The next lemma is an immediate consequence of Theorem 3.3.

LEMMA 3.5. *Index*(ι) *is given as follows*:

$$\text{index}(\iota) = \sum_{i=1}^{t} \sum_{\substack{\lambda \in D(H) \\ a_\lambda > a_i}} \dim \mathrm{Hom}_{H \cap L}(V_\lambda, (\mathfrak{m}_{\mathrm{i}}^{\perp})^{\mathbf{C}}) \dim V_\lambda, \tag{3}$$

where we follow the notation in Theorem 3.3.

Also we apply (2) to Lie group case: Let $\iota : G \to U$ and $\iota^\perp : G^\perp \to U$ be inclusion maps of a reflective submanifold $G = H^*/H$ and the orthogonal complement $G^\perp = H'/H$ in $U = U^*/U$, where H^* and U^* denote $H^* = H \times H$ and $U^* = U \times U$, respectively. Here we may take $H = U^\sigma$ and $H' = U^{\tau\sigma}$ by Proposition 1.10 and Proposition 1.11. The next lemma is an immediate consequence of Lemma 3.4 and Theorem 3.3.

LEMMA 3.6. *Index*(ι) *is given as follows*:

$$\text{index}(\iota) = \sum_{i=1}^{k} \sum_{\substack{\lambda, \mu \in D(H) \\ a_\lambda + a_\mu > a_i}} \dim \mathrm{Hom}_H(V_\lambda \otimes V_\mu, (\mathfrak{h}_{\mathrm{i}}^{\perp})^{\mathbf{C}}) \dim(V_\lambda \otimes V_\mu), \tag{4}$$

where we follow the notation in Theorem 3.3 *and* a_i *is the eigenvalue for the Casimir operator of each representation* $(\rho_i, \mathfrak{h}_{\mathrm{i}}^{\perp})$, *where* $\mathfrak{h}^\perp = \sum_{i=1}^{k} \mathfrak{h}_{\mathrm{i}}^{\perp}$ *is a simple* H*-module decomposition.*

4 Stability of Totally Geodesic Submanifolds with Cohomogeneity One Actions

In this section, we first introduce Freudenthal formula for complex irreducible representation of a Lie group.

THEOREM 4.1. *Let* (V, ρ) *be a complex irreducible representation of G. Then the eigenvalue* a_λ *of the Casimir operator* $\rho(C)$ *with respect to* $\langle\,,\rangle$ *is given by the following*:

$$a_\lambda = -\langle \lambda, \lambda + 2\delta(G) \rangle,$$

where $\delta(G)$ *is half the sum of positive roots of G and* $\langle\,,\rangle$ *is the canonical inner product on* $\mathfrak{g}$.

Next, by Theorem 4.1 we calculate the eigenvalue a_λ of the Casimir operator with respect to the canonical inner product on $\mathfrak{g}$.

NOTATION 4.2. We follow the notation in [2] concerning the numbering of the fundamental weights.

Type A_r: $\mathfrak{g} = \mathfrak{su}(r+1)$ $(r \geq 1)$,

$$a_{\varpi_i} = -\frac{i(r+2)(r+1-i)}{r+1},$$

where we here note that $a_{\varpi_1} = a_{\varpi_r} > a_{\varpi_2} = a_{\varpi_{r-1}} > \cdots > a_{\varpi_{[r/2]}}$.

Type B_r: $\mathfrak{g} = \mathfrak{so}(2r+1)$ $(r \geq 2)$,

$$a_{\varpi_i} = -i(2r-i+1), \quad (1 \leq i \leq r-1),$$

$$a_{\varpi_r} = -\frac{r(2r+1)}{4}$$

where we here note that $a_{\varpi_1} > a_{\varpi_2} > \cdots > a_{\varpi_{r-1}}$.

Type C_r: $\mathfrak{g} = \mathfrak{sp}(r)$ $(r \geq 3)$,

$$a_{\varpi_i} = -i(2r-i+2),$$

where we here note that $a_{\varpi_1} > a_{\varpi_2} > \cdots > a_{\varpi_r}$.

Type D_r: $\mathfrak{g} = \mathfrak{so}(2r)$ $(r \geq 4)$,

$$a_{\varpi_i} = -i(2r-i), \quad (1 \leq i \leq r-2),$$

$$a_{\varpi_{r-1}} = a_{\varpi_r} = -\frac{r(2r-1)}{4}$$

where we here note that $a_{\varpi_1} > a_{\varpi_2} > \cdots > a_{\varpi_{r-2}}$.

Type E_6: $\mathfrak{g} = \mathfrak{e}_6$,

$$a_{\varpi_1} = a_{\varpi_6} > a_{\varpi_2} > a_{\varpi_3} = a_{\varpi_5} > a_{\varpi_4}.$$

Type E_7: $\mathfrak{g} = \mathfrak{e}_7$,

$$a_{\varpi_7} > a_{\varpi_1} > a_{\varpi_2} > a_{\varpi_6} > a_{\varpi_3} > a_{\varpi_5} > a_{\varpi_4}.$$

Type E_8: $\mathfrak{g} = \mathfrak{e}_8$,

$$a_{\varpi_8} > a_{\varpi_1} > a_{\varpi_7} > a_{\varpi_2} > a_{\varpi_6} > a_{\varpi_3} > a_{\varpi_5} > a_{\varpi_4}.$$

Type F_4: $\mathfrak{g} = \mathfrak{f}_4$,

$$a_{\varpi_4} > a_{\varpi_1} > a_{\varpi_3} > a_{\varpi_2}.$$

Type G_2: $\mathfrak{g} = \mathfrak{g}_2$,

$$a_{\varpi_1} > a_{\varpi_2}.$$

Remark 4.3. We note that the absolute value of a_{ϖ_1} is the minimum among a_{ϖ_i} $(1 \leqq i \leqq r)$ for type A_r, B_r, C_r, D_r $(r \geqq 4)$ and G_2.

Let $\iota : M \to N$ be a totally geodesic imbedding. Now, we consider the following cases.

1. The case that N is not a Lie group.
2. The case that N is a Lie group.

Case 1. Let M be a reflective submanifold with a cohomogeneity one action on a compact irreducible simply connected symmetric space $N = U/L$. By Proposition 2.3, $M^{\perp}$ is a compact symmetric space of rank one. Here we may take $M = H/H \cap L$ and $M^{\perp} = H'/H \cap L$ by Proposition 1.9 and assume that $o \in M$. In order to study the stability of M in N, we use (3) in Section 3 in this case. Since (U, H) is a compact symmetric pair, the representation of H on $T_o U/H \cong \mathfrak{h}^{\perp}$ is equivalent to the isotropy representation of U/H. On the other hand, $M^{\perp}$ is a compact symmetric space of rank one and the representation of $H \cap L$ on $\mathfrak{m}^{\perp}$ is equivalent to the isotropy representation of $M^{\perp}$.

Theorem 4.4 *Under the assumption of the case* 1, *we assume that the Lie group H has a rank greater than four and that the restriction of the isotropy representation of U/H to G is equivalent to $\varpi_1(G)$, where G denotes some Lie subgroup of H which was shown in Lemma* 1.13. *Then the index of the inclusion map ι is equal to zero.*

Proof. By the assumption, $M = H/H \cap L$ is a totally geodesic singular orbit of a Hermann action of H. We will consider the following cases.

(i) U/H is a Hermitian symmetric space.
(ii) Both U/H and $M^{\perp}$ are quaternionic Kähler symmetric spaces.
(iii) U/H is a compact symmetric space except for (i) and (ii).

Case (i).

Since U/H is a Hermitian symmetric space, the center of H is one-dimensional. Thus, we denote the isotropy group H by $H = U(1) \cdot \hat{H}$. We may take G as the semisimple part $\hat{H}$ by Lemma 1.13. Also the Hermann action H on N gives rise to the following G-module decomposition:

$$\mathfrak{g}^{\perp} = \mathfrak{g}_1^{\perp} \oplus \mathfrak{g}_2^{\perp},$$

where $\mathfrak{g}^{\perp}$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{u}$, $\mathfrak{g}_1^{\perp} \cong \mathbf{R}$ and $\mathfrak{g}_2^{\perp} \cong T_oU/H$. By the assumption, T_oU/H is isomorphic to $V_{\varpi_1(G)}$ as a G-module. In order to count $\dim \mathrm{Hom}_K(V_\lambda, (\mathfrak{m}_i^{\perp})^{\mathbf{C}})$ we must find the representation λ which satisfies the inequality $a_\lambda > a_i$ $(i = 1, 2)$ for all $\lambda \in D(G)$. In this case, the representation which satisfies the above condition is a trivial representation because a_1 is equal to zero and the absolute value of $a_2 = a_{\varpi_1(G)}$ is less than or equal to $a_{\varpi_j(G)}$ $(j \geq 2)$ except for type F_4, E_7 and E_8. Also the orthogonal complement $M^{\perp}$ of M is a compact symmetric space of rank one. Thus the isotropy representation of $M^{\perp}$ is isomorphic to some isotropy representation in Table 3. Since $\mathfrak{m}_1^{\perp} = \{0\}$ and $\mathfrak{m}_2^{\perp} \cong T_oM^{\perp}$ as a K-module, we conclude the following:

$$\mathrm{index}(\imath) = \dim \mathrm{Hom}_K(\mathbf{C}, (\mathfrak{m}_2^{\perp})^{\mathbf{C}}).$$

Thus $\mathrm{index}(\imath) = 0$.

Case (ii).

Since U/H is a quaternionic Kähler symmetric space, the isotropy group H contains a simple normal subgroup isomorphic to $Sp(1)$. Thus, we denote the isotropy group H by $H = Sp(1) \cdot \hat{H}$. Similarly in case (i), we can take G as $\hat{H}$ by Lemma 1.13 and we have the following decomposition:

$$\mathfrak{g}^{\perp} = \mathfrak{g}_1^{\perp} \oplus \mathfrak{g}_2^{\perp} \oplus \mathfrak{g}_3^{\perp} \oplus \mathfrak{g}_4^{\perp} \oplus \mathfrak{g}_5^{\perp},$$

where $\mathfrak{g}^{\perp}$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{u}$, $\mathfrak{g}_i^{\perp} \cong \mathbf{R}$ $(i = 1, 2, 3)$ and $\mathfrak{g}_4^{\perp} \oplus \mathfrak{g}_5^{\perp} \cong T_oU/H$ and $\mathfrak{g}_4^{\perp} \cong \mathfrak{g}_5^{\perp}$. By the assumption, T_oU/H is isomorphic to $V_{\varpi_1(G)}$ as a G-module. In order to count $\dim \mathrm{Hom}_K(V_\lambda, (\mathfrak{m}_i^{\perp})^{\mathbf{C}})$ we must find the representation λ which satisfies the inequality $a_\lambda > a_i$ $(1 \leq i \leq 5)$ for all $\lambda \in D(G)$. In this case, the representation which satisfies the above condition is a trivial representation because $a_i (i = 1, 2, 3)$ is equal to zero and the absolute value of $a_i = a_{\varpi_1(G)}$ $(i = 4, 5)$ is less than or equal to $a_{\varpi_j(G)}$ $(j \geq 2)$ except for type F_4, E_7 and E_8. Also the orthogonal complement $M^{\perp}$ of M is a quaternionic projective space. Thus the isotropy representation of $M^{\perp} = \mathbf{H}P^n$ is isomorphic to the representation $\varpi_1(C_1) + \varpi_1(C_n)$ in Table 3. Since $\mathfrak{m}_i^{\perp} = \{0\} (i = 1, 2, 3)$ and $\mathfrak{m}_j^{\perp} \cong V_{\varpi_1(C_n)} (j = 4, 5)$ as a K-module, we conclude the following:

$$\mathrm{index}(\imath) = \dim \mathrm{Hom}_K(\mathbf{C}, (\mathfrak{m}_4^{\perp})^{\mathbf{C}}) + \dim \mathrm{Hom}_K(\mathbf{C}, (\mathfrak{m}_5^{\perp})^{\mathbf{C}}).$$

Thus $\mathrm{index}(\imath) = 0$.

Case (iii).

Since U/H is a compact symmetric space except for (i) and (ii), the isotropy group H is a simple Lie group. In this case, we can take $G = H$ by Lemma 1.13. Thus $\mathfrak{g}^{\perp}$ is isomorphic to $T_o U/H$ and is a simple G-module. By the assumption the representation G on $\mathfrak{g}^{\perp}$ is equivalent to $\varpi_1(G)$. In order to count $\dim \mathrm{Hom}_K(V_\lambda, (\mathfrak{m}^{\perp})^{\mathbf{C}})$ we must find the representation λ which satisfies the inequality $a_\lambda > a$ for all $\lambda \in D(G)$. In this case, the representation is a trivial representation because the absolute value of $a = a_{\varpi_1(G)}$ is less than or equal to $a_{\varpi_j(G)}$ $(j \geq 2)$ except for type F_4, E_7 and E_8. Also $\mathfrak{m}^{\perp}$ is a simple K-module, therefore we have the following:

$$\mathrm{index}(\imath) = \dim \mathrm{Hom}_K(\mathbf{C}, (\mathfrak{m}^{\perp})^{\mathbf{C}}).$$

Thus $\mathrm{index}(\imath) = 0$. □

Case 2. Let G be the connected component of the fixed-point set of some involutive automorphism of U and let G be a singular orbit of a cohomogeneity one action on a compact simply connected Lie group $U = U^*/U$, where U^* denotes $U \times U$. By Proposition 2.3, $G^{\perp}$ is a compact symmetric space of rank one. Here we may take $G = H^*/H$ and $G^{\perp} = H'/H$ by Proposition 1.9. In order to study the stability of G in U, we use (4) in Section 3 in this case. Since (U^*, H^*) is a compact symmetric pair, the representation of H^* on $T_o U^*/H^* \cong \mathfrak{h}^{\perp} \oplus \mathfrak{h}^{\perp}$ is equivalent to the isotropy representation of $U/H \times U/H$. On the other hand, $G^{\perp}$ is a compact symmetric space of rank one and the representation of H on $\mathfrak{h}^{\perp}$ is equivalent to the isotropy representation of a compact symmetric space of rank one.

Corollary 4.5. *Under the assumption of the case* 2, *we assume that the Lie group H has a rank greater than four and that the restriction of the isotropy representation of U/H to G is equivalent to $\varpi_1(G)$, where G denotes some Lie subgroup of H which was shown in Lemma* 1.13. *Then the index of the inclusion map $\imath : G \to U$ is equal to zero.*

Proof. We note that each irreducible part of the representation of H^* on $T_o U^*/H^*$ is isomorphic to the representation of H on $T_o U/H$. Thus we conclude that this case is similar to the case 1. □

Now, we check whether each case in Table 2 satisfies the assumption in Theorem 4.4 (or Corollary 4.5) or not. The following cases satisfy the assumption: (1), ..., (5), (7), ..., (13), (15), ... (20), (22), ... (27), (29), (30). These cases are stable. Among these cases there are some exceptions which do not satisfy the assumption of rank. We examine their stability case by case.

Under the low rank assumptions, we determine the stability of $M = G/K$ in $N = U/L$. That is to say $1 \leq \mathrm{rank}(G) \leq 4$. In Table 2 the cases of (1), (2), (3), (4), (5) and (29) satisfy this condition.

Case (2).

For an inclusion map $\iota : G_k^o(\mathbf{R}^{n-1}) \to G_k^o(\mathbf{R}^n)$, we discuss $6 \leqq n \leqq 9$.

When $n = 6$, we consider the following cases:

(i) $\iota : G_3^o(\mathbf{R}^5) \to G_3^o(\mathbf{R}^6)$
(ii) $\iota : G_4^o(\mathbf{R}^5) \to G_4^o(\mathbf{R}^6)$

Case (i).

Since $G_3^o(\mathbf{R}^5)$ is a reflective submanifold in $G_3^o(\mathbf{R}^6)$, $G_3^o(\mathbf{R}^5)$ is a totally geodesic orbit of Hermann action of $H = SO(5)$ by Proposition 1.9. The Hermann action gives rise to a homomorphism $\rho : \mathfrak{g} \to \mathfrak{u}$, where $\mathfrak{g} = \mathfrak{so}(5)$, $\mathfrak{u} = \mathfrak{so}(6)$. We have $\mathfrak{u} = \rho(\mathfrak{g}) \oplus \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ is the orthogonal complement of $\rho(\mathfrak{g})$ in $\mathfrak{u}$. $\mathfrak{g}^\perp$ is a $\mathfrak{g}$-module, $\mathfrak{g}^\perp \cong T_o S^5 \cong V_{\varpi_1(SO(5))}$. Thus we have the index(ι):

$$\mathrm{index}(\iota) = \sum_{\lambda \in \{0, \varpi_2(B_2)\}} \dim \mathrm{Hom}_{SO(3)\times SO(2)}(V_\lambda, (\mathfrak{m}^\perp)^{\mathbf{C}}) \dim V_\lambda.$$

Since $\mathfrak{m}^\perp$ is isomorphic to $T_o S^3$, we obtain $(\mathfrak{m}^\perp)^{\mathbf{C}} \cong V_{\varpi_1(SO(3))}$ as a $SO(3)$-module. Also we have the following decomposition as a $SO(3)$-module: $V_{\varpi_2(B_2)} \cong V_{\varpi_1(C_2)} = V_{3\varpi_1(A_1)}$. Therefore $\mathrm{index}(\iota) = 0$.

Case (ii).

Because $G_4^o(\mathbf{R}^5) \cong S^4$ is a symmetric R-space of $G_4^o(\mathbf{R}^6) \cong G_2^o(\mathbf{R}^6)$, it is stable ([12]).

When $n = 7$, we consider the following cases:

(i) $\iota : G_3^o(\mathbf{R}^6) \to G_3^o(\mathbf{R}^7)$
(ii) $\iota : G_4^o(\mathbf{R}^6) \to G_4^o(\mathbf{R}^7)$
(iii) $\iota : G_5^o(\mathbf{R}^6) \to G_5^o(\mathbf{R}^7)$

Case (i).

Since $G_3^o(\mathbf{R}^6)$ is a reflective submanifold in $G_3^o(\mathbf{R}^7)$, $G_3^o(\mathbf{R}^6)$ is a totally geodesic orbit of Hermann action of $H = SO(6)$ by Proposition 1.9. The Hermann action gives rise to a homomorphism $\rho : \mathfrak{g} \to \mathfrak{u}$, where $\mathfrak{g} = \mathfrak{so}(6)$, $\mathfrak{u} = \mathfrak{so}(7)$. We have $\mathfrak{u} = \rho(\mathfrak{g}) \oplus \mathfrak{g}^{\perp}$, where $\mathfrak{g}^{\perp}$ is the orthogonal complement of $\rho(\mathfrak{g})$ in $\mathfrak{u}$. $\mathfrak{g}^{\perp}$ is a $\mathfrak{g}$-module, $\mathfrak{g}^{\perp} \cong T_oS^6 \cong V_{\varpi_1(SO(6))}$. Thus we have the index($\imath$):

$$\mathrm{index}(\imath) = \sum_{\lambda \in \{0, \varpi_2(D_3)\}} \dim \mathrm{Hom}_{SO(3)\times SO(3)}(V_\lambda, (\mathfrak{m}^{\perp})^{\mathbf{C}}) \dim V_\lambda.$$

Since $\mathfrak{m}^{\perp}$ is isomorphic to T_oS^3, we obtain $(\mathfrak{m}^{\perp})^{\mathbf{C}} \cong V_{\varpi_1(SO(3))}$. Also we have the following decomposition as a K-module: $V_{\varpi_2(D_3)} \cong V_{\varpi_3(A_3)} = V_{\varpi_1(A_1)} \otimes V_{\varpi_1(A_1)}$. Therefore $\mathrm{index}(\imath) = 0$.

Case (ii).

Since $G_4^o(\mathbf{R}^7)$ is a quaternionic Kähler manifold and $G_4^o(\mathbf{R}^6)$ is a quaternionic Kähler submanifold of $G_4^o(\mathbf{R}^7)$, $G_4^o(\mathbf{R}^6)$ is stable ([14]).

Case (iii).

Because $G_5^o(\mathbf{R}^6) \cong S^5$ is a symmetric R-space of $G_5^o(\mathbf{R}^7) \cong G_2^o(\mathbf{R}^7)$, it is stable ([12]).

When $n = 8$, we consider the following cases:

(i) $\imath : G_3^o(\mathbf{R}^7) \to G_3^o(\mathbf{R}^8)$
(ii) $\imath : G_4^o(\mathbf{R}^7) \to G_4^o(\mathbf{R}^8)$
(iii) $\imath : G_5^o(\mathbf{R}^7) \to G_5^o(\mathbf{R}^8)$
(iv) $\imath : G_6^o(\mathbf{R}^7) \to G_6^o(\mathbf{R}^8)$

Case (i).

Since $G_3^o(\mathbf{R}^7)$ is a reflective submanifold in $G_3^o(\mathbf{R}^8)$, $G_3^o(\mathbf{R}^7)$ is a totally geodesic orbit of Hermann action of $H = SO(7)$ by Proposition 1.9. The Hermann action gives rise to a homomorphism $\rho : \mathfrak{g} \to \mathfrak{u}$, where $\mathfrak{g} = \mathfrak{so}(7)$, $\mathfrak{u} = \mathfrak{so}(8)$. We have $\mathfrak{u} = \rho(\mathfrak{g}) \oplus \mathfrak{g}^{\perp}$, where $\mathfrak{g}^{\perp}$ is the orthogonal complement of $\rho(\mathfrak{g})$ in $\mathfrak{u}$. $\mathfrak{g}^{\perp}$ is a $\mathfrak{g}$-module, $\mathfrak{g}^{\perp} \cong T_oS^7 \cong V_{\varpi_1(SO(7))}$. Thus we have the index($\imath$):

$$\mathrm{index}(\imath) = \sum_{\lambda \in \{0, \varpi_3(B_3)\}} \dim \mathrm{Hom}_{SO(3)\times SO(4)}(V_\lambda, (\mathfrak{m}^{\perp})^{\mathbf{C}}) \dim V_\lambda.$$

Because G is isomorphic to $SO(7)$, the spin representation $\varpi_3(B_3)$ is not a representation of $SO(7)$. Since $\mathfrak{m}^{\perp}$ is isomorphic with T_oS^3, we obtain $(\mathfrak{m}^{\perp})^{\mathbf{C}} \cong V_{\varpi_1(SO(3))}$. We conclude the index($\imath$):

$$\mathrm{index}(\imath) = \dim \mathrm{Hom}_{SO(3)\times SO(4)}(\mathbf{C}, (\mathfrak{m}^{\perp})^{\mathbf{C}}).$$

Therefore $\mathrm{index}(\imath) = 0$.

Case (ii).

Since $G_4^o(\mathbf{R}^8)$ is a quaternionic Kähler manifold and $G_4^o(\mathbf{R}^7)$ is a quaternionic Kähler submanifold of $G_4^o(\mathbf{R}^8)$, $G_4^o(\mathbf{R}^7)$ is stable ([14]).

Case (iii).

The case is similar to the case (i). Thus $G_5^o(\mathbf{R}^7)$ is stable in $G_5^o(\mathbf{R}^8)$.

Case (iv).

Because $G_6^o(\mathbf{R}^7) \cong S^6$ is a symmetric R-space of $G_6^o(\mathbf{R}^8) \cong G_2^o(\mathbf{R}^8)$, it is stable ([12]).

When $n = 9$, clearly the inclusion map $\iota : G_k^o(\mathbf{R}^8) \to G_k^o(\mathbf{R}^9)$ $(3 \leqq k \leqq 6)$ is stable.

Case (1).

We can conclude that these cases are stable similarly to the case (2).

Case (3).

This case is a special case of (1). Therefore this case is stable.

Case (4), Case (5).

Both the case (4) and the case (5) are cases of a complex submanifold in a Kähler manifold. Thus these cases are stable.

Case (29).

For an inclusion map $\iota : Spin(n-1) \to Spin(n)$, we discuss $n = 5, 7, 8, 9$.

When $n = 5$, we consider the inclusion map $\iota : Spin(4) \to Spin(5)$. Since $Spin(5)$ is isomorphic to $Sp(2)$ and $Spin(4)$ is isomorphic to $S^3 \times S^3$, it is stable ([13]).

When $n = 7$, we consider the inclusion map $\iota : Spin(6) \to Spin(7)$.

We here note that $Spin(6)$ is isomorphic to $SU(4)$. Since $Spin(6)$ is a reflective submanifold in $Spin(7)$, $Spin(6)$ is a totally geodesic orbit of Hermann action of $H^* = Spin(6) \times Spin(6)$ by Proposition 1.9. The Hermann action gives rise to a homomorphism $\rho : \mathfrak{h}^* \to \mathfrak{u}^*$, where $\mathfrak{h}^* = \mathfrak{so}(6) \oplus \mathfrak{so}(6)$, $\mathfrak{u}^* = \mathfrak{so}(7) \oplus \mathfrak{so}(7)$. We have $\mathfrak{u}^* = \rho(\mathfrak{h}^*) \oplus (\mathfrak{h}^*)^\perp$, where $(\mathfrak{h}^*)^\perp$ is the orthogonal complement of $\rho(\mathfrak{h}^*)$ in $\mathfrak{u}^*$ and $(\mathfrak{h}^*)^\perp = \mathfrak{h}^\perp \oplus \mathfrak{h}^\perp$. Also $\mathfrak{h}^\perp$ is a simple $\mathfrak{h}$-module and $\mathfrak{h}^\perp \cong T_o S^6 \cong V_{\varpi_1(SO(6))} \cong V_{\varpi_2(SU(4))}$. We conclude the index$(\iota)$:

$$\mathrm{index}(\iota) = \sum_{\lambda \in \{0, 0+\varpi_1(A_3), \varpi_1(A_3)+0\}} \dim \mathrm{Hom}_{SU(4)}(V_\lambda, (\mathfrak{h}^\perp)^{\mathbf{C}}) \dim V_\lambda.$$

Since $\mathfrak{h}^\perp$ is isomorphic to T_oS^6, we obtain $(\mathfrak{h}^\perp)^{\mathbf{C}} \cong V_{\varpi_2(SU(4))}$ as a $SU(4)$-module. As for $\lambda = 0 + \varpi_1(A_3)$ and $\lambda' = \varpi_1(A_3) + 0$, we have $V_\lambda = V_{\varpi_1(A_3)}$ and $V_{\lambda'} = V_{\varpi_1(A_3)}$ as a $SU(4)$-module. Therefore $\mathrm{index}(\iota) = 0$.

When $n = 8$, we consider the inclusion map $\iota : Spin(7) \to Spin(8)$.

Since $Spin(7)$ is a reflective submanifold in $Spin(8)$, $Spin(7)$ is a totally geodesic orbit of Hermann action of $H^* = Spin(7) \times Spin(7)$ by Proposition 1.9. The Hermann action gives rise to a homomorphism $\rho : \mathfrak{h}^* \to \mathfrak{u}^*$, where $\mathfrak{h}^* = \mathfrak{so}(7) \oplus \mathfrak{so}(7)$, $\mathfrak{u}^* = \mathfrak{so}(8) \oplus \mathfrak{so}(8)$. We have $\mathfrak{u}^* = \rho(\mathfrak{h}^*) \oplus (\mathfrak{h}^*)^\perp$, where $(\mathfrak{h}^*)^\perp$ is the orthogonal complement of $\rho(\mathfrak{h}^*)$ in $\mathfrak{u}^*$ and $(\mathfrak{h}^*)^\perp = \mathfrak{h}^\perp \oplus \mathfrak{h}^\perp$. Also $\mathfrak{h}^\perp$ is a simple $\mathfrak{h}$-module and $\mathfrak{h}^\perp \cong T_oS^7 \cong V_{\varpi_1(SO(7))}$. Thus we have the index(ι):

$$\mathrm{index}(\iota) = \sum_{\lambda \in \{0, 0+\varpi_3(SO(7)), \varpi_3(SO(7))+0\}} \dim \mathrm{Hom}_{SO(7)}(V_\lambda, (\mathfrak{h}^\perp)^{\mathbf{C}}) \dim V_\lambda.$$

Since $\mathfrak{h}^\perp$ is isomorphic to T_oS^7, we obtain $(\mathfrak{h}^\perp)^{\mathbf{C}} \cong V_{\varpi_1(SO(7))}$ as a $SO(7)$-module. As for $\lambda = 0 + \varpi_3(SO(7))$ and $\lambda' = \varpi_3(SO(7)) + 0$, we have $V_\lambda = V_{\varpi_3(SO(7))}$ and $V_{\lambda'} = V_{\varpi_3(SO(7))}$ as a $SO(7)$-module. Therefore $\mathrm{index}(\iota) = 0$.

When $n = 9$, clearly the inclusion map $\iota : Spin(8) \to Spin(9)$ is stable.

Now we obtain the following theorem.

Theorem 4.6. *Under the assumption of case* 1 *and* 2, *we assume that the restriction of the isotropy representation of U/H to G is equivalent to $\varpi_1(G)$, where G denotes some Lie subgroup of H which was shown in Lemma* 1.13. *Then the index of the inclusion map $\iota : M \to N$ is equal to zero.*

Also we examine the stability of cases in Table 2 which do not satisfy the assumption in Theorem 4.4 (or Corollary 4.5).

Case (6).

Because $G_2^o(\mathbf{R}^{2n})$ is a Hermitian symmetric space, we can conclude that $\mathbf{C}P^{n-1}$ is stable.

Case (14).

Because $G_2(\mathbf{C}^{2n})$ is a quaternionic Kähler symmetric space, we can conclude that $\mathbf{H}P^{n-1}$ is stable by [14].

Case (21).

In this case, it is unstable [5].

Case (28).

Because an inclusion map $f : AI(3) \to SU(3)$ is the Cartan embedding, it is unstable by [8].

Case (31).

Since *EII* is a quaternionic Kähler manifold and *FI* is a quaternionic Kähler submanifold of *EII*, *FI* is stable ([14]).

Case (32).

Because $\mathbf{O}P^2$ is a symmetric *R*-space of *EIII*, it is stable ([12]).

Case (33).

$S^1 \cdot S^9$ is the meridian of *EIII* ([9]). Thus $S^1 \cdot S^9$ is stable ([13]).

Case (34).

In this case, it is unstable [5].

Case (35).

Since *FI* is a quaternionic Kähler manifold and $G_4^o(\mathbf{R}^9)$ is a quaternionic Kähler submanifold of *FI*, $G_4^o(\mathbf{R}^9)$ is stable ([14]).

Case (36).

Because *Spin*(9) is a Lie subgroup of Dynkin index 1 in F_4, *Spin*(9) is stable by [7].

Now we obtain the following theorem.

Theorem 4.7. *All of the stability of totally geodesic singular orbits which are obtained by the cohomogeneity one actions on compact simply connected irreducible symmetric spaces are given in Table* 2. *The cases whose numbers are attached the symbol* * *are unstable and the other cases are stable.*

Table 2: A totally geodesic singular orbit $M = G/K$ of a cohomogeneity one action on a simply connected irreducible compact symmetric space $N = U/L$ associated with a Hermann action H and the orthogonal complement $M^{\perp}$ and the isotropy representation of U/H

	N	M	$M^{\perp}$	U/H	isotropy representation of U/H
(1)	$G_k^o(\mathbf{R}^n)$	$G_{k-1}^o(\mathbf{R}^{n-1})$	S^{n-k}	S^{n-1}	$\varpi_1(SO(n-1))$
(2)	$G_k^o(\mathbf{R}^n)$	$G_k^o(\mathbf{R}^{n-1})$	S^k	S^{n-1}	$\varpi_1(SO(n-1))$
(3)	$G_k^o(\mathbf{R}^{2k})$	$G_{k-1}^o(\mathbf{R}^{2k-1})$	S^k	S^{2k}	$\varpi_1(SO(2k))$
(4)	$G_2^o(\mathbf{R}^{2n})$	S^{2n-2}	S^{2n-2}	S^{2n-1}	$\varpi_1(SO(2n-1))$
(5)	$G_2^o(\mathbf{R}^{2n})$	$G_2^o(\mathbf{R}^{2n-1})$	S^2	S^{2n-1}	$\varpi_1(SO(2n-1))$
(6)	$G_2^o(\mathbf{R}^{2n})$	$\mathbf{C}P^{n-1}$	$\mathbf{C}P^{n-1}$	$DIII(n)$	$\varpi_2(A_{n-1})$
(7)	$G_3^o(\mathbf{R}^6)$	$G_3^o(\mathbf{R}^5)$	S^3	S^5	$\varpi_1(SO(5))$
(8)	$G_3^o(\mathbf{R}^6)$	$S^1 \cdot AI(3)$	S^3	S^5	$\varpi_1(SO(5))$
(9)	$G_k(\mathbf{C}^n)$	$G_{k-1}(\mathbf{C}^{n-1})$	$\mathbf{C}P^{n-k}$	$\mathbf{C}P^{n-1}$	$T + \varpi_1(A_{n-2})$
(10)	$G_k(\mathbf{C}^n)$	$G_k(\mathbf{C}^{n-1})$	$\mathbf{C}P^{2k}$	$\mathbf{C}P^{n-1}$	$T + \varpi_1(A_{n-2})$
(11)	$G_k(\mathbf{C}^{2k})$	$G_k(\mathbf{C}^{2k-1})$	$\mathbf{C}P^k$	$\mathbf{C}P^{2k-1}$	$T + \varpi_1(A_{2k-2})$
(12)	$G_2(\mathbf{C}^{2n})$	$G_2(\mathbf{C}^{2n-1})$	$\mathbf{C}P^2$	$\mathbf{C}P^{2n-1}$	$T + \varpi_1(A_{2n-2})$
(13)	$G_2(\mathbf{C}^{2n})$	$\mathbf{C}P^{2n-2}$	$\mathbf{C}P^{2n-2}$	$\mathbf{C}P^{2n-1}$	$T + \varpi_1(A_{2n-2})$
(14)	$G_2(\mathbf{C}^{2n})$	$\mathbf{H}P^{n-1}$	$\mathbf{H}P^{n-1}$	$AII(n)$	$\varpi_2(C_n)$
(15)	$G_k(\mathbf{H}^n)$	$G_{k-1}(\mathbf{H}^{n-1})$	$\mathbf{H}P^{n-k}$	$\mathbf{H}P^{n-1}$	$\varpi_1(C_1) + \varpi_1(C_{n-1})$
(16)	$G_k(\mathbf{H}^n)$	$G_k(\mathbf{H}^{n-1})$	$\mathbf{H}P^k$	$\mathbf{H}P^{n-1}$	$\varpi_1(C_1) + \varpi_1(C_{n-1})$
(17)	$G_k(\mathbf{H}^{2k})$	$G_k(\mathbf{H}^{2k-1})$	$\mathbf{H}P^k$	$\mathbf{H}P^{2k-1}$	$\varpi_1(C_1) + \varpi_1(C_{2k-1})$
(18)	$AI(n)$	$S^1 \cdot AI(n-1)$	$\mathbf{R}P^{n-1}$	$\mathbf{C}P^{n-1}$	$T + \varpi_1(A_{n-2})$
(19)	$AII(n)$	$S^1 \cdot AII(n-1)$	$\mathbf{H}P^{n-1}$	$G_2(\mathbf{C}^{2n})$	$T + \varpi_1(A_1) + \varpi_1(A_{2n-3})$
(20)	$AII(3)$	$S^1 \cdot S^5$	$\mathbf{H}P^2$	$G_2(\mathbf{C}^6)$	$T + \varpi_1(A_1) + \varpi_1(A_3)$
(21)*	$AII(3)$	$SU(3)$	$\mathbf{C}P^3$	$G_3(\mathbf{C}^6)$	$T + \varpi_1(A_2) + \varpi_1(A_2)$
(22)	$DIII(n)$	$DIII(n-1)$	$\mathbf{C}P^{n-1}$	$G_2^o(\mathbf{R}^{2n})$	$\varpi_1(SO(2)) + \varpi_1(SO(2n-2))$
(23)	$CI(n)$	$S^2 \times CI(n-1)$	$\mathbf{C}P^{n-1}$	$\mathbf{H}P^{n-1}$	$\varpi_1(C_1) + \varpi_1(C_{n-1})$
(24)	$SU(n)$	$S(U(1) \times U(n-1))$	$\mathbf{C}P^{n-1}$	$\mathbf{C}P^{n-1}$	$T + \varpi_1(A_{n-2})$
(25)	$SU(4)$	$S(U(1) \times U(3))$	$\mathbf{C}P^3$	$\mathbf{C}P^3$	$T + \varpi_1(A_2)$
(26)	$SU(4)$	$Sp(2)$	S^5	S^5	$\varpi_1(SO(5))$

Table 2: Continued

	N	M	$M^{\perp}$	U/H	isotropy representation of U/H
(27)	$SU(3)$	$S^1 \cdot S^3$	$\mathbf{C}P^2$	$\mathbf{C}P^2$	$T + \varpi_1(A_1)$
(28)*	$SU(3)$	$AI(3)$	$\mathbf{R}P^3$	$SU(3)$	$(\varpi_1 + \varpi_2)(A_2)$
(29)	$Spin(n)$	$Spin(n-1)$	S^{n-1}	S^{n-1}	$\varpi_1(SO(n-1))$
(30)	$Sp(n)$	$Sp(n-1) \times Sp(1)$	$\mathbf{H}P^{n-1}$	$\mathbf{H}P^{n-1}$	$\varpi_1(C_1) + \varpi_1(C_{n-1})$
(31)	EII	FI	$\mathbf{H}P^3$	EIV	$\varpi_4(F_4)$
(32)	$EIII$	$\mathbf{O}P^2$	$\mathbf{O}P^2$	EIV	$\varpi_4(F_4)$
(33)	EIV	$S^1 \cdot S^9$	$\mathbf{O}P^2$	$EIII$	$T + \varpi_5(D_5)$
(34)*	EIV	$AII(3)$	$\mathbf{H}P^3$	EII	$\varpi_1(A_1) + \varpi_3(A_5)$
(35)	FI	$G_4^o(\mathbf{R}^9)$	$\mathbf{H}P^2$	$\mathbf{O}P^2$	$\varpi_4(B_4)$
(36)	F_4	$Spin(9)$	$\mathbf{O}P^2$	$\mathbf{O}P^2$	$\varpi_4(B_4)$

Table 3: The isotropy representations of compact symmetric spaces of rank one

M	isotropy representation
$S^n \quad (n \geq 2)$	$\varpi_1(SO(n))$
$\mathbf{R}P^n \quad (n \geq 2)$	$\varpi_1(SO(n))$
$\mathbf{C}P^n \quad (n \geq 2)$	$T + \varpi_1(A_{n-1})$
$\mathbf{H}P^n \quad (n \geq 2)$	$\varpi_1(C_1) + \varpi_1(C_n)$
$\mathbf{O}P^2$	$\varpi_4(B_4)$

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