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A CHARACTERIZATION OF TILING GROUPS

By

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Abstract. For one dimensional tilings, we can define associated groups. And it is known that those groups has Gauss decomposition. We will show one characterization of those groups.

1 Introduction

In this paper, we will consider an algebraic characterization of the group G defined by a tiling $\mathcal{T}$. It is known that we can construct groups and Lie algebras with Gauss decompositions from tilings (cf. [5]). We will treat one-dimensional tilings in the present paper. We regard one-dimensional tilings as sequence of letters. We will define finite subwords of a tiling (*Section* 2). And we will introduce a new tiling generated $\mathcal{T}^*$ by a given tiling $\mathcal{T}$. Then one can basically treat any kind of tiling by this process. In addition, by our definition of $\mathcal{T}^*$, we can keep all information of an original tiling $\mathcal{T}$ (*Section* 3). Then we will construct tiling monoids (*Section* 4), tiling bialgebras (*Section* 5), tiling groups (*Section* 6). Then we will define an abstract group $\tilde{G}$ satisfying three relations, and show that $\tilde{G}$ has a Gauss decomposition (*Section* 7). And we will get one characterization of G (*Section* 8).

2 Tiling

First we define finite subwords of the tiling. Let $\mathbf{R}$ be the real line. A tile in $\mathbf{R}$ is a connected closed bounded subset of $\mathbf{R}$, namely a closed interval $[a, b]$ whose interior is nonempty. A tiling $\mathcal{T}$ of $\mathbf{R}$ is an infinite set of tiles which covers $\mathbf{R}$ overlapping, at most, at their boundaries. Let $W(\mathcal{T})$ be the set of all finite subwords in $\mathcal{T}$. If $w = X_1X_2\cdots X_r \in W(\mathcal{T})$, then $l(w) = r$ is called the length of w. Let $W_r(\mathcal{T})$ be the set of all finite subwords with length r. Put $\Omega = \Omega(\mathcal{T}) = W_1(\mathcal{T})$, the set of all letters appearing in $\mathcal{T}$. For convenience, we assume that Ω is finite.

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3 Division into 3 Parts

Now we introduce a certain substitution which divides a tile into 3 parts. For a given tiling $\mathscr{T}$, we define a substitution σ as follows

$$\sigma : X \to X'X''X''' \quad ({}^{\forall}X \in \Omega).$$

Here the letters X', X'' and X''' are totally new symbols. That is, the tiling $\mathscr{T}$:

$$\cdots XYZ \cdots$$

is changed into

$$\cdots X'X''X'''Y'Y''Y'''Z'Z''Z''' \cdots$$

by σ. And a finite subword

$$w = X_1X_2 \cdots X_r \in W$$

is changed into

$$\sigma(w) = X_1'X_1''X_1'''X_2'X_2''X_2''' \cdots X_r'X_r''X_r'''.$$

Hence, the substitution σ creates a new tiling $\mathscr{T}^*$ from $\mathscr{T}$. By the definition, $|\Omega(\mathscr{T}^*)| = 3 \times |\Omega(\mathscr{T})|$. That is, $\Omega(\mathscr{T}^*) = \{X', X'', X''' \mid X \in \Omega(\mathscr{T})\}$ without any redundancy. And put $V^* = \sigma(W(\mathscr{T})) = \{\sigma(w) \mid w \in W(\mathscr{T})\} \subset W(\mathscr{T}^*)$. Then we can express $v \in V^*$ as follows

$$v = X_1'X_1''X_1'''X_2'X_2''X_2''' \cdots X_r'X_r''X_r'''.$$

4 Tiling Monoids

For $w = X_1X_2 \cdots X_r \in V^*$, we choose two positions (i, j) with $1 \le i, j \le r$ and attach the labels 1 and 2 at X_i and X_j as $\overset{1}{X_i}$ and $\overset{2}{X_j}$ respectively. We note that each of $i \le j$, $i = j$, $i \ge j$ is allowed. If $i = j$, then we denote by $\overset{12}{X_i}$ to show that X_i has two labels 1 and 2 simultaneously. We call

$$X_1X_2 \cdots \overset{1}{X_i} \cdots \overset{2}{X_j} \cdots X_r$$

a doubly pointed words obtained from V^*. And we write this double pointed words as $w(i, j)$ if necessary. Then $D = D(\mathscr{T}^*)$ denotes the set of all doubly pointed words obtained from V^*. Let $M = M(\mathscr{T}^*) = D(\mathscr{T}^*) \cup \{\mathbf{z}, \boldsymbol{\varepsilon}\}$, where $\mathbf{z}$ and $\boldsymbol{\varepsilon}$ are just independent abstract symbols. Now we will introduce a binary operation on M. Let

$$\mathbf{x} = X_1 X_2 \cdots \overset{1}{X_i} \cdots \overset{2}{X_j} \cdots X_r$$

$$\mathbf{y} = Y_1 Y_2 \cdots \overset{1}{Y_k} \cdots \overset{2}{Y_l} \cdots Y_s$$

be two elements of $D(\mathscr{T}^*)$. Put $a = \min\{j,k\}$, $b = \min\{r-j, s-k\}$, $m = \max\{j,k\} - \min\{j,k\}$, $n = \max\{r-j, s-k\} - \min\{r-j, s-k\}$, and set

$$q = a + b = \frac{(r+s)-(m+n)}{2}.$$

If

$$(*)\begin{cases} X_{j-a+1} & = & Y_{k-a+1} \\ \vdots & \vdots & \vdots \\ X_j & = & Y_k \\ \vdots & \vdots & \vdots \\ X_{j+b} & = & Y_{k+b} \end{cases}$$

then we define a new word

$$Z_1 \cdots Z_m Z_{m+1} \cdots Z_{m+q} Z_{m+q+1} \cdots Z_{m+q+n},$$

where

$$\begin{cases} Z_p & (1 \le p \le m) = \begin{cases} X_p & \text{if } j > k \\ Y_p & \text{if } j < k \end{cases} \\ Z_{m+p} & (1 \le p \le q) \ = X_{j-a+p} (= Y_{k-a+p}) \\ Z_{m+p+q} & (1 \le p \le n) \ = \begin{cases} X_{j+b+p} & \text{if } r-j > s-k \\ Y_{k+b+p} & \text{if } r-j < s-k \end{cases}. \end{cases}$$

Put

$$i' = \begin{cases} i & \text{if } j \ge k \\ m+i & \text{if } j < k \end{cases}, \quad j' = \begin{cases} m+l & \text{if } j > k \\ l & \text{if } j \le k \end{cases}, \quad r' = m+q+n.$$

If $(*)$ holds and new word $Z_1 Z_2 \cdots Z_{r'}$ belongs to V^*, then we define

$$\mathbf{xy} = Z_1 Z_2 \cdots \overset{1}{Z_{i'}} \cdots \overset{2}{Z_{j'}} \cdots Z_{r'} \in D(\mathscr{T}^*),$$

otherwise we define $\mathbf{xy} = \mathbf{z}$. Also we define $\mathbf{mz} = \mathbf{zm} = \mathbf{z}$ as well as $\mathbf{m\varepsilon} = \mathbf{\varepsilon m} = \mathbf{m}$ for all $\mathbf{m} \in M$. Then, the set M becomes a monoid with the above operation. We call M the tiling monoid of a given tiling $\mathscr{T}$. In another sense, M can also be regarded as an inverse monoid with zero (cf. [9]).

It might be better for the readers to see several examples of our product here. Fibonacci tiling $\mathscr{F}$ is one-dimensional tiling made by next substitution

$$\tau : \begin{array}{ccc} A & \to & AB \\ B & \to & A \end{array},$$

and we can write

$$\mathscr{F} = ABAABABA \cdots.$$

Therefore, $\mathscr{F}^*$ is as follows

$$\mathscr{F}^* = A'A''A'''B'B''B''' \cdots.$$

Then it is recognised that

$$V^* = \{A'A''A'''B'B''B''', B'B''B'''A'A''A''', A'A''A'''A'A''A''', \dots\}.$$

Let

$$\mathbf{x} = \overset{1}{A'}A''A'''B'\overset{2}{B''}B'''$$
$$\mathbf{y} = \overset{2}{A'}A''A'''B'\overset{1}{B''}B'''$$
$$\mathbf{v} = B'\overset{1}{B''}B'''A'A''A'''A'\overset{2}{A''}A'''$$

be elements of $D(\mathscr{F}^*)$. Then we have

$$\mathbf{xy} = \overset{12}{A'}A''A'''B'B''B'''$$
$$\mathbf{yx} = A'A''A'''B'\overset{12}{B''}B'''$$
$$\mathbf{xv} = \overset{1}{A'}A''A'''B'B''B'''A'A''A'''A'\overset{2}{A''}A'''$$
$$\mathbf{vx} = \mathbf{z}.$$

And let

$$\mathbf{w} = A'\overset{1}{A''}A'''A'A''\overset{2}{A'''}$$

be the element of $D(\mathscr{F}^*)$. Because $A'A''A'''A'A''A'''A'A''A''' \notin V^*$, we have

$$\mathbf{xw} = \mathbf{z}.$$

5 Tiling Bialgebra

Let $A = \mathbf{C}[M] = \bigoplus_{\mathbf{m} \in M} \mathbf{Cm}$ be the monoid algebra of M over $\mathbf{C}$. Then $\mathbf{Cz}$ is a two-sided ideal of A. And we set $B = B(\mathscr{T}) = A/\mathbf{Cz}$. Then, B is sometimes called the tiling bialgebra (cf. [1], [10]) of $\mathscr{T}$. For a subset $V^* \subset W$, we define $E = E(V^*)$ to be the subset of D consisting of all doubly pointed words obtained from V^* with the pointed positions of type $(i, i+1)$ for all $i \geq 1$. And $F = F(V^*)$ the subset of D consisting of all doubly pointed words obtained from V^* with the pointed positions of type $(i+1, i)$ for all $i \geq 1$. Therefore, we can write E, F as follows

$$E = \{w(i, i+1) \in D \mid w \in V^*, 1 \leq i < l(w)\}$$

$$F = \{w(i+1, i) \in D \mid w \in V^*, 1 \leq i < l(w)\}.$$

6 Tiling group

For each $t \in \mathbf{C}$ and $\xi \in E \cup F$, we put $x_\xi(t) = 1 + t\xi \in B(\mathscr{T}^*)^\times$, where $B(\mathscr{T}^*)^\times$ is the multiplicative group of all units in $B(\mathscr{T}^*)$. Let G be the subgroup of $B(\mathscr{T}^*)$ generated by $x_\xi(t)$ for all $\xi \in E \cup F$ and $t \in \mathbf{C}$. We call G the tiling group associated with an original tiling $\mathscr{T}$. And for each $\xi \in E \cup F$ and $u \in \mathbf{C}^\times$, we set

$$w_\xi(u) = x_\xi(u) x_{\hat{\xi}}(-u^{-1}) x_\xi(u)$$

$$h_\xi(u) = w_\xi(u) w_\xi(-1).$$

Then we define subgroups of G as follows

$$G_+ = \langle x_e(t) \mid e \in E, t \in \mathbf{C} \rangle$$

$$G_0 = \langle h_\xi(u) \mid \xi \in E \cup F, u \in \mathbf{C}^\times \rangle$$

$$G_- = \langle x_f(t) \mid f \in F, t \in \mathbf{C} \rangle.$$

Then we can obtain the following result

$$G = G_\pm G_\mp G_0 G_\pm.$$

This relation is called the Gauss decomposition.

7 Gauss Decomposition of $\tilde{G}$

Now we define $(R1)$, $(R2)$ and $(R3)$ as follows

- $(R1)$ $\tilde{x}_\xi(t)\tilde{x}_\xi(t') = \tilde{x}_\xi(t + t')$ $(t, t' \in \mathbf{C})$
- $(R2)$ $\tilde{x}_{\xi_1}(t_1) \cdots \tilde{x}_{\xi_r}(t_r)\tilde{x}_\eta(t) = \tilde{x}_{\zeta_1}(u_1) \cdots \tilde{x}_{\zeta_s}(u_s)\tilde{x}_{\xi_1}(t_1) \cdots \tilde{x}_{\xi_r}(t_r)$

$$\text{if} \sum_{m,n=0}^{r} \sum_{\substack{1 \le k_1 < \cdots < k_m \le r \\ 1 \le l_1 < \cdots < l_n \le r}} (-1)^n t_{k_1} \cdots t_{k_m} t_{l_1} \cdots t_{l_n} t \xi_{k_1} \cdots \xi_{k_m} \eta \xi_{l_n} \cdots \xi_{l_1}$$

$$= u_1 \zeta_1 + \cdots + u_s \zeta_s$$

$$(t_i, t, u_j \in \mathbf{C}, \xi_i, \eta, \zeta_j \in E \cup F, \zeta_i \zeta_j = \zeta_j \zeta_i)$$

- $(R3)$ $\tilde{h}_\xi(u) \tilde{h}_\xi(t) = \tilde{h}_\xi(ut)$

$$(\tilde{h}_\xi(t) = \tilde{x}_\xi(t) \tilde{x}_{\hat{\xi}}(-t^{-1}) \tilde{x}_\xi(t-1) \tilde{x}_{\hat{\xi}}(1) \tilde{x}_\xi(-1), \xi \in E \cup F, t, u \in \mathbf{C}^\times)$$

Then we define $\tilde{G}$ generated by $\tilde{x}_\xi(t)$, $\xi \in E \cup F$, $t \in \mathbf{C}$ with relations $(R1)$, $(R2)$ and $(R3)$. And we define three subgroups of $\tilde{G}$ as follows:

$$\tilde{G}_+ = \langle \tilde{x}_\xi(t) \mid \xi \in E, t \in \mathbf{C} \rangle$$

$$\tilde{G}_0 = \langle \tilde{h}_\xi(u) \mid \xi \in E \cup F, u \in \mathbf{C}^\times \rangle$$

$$\tilde{G}_- = \langle \tilde{x}_\xi(t) \mid \xi \in F, t \in \mathbf{C} \rangle.$$

Then we obtain the following Theorem.

Theorem 1. *We define $\tilde{G}$, $\tilde{G}_\pm$, $\tilde{G}_0$ as above. Then we have*

$$\tilde{G} = \tilde{G}_\pm \tilde{G}_\mp \tilde{G}_0 \tilde{G}_\pm.$$

For the proof of this theorem, we show some lemmas.

Lemma 1. *For each $\xi \in E \cup F$, we have*

$$\langle \tilde{x}_\xi(t), \tilde{x}_{\hat{\xi}}(t) \mid t \in \mathbf{C} \rangle \simeq SL(2, \mathbf{C}).$$

Proof. We give $\tilde{x}_\xi(t), \tilde{x}_{\hat{\xi}}(t) \in \tilde{G}$ the next correspondence

$$\tilde{x}_\xi(t) \leftrightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \tilde{x}_{\hat{\xi}}(t) \leftrightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Then we have this lemma. □

Before the next lemma, we define an operation. For $\alpha \in W_2(\mathscr{T}^*)$ with $\alpha = XY$ and $\xi \in E \cup F$, we say $\xi \vdash \alpha$ if and only if

$$\xi = Z_1 Z_2 \cdots \overset{i}{X} \overset{j}{Y} \cdots Z_r$$

with $\{i, j\} = \{1, 2\}$. Let

$$\tilde{U}_{\alpha,+} = \langle \tilde{x}_\xi(t) \mid t \in \mathbf{C}, \xi \in E, \xi \vdash \alpha \rangle$$

$$\tilde{U}_{\alpha,-} = \langle \tilde{x}_\xi(t) \mid t \in \mathbf{C}, \xi \in F, \xi \vdash \alpha \rangle$$

$$\tilde{T}_\alpha = \langle \tilde{h}_\xi(u) \mid u \in \mathbf{C}^\times, \xi \in E \cup F, \xi \vdash \alpha \rangle$$

$$\tilde{G}_\alpha = \langle \tilde{U}_{\alpha,\pm} \rangle$$

for each $\alpha \in W_2(\mathscr{T}^*)$.

Lemma 2. *For $\tilde{h}_\xi(u) \in \tilde{T}_\alpha$, we have the next relation*

$$\tilde{h}_\xi(u)\tilde{U}_{\alpha,\pm}\tilde{h}_\xi(u)^{-1} = \tilde{U}_{\alpha,\pm}.$$

Proof. For each $\eta \in E \cup F$ with $\eta \vdash \alpha$, we have the next relation

$$\tilde{h}_\xi(u)\eta\tilde{h}_\xi(u)^{-1} = \sum_{i=1}^{s} u_i\eta_i \quad (\eta_i \vdash \alpha).$$

Then by $(R2)$

$$\tilde{h}_\xi(u)\tilde{x}_\eta(t) = \tilde{x}_{\eta_1}(u_1)\cdots\tilde{x}_{\eta_s}(u_s)\tilde{h}_\xi(u).$$

So the relation

$$\tilde{h}_\xi(u)\tilde{x}_\eta(t)\tilde{h}_\xi(u)^{-1} \in \tilde{U}_{\alpha,+}$$

is obtained from $(R2)$. Therefore we prove this lemma. □

Lemma 3. *Let $\alpha \in W_2(\mathscr{T}^*)$. Then we have*

$$\tilde{G}_\alpha = \tilde{U}_{\alpha,\pm}\tilde{U}_{\alpha,\mp}\tilde{T}_\alpha\tilde{U}_{\alpha,\pm}.$$

Proof. Let $\tilde{g} \in \tilde{G}_\alpha$ and we set

$$\tilde{g} = \tilde{x}_{\xi_1}(t_1)\tilde{x}_{\xi_2}(t_2)\cdots\tilde{x}_{\xi_r}(t_r)$$

with $\xi_i \in E \cup F$ and $t_i \in \mathbf{C}$ for $i = 1, 2, \dots r$. Then we put

$$B(\tilde{g}) = \langle \xi_i, \hat{\xi}_i \mid 1 \le i \le r \rangle.$$

And we define $E(\tilde{g}) = E \cap B(\tilde{g})$, $F(\tilde{g}) = F \cap B(\tilde{g})$. Let $\tilde{G}(\tilde{g})$ be a subgroup of $\tilde{G}$ generated by $\tilde{x}_\xi(t)$ for all $\xi \in E(\tilde{g}) \cup F(\tilde{g})$ and $t \in \mathbf{C}$. Then we have that $\tilde{G}(\tilde{g})$ is isomorphic to the direct product of finite copies of $SL(2, \mathbf{C})$. And we set

$$\tilde{U}_+(\tilde{g}) = \langle \tilde{x}_\xi(t) \mid \xi \in E(\tilde{g}), t \in \mathbf{C} \rangle$$
$$\tilde{T}(\tilde{g}) = \langle \tilde{h}_\xi(u) \mid \xi \in E(\tilde{g}) \cup F(\tilde{g}), u \in \mathbf{C}^\times \rangle$$
$$\tilde{U}_-(\tilde{g}) = \langle \tilde{x}_\xi(t) \mid \xi \in F(\tilde{g}), t \in \mathbf{C} \rangle.$$

Then we obtain $\tilde{G}(\tilde{g}) = \tilde{U}_+(\tilde{g})\tilde{U}_-(\tilde{g})\tilde{T}(\tilde{g})\tilde{U}_+(\tilde{g})$. Therefore we see

$$\tilde{g} \in \tilde{G}(\tilde{g}) \subset \tilde{U}_{\alpha,+}\tilde{U}_{\alpha,-}\tilde{T}_\alpha\tilde{U}_{\alpha,+}.$$

And this relation implies $\tilde{G}_\alpha = \tilde{U}_{\alpha,+}\tilde{U}_{\alpha,-}\tilde{T}_\alpha\tilde{U}_{\alpha,+}$. Similarly we can obtain $\tilde{G}_\alpha = \tilde{U}_{\alpha,-}\tilde{U}_{\alpha,+}\tilde{T}_\alpha\tilde{U}_{\alpha,-}$. □

Now we define

$$\tilde{U}'_{\alpha,\pm} = \langle \tilde{x}\tilde{U}_{\beta,\pm}\tilde{x}^{-1} \mid \tilde{x} \in \tilde{U}_{\alpha,\pm}, \beta \in W_2(\mathscr{T}^*), \beta \neq \alpha \rangle$$
$$\tilde{T}'_\alpha = \langle \tilde{T}_\beta \mid \beta \in W_2(\mathscr{T}^*) \rangle.$$

Then we obtain the following.

Lemma 4.

$$(1) \quad \tilde{G}_\pm = \tilde{U}_{\alpha,\pm}\tilde{U}'_{\alpha,\pm} = \tilde{U}'_{\alpha,\pm}\tilde{U}_{\alpha,\pm}$$
$$(2) \quad \tilde{G}_0 = \tilde{T}_\alpha\tilde{T}'_\alpha = \tilde{T}'_\alpha\tilde{T}_\alpha.$$

Proof. (1) follows the definition of $\tilde{U}_{\alpha,\pm}$. (2) follows from $(R2)$. □

Then we can prove Theorem 1.

Proof of Theorem 1. First we put $\tilde{\mathfrak{X}} = \tilde{G}_+\tilde{G}_-\tilde{G}_0\tilde{G}_+$. Let $\xi \in E \cup F$ and $t \in \mathbf{C}$. Then there is $\alpha \in W_2(\mathscr{T}^*)$ such that $\xi \vdash \alpha$. If $\xi \in E$, then $\tilde{x}_\xi(t)\tilde{\mathfrak{X}} = \tilde{\mathfrak{X}}$. If $\xi \in F$, then we have

$$\begin{aligned}
\tilde{x}_\xi(t)\tilde{\mathfrak{X}} &\in \tilde{U}_{\alpha,-}\tilde{\mathfrak{X}} \\
&= \tilde{U}_{\alpha,-}(\tilde{G}_+\tilde{G}_-\tilde{G}_0\tilde{G}_+) \\
&= \tilde{U}_{\alpha,-}(\tilde{U}'_{\alpha,+}\tilde{U}_{\alpha,+})(\tilde{U}'_{\alpha,-}\tilde{U}_{\alpha,-})(\tilde{T}_\alpha\tilde{T}'_\alpha)(\tilde{U}_{\alpha,+}\tilde{U}'_{\alpha,+}) \\
&= \tilde{U}'_{\alpha,+}\tilde{U}_{\alpha,-}\tilde{U}'_{\alpha,-}\tilde{U}_{\alpha,+}\tilde{U}_{\alpha,-}\tilde{T}_\alpha\tilde{U}_{\alpha,+}\tilde{T}'_\alpha\tilde{U}'_{\alpha,+}
\end{aligned}$$

$$= \tilde{U}'_{\alpha,+}\tilde{U}'_{\alpha,-}(\tilde{U}_{\alpha,-}\tilde{U}_{\alpha,+}\tilde{U}_{\alpha,-}\tilde{T}_{\alpha}\tilde{U}_{\alpha,+})\tilde{T}'_{\alpha}\tilde{U}'_{\alpha,+}$$

$$= \tilde{U}'_{\alpha,+}\tilde{U}'_{\alpha,-}(\tilde{U}_{\alpha,+}\tilde{U}_{\alpha,-}\tilde{T}_{\alpha}\tilde{U}_{\alpha,+})\tilde{T}'_{\alpha}\tilde{U}'_{\alpha,+}$$

$$= \tilde{U}'_{\alpha,+}\tilde{U}_{\alpha,+}\tilde{U}'_{\alpha,-}\tilde{U}_{\alpha,-}\tilde{T}_{\alpha}\tilde{T}'_{\alpha}\tilde{U}_{\alpha,+}\tilde{U}'_{\alpha,+}$$

$$= \tilde{G}_+\tilde{G}_-\tilde{G}_0\tilde{G}_+$$

$$= \tilde{\mathfrak{X}}.$$

Therefore $\tilde{G}\tilde{\mathfrak{X}} = \tilde{\mathfrak{X}}$. This relation shows $\tilde{G} = \tilde{\mathfrak{X}}$. Similarly we can establish $\tilde{G} = \tilde{G}_-\tilde{G}_+\tilde{G}_0\tilde{G}_-$. Therefore, we have finished to prove theorem.

8 Characterization of G

Here, we put $\pi : \tilde{G} \to G$: epimorphism. Then we obtain some lemmas.

Lemma 5. *We set* $B_+ = \langle \xi \mid \xi \in E \rangle$, $Z(B)_+ = B_+ \cap Z(B)$. *Then we have*

$$Z(B)_+ = 0.$$

Proof. Let $z \in Z(B)_+$. Suppose $z \neq 0$. We write $z = \sum_i t_i x_i$, where $x_i \in M_+$, $t_i \in \mathbf{C}$, $t_i \neq 0$. Then we choose x_0 such that $l(x_0)$ is minimal in the $l(x_i)$ for all i. And we set $h_0 = x_0\hat{x}_0$, $M_+ = M \cap B_+$, then we have

$$0 = [h_0, z] = t_0 x_0 + \sum_{\substack{l(x'_j) \geq l(x_0) \\ x'_j \neq x_0}} t'_j x'_j \quad (x'_j \in M_+, t'_j \in \mathbf{C}).$$

Because it contradicts $t_0 \neq 0$, we obtain

$$Z(B)_+ = 0.$$ □

Similarly we can prove $Z(B)_- = 0$.

Lemma 6. *Let* $Z(\tilde{G})$ *be the center of* $\tilde{G}$. *Then we have*

$$\ker \pi \subseteq Z(\tilde{G}).$$

Proof. We put $\tilde{g} = \tilde{x}_{\xi_1}(t_1) \cdots \tilde{x}_{\xi_r}(t_r) \in \ker \pi$. Then $\pi(\tilde{g}) = 1$. So we get $\pi(\tilde{g})\eta\pi(\tilde{g}) = \eta$. And then we obtain $\tilde{g}\tilde{x}_\eta(t) = \tilde{x}_\eta(t)\tilde{g}$. Therefore $\tilde{g} \in Z(\tilde{G})$. □

Lemma 7.

$$\tilde{G}_0 \simeq G_0.$$

Proof. We put $\pi : \tilde{G} \to G$: epimorphism. For each $\tilde{g}_0 \in \ker \pi \cap \tilde{G}_0$, we can write

$$\tilde{g}_0 = \tilde{h}_{\xi_1}(u_1)\tilde{h}_{\xi_2}(u_2)\cdots\tilde{h}_{\xi_k}(u_k) \quad (\xi \in E \cup F, u_i \in \mathbf{C}).$$

Then by Lemma 4 and $(R3)$, we can assume $\xi_i \neq \xi_j$ $(i \neq j)$ and $l(\xi_1) \leq l(\xi_2) \leq \cdots$. Then

$$\begin{aligned}
1 = \pi(\tilde{g}_0) &= (1 + (u_1 - 1)\xi_1\hat{\xi}_1 + (u_1^{-1} - 1)\hat{\xi}_1\xi_1) \\
&\quad \cdots (1 + (u_k - 1)\xi_k\hat{\xi}_k + (u_k^{-1} - 1)\hat{\xi}_k\xi_k) \\
&= 1 + \sum_{i=1}^{k} (u_i - 1)\xi_i\hat{\xi}_i + \sum_{i=1}^{k} (u_i^{-1} - 1)\hat{\xi}_i\xi_i \\
&\quad + \sum_{i,j=1, i\neq j}^{k} (u_i - 1)(u_j - 1)\xi_i\hat{\xi}_i\xi_j\hat{\xi}_j \\
&\quad + \sum_{i,j=1, i\neq j}^{k} (u_i - 1)(u_j^{-1} - 1)\xi_i\hat{\xi}_i\hat{\xi}_j\xi_j \\
&\quad + \sum_{i,j=1, i\neq j}^{k} (u_i^{-1} - 1)(u_j - 1)\hat{\xi}_i\xi_i\xi_j\hat{\xi}_j \\
&\quad + \sum_{i,j=1, i\neq j}^{k} (u_i^{-1} - 1)(u_j^{-1} - 1)\hat{\xi}_i\xi_i\hat{\xi}_j\xi_j \\
&\quad + \cdots + (u_1 - 1)\cdots(u_k - 1)\xi_1\hat{\xi}_1\cdots\xi_k\hat{\xi}_k \\
&\quad + \cdots + (u_1^{-1} - 1)\cdots(u_k^{-1} - 1)\hat{\xi}_1\xi_1\cdots\hat{\xi}_k\xi_k \\
&= 1 + (u_1 - 1)\xi_1\hat{\xi}_1 + (u_1^{-1} - 1)\hat{\xi}_1\xi_1 \\
&\quad + \sum_{\xi \in E, \xi \neq \xi_1} t_\xi\xi\hat{\xi} + \sum_{\xi \in E, \xi \neq \xi_1} t'_\xi\hat{\xi}\xi \quad (\xi \in E,\ t_\xi, t'_\xi \in \mathbf{C}).
\end{aligned}$$

Therefore $u_1 = 1$. Similarly, we obtain $u_i = 1$. So we have $\tilde{g}_0 = 1$. Therefore

$$\tilde{G}_0 \simeq G_0. \qquad \square$$

We define $Z(\tilde{G})_\pm = Z(\tilde{G}) \cap G_\pm$. Then we have the following lemma.

Lemma 8.

$$\ker \pi = Z(\tilde{G})_+ Z(\tilde{G})_-.$$

Proof. By Theorem 1, we can write

$$\tilde{G} = \tilde{G}_{\pm}\tilde{G}_{\mp}\tilde{G}_0\tilde{G}_{\pm}$$
$$= \bigcup_{\tilde{g}\in\tilde{G}_{\pm}} \tilde{g}\tilde{G}_{\mp}\tilde{G}_0\tilde{G}_{\pm}\tilde{g}^{-1}.$$

Therefore for all $\tilde{g}' \in Z(\tilde{G})$, we have $\tilde{g}\tilde{g}'\tilde{g}^{-1} \in \tilde{G}_{\mp}\tilde{G}_0\tilde{G}_{\pm}$ for each $\tilde{g} \in \tilde{G}_{\mp}$. Therefore we obtain

$$Z(\tilde{G}) \subseteq \tilde{G}_{\mp}\tilde{G}_0\tilde{G}_{\pm}.$$

Then for $\tilde{g} \in Z(\tilde{G})$, we can write

$$\tilde{g} = \tilde{g}_-\tilde{g}_0\tilde{g}_+ \in \tilde{G}_-\tilde{G}_0\tilde{G}_+.$$

Therefore

$$\pi(\tilde{g}) = \pi(\tilde{g}_-)\pi(\tilde{g}_0)\pi(\tilde{g}_+).$$

Now let

$$B_i = \bigoplus_{w(j,j+i)\in B} \mathbf{C}w(j, j+i).$$

We set

$$\pi(\tilde{g}_-) = 1 + b_-$$
$$\pi(\tilde{g}_0) = b_0$$
$$\pi(\tilde{g}_+) = 1 + b_+$$
$$\left(\begin{matrix} b_- = b_{-1} + b_{-2} + \cdots + b_{-r} \\ b_+ = b_1 + b_2 + \cdots + b_s \end{matrix}, b_i \in B_i\right).$$

Then we can rewrite

$$\pi(\tilde{g}) = (1 + b_-)b_0(1 + b_+)$$
$$= b_{-r}b_0 + \cdots + b_0b_s.$$

Because $\pi(\tilde{g}) = 1$, $b_{-r}b_0 = b_0b_s = 0$. So $b_{-r} = b_s = 0$. Then we have $b_{\pm} = 0$. Therefore we obtain $\tilde{g}_-\tilde{g}_+ \in \ker\pi$. Therefore

$$\ker\pi = Z(\tilde{G})_+Z(\tilde{G})_-. \qquad \square$$

Thus we have G and $\tilde{G}$.

Theorem 2.

$$\tilde{G}/(Z(\tilde{G})_+ Z(\tilde{G})_-) \simeq G.$$

Proof. By the definition, we obtain $\tilde{G}/\ker \pi \simeq G$. Therefore we have finished to prove this theorem. □

Therefore we get one characterization of G.

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