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CURVATURE PINCHING FOR KAEHLER SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE*

By

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Abstract. A complete classification for a compact Kaehler submanifold M_n in $P_{n+p}(C)$ with the scalar curvature $\rho \geq n^2$ is given, so that a conjecture of K. Ogiue is resolved partially.

1 Introduction

Let $P_{n+p}(C)$ be an $(n+p)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. There are a number of conjectures for Kaehler submanifolds in $P_{n+p}(C)$ suggested by K. Ogiue ([8]); some have been resolved under a suitable topological restriction (e.g. M_n is complete) (cf. [1], [2], [8], [9], [10], [11], [12], [13], [14], [16] and [17]). In this direction, one of the open problems so far is as follows:

Conjecture (K. Ogiue). Let M_n be an n-dimensional complete submanifold immersed in $P_{n+p}(C)$. If $\rho > n^2$, is M totally geodesic in $P_{n+p}(C)$?

In the case that M_n is a complete Kaehler submanifold immersed in $P_{n+p}(C)$ which has the Ricci curvature $S > \frac{n}{2}$, it was proved in [9] that such a submanifold M_n is totally geodesic in $P_{n+p}(C)$ ([9]). Recently, in the case of M_n has $S \geq \frac{n}{2}$ Suh and Yang ([12]) proved that such one is parallel, i.e., either totally geodesic or congruent to one of Q_n and $P_1(C) \times P_1(C)$. Also, the case that the scalar curvature $\rho > n(n+1) - \frac{n+2}{3}$ was studied by Tanno [15], and he proved that M is totally geodesic in $P_{n+p}(C)$.

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In the present paper we would like to consider the case that M_n is compact and $\rho > n^2$, so that the above conjecture is resolved partially. The main result is the following:

Theorem. *Let M_n be an n-dimensional compact Kaehler submanifold immersed in $P_{n+p}(C)$. Then $\rho \geq n^2$ if and only if M is either totally geodesic in $P_{n+p}(C)$ or $\rho = n^2$. In the latter case M^n is imbedded submanifold congruent to the standard imbedding of one of the following submanifolds: $P_1(C) \times P_1(C)$ and the complex quadric Q_n, $n \geq 3$.*

Hence, we have the following (see [8], p662–p663):

Corollary. *Let M_n be an n-dimensional compact Kaehler submanifold immersed in $P_{n+p}(C)$. If $\rho > n^2$, then M is totally geodesic in $P_{n+p}(C)$.*

2 Preliminaries

Let M_n be a compact Kaehler submanifold of complex dimension n, immersed in the complex projective space $P_{n+p}(C)$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature 1. We denote by UM the unit tangent bundle over M and by UM_x its fibre over $x \in M$ and by J and $\langle\,,\rangle$ the complex structure and the Fubini-Study metric. Let ∇ and h be the Riemannian connection and the second fundamental form of the immersion, respectively. A and $\nabla^{\perp}$ are the Weingarten endomorphism and the normal connection. The first and the second covariant derivatives of the normal valued tensor h are given by

$$(\nabla h)(X, Y, Z) = \nabla_X^{\perp}(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

and

$$\begin{aligned}(\nabla^2 h)(X, Y, Z, W) = \nabla_X^{\perp}((\nabla h)(Y, Z, W)) - (\nabla h)(\nabla_X Y, Z, W) \\ - (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y, Z, \nabla_X W),\end{aligned}$$

respectively, for any vector fields X, Y, Z and W tangent to M_n.

Let R and $R^{\perp}$ denote the curvature tensor associated with ∇ and $\nabla^{\perp}$, respectively. Then h and ∇h are symmetric and for $\nabla^2 h$ we have the Ricci-identity

$$\begin{aligned}&(\nabla^2 h)(X, Y, Z, W) - (\nabla^2 h)(Y, X, Z, W) \\ &\quad = R^{\perp}(X, Y)h(Z, W) - h(R(X, Y)Z, W) - h(Z, R(X, Y)W).\end{aligned}$$

We also consider the relations

$$h(JX, Y) = Jh(X, Y) \quad \text{and} \quad A_{J\xi} = JA_\xi = -A_\xi J,$$

where ξ is a normal vector to M_n.

If S and ρ is the Ricci tensor of M and the scalar curvature of M, respectively, since M is a complex Kaehler submanifold in $P_{n+p}(C)$, then from the Gauss equation we have

$$S(v, w) = \frac{n+1}{2}\langle v, w\rangle - \sum_{i=1}^{2n}\langle A_{h(v, e_i)}e_i, w\rangle, \tag{1}$$

$$\rho = n(n+1) - |h|^2. \tag{2}$$

Now, let $v \in UM_x$, $x \in M$. If $e_2, \dots, e_{2n}$ are orthonormal vectors in UM_x orthogonal to v, then we can consider $\{e_2, \dots, e_{2n}\}$ as an orthonormal basis of $T_v(UM_x)$. We remark that $\{v = e_1, e_2, \dots, e_{2n}\}$ is an orthonormal basis of T_xM. We denote the Laplacian of $UM_x \cong S^{2n-1}$ by Δ.

Define a function f_1 on UM_x, $x \in M$, by

$$f_1(v) = \sum_{i,j=1}^{2n}\langle A_{h(e_i, e_j)}e_j, A_{h(v, v)}e_i\rangle.$$

Noting that $\nabla_{e_k}v = -e_k$, $\nabla_{e_k}e_\ell = \delta_{k\ell}v$, $k, \ell = 2, \dots, 2n$, we have

$$\begin{aligned}(\Delta f_1)(v) &= \sum_{k=2}^{2n}(\nabla f_1)(v, e_k, e_k)\\ &= -2\sum_{k=2}^{2n}\nabla_{e_k}\left(\sum_{i,j=1}^{2n}\langle A_{h(e_i,e_j)}e_j, A_{h(e_k,v)}e_i\rangle\right)\\ &= -2\sum_{k=2}^{2n}f_1(v) + 2\sum_{k=2}^{2n}f_1(e_k).\end{aligned}$$

Using the minimality of M we can prove that

$$\begin{aligned}(\Delta f_1)(v) &= -2(2n-1)f_1(v) + 2\sum_{k=2}^{2n}\langle A_{h(e_i,e_j)}e_j, A_{h(e_k,e_k)}e_i\rangle\\ &= -4nf_1(v).\end{aligned} \tag{3}$$

For more details on this, see [7], [10]. Similarly, define f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10} and f_{11} by

$$f_2(v) = \sum \langle A_{h(v,v)}v, A_{h(v,e_i)}e_i \rangle,$$

$$f_3(v) = \sum \langle A_{h(e_i,e_j)}e_j, A_{h(v,e_i)}v \rangle,$$

$$f_4(v) = \sum \langle A_{h(v,e_i)}e_i, A_{h(v,e_j)}e_j \rangle,$$

$$f_5(v) = \sum \langle A_{h(v,v)}e_i, A_{h(v,v)}e_i \rangle,$$

$$f_6(v) = \sum \langle A_{h(e_j,v)}e_i, A_{h(e_j,v)}e_i \rangle$$

$$f_7(v) = |h(v,v)|^2,$$

$$f_8(v) = \sum \langle A_{h(v,e_i)}e_i, v \rangle |h(v,v)|^2,$$

$$f_9(v) = \left(\sum \langle A_{h(v,e_i)}e_i, v \rangle \right)^2,$$

$$f_{10}(v) = \sum \langle A_{h(v,e_i)}e_i, v \rangle$$

$$f_{11}(v) = |h|^2 |h(v,v)|^2,$$

respectively. Then we obtain

$$(\Delta f_2)(v) = -4(2n+2)f_2(v) + 4f_3(v) + 4f_4(v) + 2f_1(v), \tag{4}$$

$$(\Delta f_3)(v) = -4nf_3(v) + 2\sum \langle A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k \rangle, \tag{5}$$

$$(\Delta f_4)(v) = -4nf_4(v) + 2\sum \langle A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k \rangle, \tag{6}$$

$$(\Delta f_5)(v) = -4(2n+2)f_5(v) + 8\sum \langle A_{h(e_j,v)}e_i, A_{h(e_j,v)}e_i \rangle, \tag{7}$$

$$(\Delta f_6)(v) = -4nf_6(v) + 2\sum \langle A_{h(e_j,e_k)}e_i, A_{h(e_j,e_k)}e_i \rangle, \tag{8}$$

$$(\Delta f_7)(v) = -4(2n+2)f_7(v) + 8\sum \langle A_{h(v,e_i)}e_i, v \rangle, \tag{9}$$

$$(\Delta f_8)(v) = -6(2n+4)f_8(v) + 16f_2(v) + 2f_{11}(v) + 8f_9(v), \tag{10}$$

$$(\Delta f_9)(v) = -4(2n+2)f_9(v) + 8f_4(v) + 4|h|^2 \sum \langle A_{h(v,e_i)}e_i, v \rangle, \tag{11}$$

$$(\Delta f_{10})(v) = -4nf_{10}(v) + 2|h|^2, \tag{12}$$

$$(\Delta f_{11})(v) = -4(2n+2)f_{11}(v) + 8|h|^2 \sum \langle A_{h(v,e_i)}e_i, v \rangle. \tag{13}$$

Since

$$\frac{1}{2}\sum(\nabla^2 f_7)(e_i, e_i, v) = \sum\langle(\nabla^2 h)(e_i, e_i, v, v), h(v, v)\rangle$$
$$= \sum\langle(\nabla h)(e_i, v, v), (\nabla h)(e_i, v, v)\rangle,$$

we have the following (See [3], [4], [5], [6] and [7]):

Lemma. *Let M be an n-dimensional complex Kaehler submanifold of $P_{n+p}(C)$. Then for $v \in UM_x$ we have*

$$\frac{1}{2}\sum(\nabla^2 f_7)(e_i, e_i, v) = \sum|(\nabla h)(e_i, v, v)|^2 + \frac{n+2}{2}|h(v, v)|^2$$
$$+ 2\sum\langle A_{h(v,v)}e_i, A_{h(e_i,v)}v\rangle$$
$$- 2\sum\langle A_{h(v,e_i)}e_i, A_{h(v,v)}v\rangle$$
$$- \sum\langle A_{h(v,v)}e_i, A_{h(v,v)}e_i\rangle. \qquad (14)$$

3 Proof of Theorem

From (2) we have

$$\rho = n(n+1) - |h|^2.$$

Thus we have only to prove Theorem under the assumption

$$|h|^2 \leq n. \qquad (15)$$

We see the following equation holds for $v \in UM_x$, $x \in M$.

$$\sum\langle A_{h(Jv,Jv)}e_i, A_{h(e_i,Jv)}Jv\rangle = -\sum\langle A_{h(v,v)}e_i, A_{h(e_i,v)}v\rangle. \qquad (16)$$

From (14) and (16) we have

$$\frac{1}{4}\sum(\nabla^2 f_7)(e_i, e_i, v) + \frac{1}{4}\sum(\nabla^2 f_7)(e_i, e_i, Jv)$$
$$= \sum|(\nabla h)(e_i, v, v)|^2 + \frac{n+2}{2}|h(v, v)|^2$$
$$- 2\sum\langle A_{h(v,e_i)}e_i, A_{h(v,v)}v\rangle - \sum\langle A_{h(v,v)}e_i, A_{h(v,v)}e_i\rangle. \qquad (17)$$

Now, we choose an orthonormal basis $\{v = e_1, e_2, \dots, e_n\}$ such that the matrix $\sum_{\alpha=1}^{2p} A_{\xi_\alpha}^2$ is diagonalized, where $\{\xi_1, \xi_2, \dots, \xi_{2p}\}$ is any orthonormal normal basis and $1 \le \alpha \le 2p$. Then we have

$$f_2(v) = f_8(v). \tag{18}$$

In terms of (4), (5), (6), (10), (11), (13), (17) and (18) we have

$$\begin{aligned}
&\frac{1}{4}\sum(\nabla^2 f_7)(e_i, e_i, v) + \frac{1}{4}\sum(\nabla^2 f_7)(e_i, e_i, Jv) \\
&\quad + \frac{1}{6n(2n+2)}(2(\Delta f_2)(v) + \frac{2}{n}(\Delta f_3)(v) - \frac{2}{n}(\Delta f_4)(v) + \frac{1}{n}(\Delta f_1)(v) \\
&\quad - (2n+2)(\Delta f_8)(v) - 2(\Delta f_9)(v) + (\Delta f_{11})(v)) \\
&= \sum |(\nabla h)(e_i, v, v)|^2 + \frac{n+2}{2}|h(v,v)|^2 - \frac{1}{n} f_{11}(v) - f_5(v) \\
&\ge \sum |(\nabla h)(e_i, v, v)|^2 + \frac{n}{2}|h(v,v)|^2 - f_5(v),
\end{aligned} \tag{19}$$

noting that (15). On the other hand, in terms of (9) and (12) we have

$$\begin{aligned}
&\frac{n}{2}\left(\frac{1}{4(2n+2)}(\Delta f_7)(v) + \frac{2}{4n(2n+2)}(\Delta f_{10})(v)\right) \\
&\quad = -\frac{n}{2}|h(v,v)|^2 + \frac{n}{2n(2n+2)}|h|^2.
\end{aligned} \tag{20}$$

Also, we have from (7) and (8)

$$\begin{aligned}
&-\frac{1}{4(2n+2)}(\Delta f_5)(v) - \frac{2}{4n(2n+2)}(\Delta f_6)(v) \\
&\quad = f_5(v) - \frac{2}{2n(2n+2)}\sum \langle A_{h(e_j, e_k)} e_i, A_{h(e_j, e_k)} e_i \rangle \\
&\quad = f_5(v) - \frac{2}{2n(2n+2)}\sum_{\alpha,\beta=1}^{2p} (\operatorname{trace} A_{\xi_\alpha} A_{\xi_\beta})^2 \\
&\quad \ge f_5(v) - \frac{1}{2n(2n+2)}|h|^4 \\
&\quad \ge f_5(v) - \frac{n}{2n(2n+2)}|h|^2,
\end{aligned} \tag{21}$$

where we used $\sum(\text{trace}\, A_{\xi_\alpha}A_{\xi_\beta})^2 \leq \frac{1}{2}|h|^4$ (See [9], p. 88) and (15), where $\{\xi_1, \xi_2, \ldots, \xi_{2p}\}$ is any orthonormal normal basis as above and $1 \leq \alpha, \beta \leq 2p$. Summing up (17), (20) and (21) and using Hopf's lemma, we have

$$\sum |(\nabla h)(e_i, v, v)|^2 = 0.$$

Thus we know that M_n is parallel. This proves Theorem (See [8], p. 662–663).

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