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RELATION BETWEEN DIFFERENTIAL POLYNOMIALS OF CERTAIN COMPLEX LINEAR DIFFERENTIAL EQUATIONS AND MEROMORPHIC FUNCTIONS OF FINITE ORDER

By

Benharrat Belaïdi and Abdallah El Farissi

Abstract. In this paper, we investigate the relationship between differential polynomials and meromorphic functions of finite order of some second order linear differential equations with meromorphic coefficients. We obtain some precise estimates.

1 Introduction and Statement of Results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [9], [13]). In addition, we will use $\lambda(f)$ and $\lambda(1/f)$ to denote respectively the exponents of convergence of the zero-sequence and the pole-sequence of a meromorphic function f, $\rho(f)$ to denote the order of growth of f, $\bar{\lambda}(f)$ and $\bar{\lambda}(1/f)$ to denote respectively the exponents of convergence of the sequence of distinct zeros and distinct poles of f.

Consider the second order linear differential equation

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = 0, \tag{1.1}$$

where $P(z)$, $Q(z)$ are nonconstant polynomials, $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ are entire functions such that $\rho(A_1) < \deg P(z)$, $\rho(A_0) < \deg Q(z)$. In [11], Ki-Ho Kwon has investigated the hyper order of solutions of (1.1) when $\deg P(z) = \deg Q(z)$. Gundersen showed in [7, p. 419] that if $\deg P(z) \neq \deg Q(z)$, then every nonconstant solution of (1.1) is of infinite order. If $\deg P(z) = \deg Q(z)$, then (1.1)

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may have nonconstant solutions of finite order. For instance $f(z) = e^z + 1$ satisfies $f'' + e^z f' - e^z f = 0$.

In [3], Z. X. Chen and K. H. Shon have investigated the case when $\deg P(z) = \deg Q(z)$ and have proved the following results:

Theorem A ([3]). *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ *be meromorphic functions with* $\rho(A_j) < 1$ $(j = 0, 1)$, *a, b be complex numbers such that* $ab \neq 0$ *and* $\arg a \neq \arg b$ *or* $a = cb$ $(0 < c < 1)$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of the equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0 \tag{1.2}$$

has infinite order.

In the same paper, Z. X. Chen and K. H. Shon have investigated the fixed points of solutions, their 1st and 2nd derivatives and the differential polynomial and have obtained:

Theorem B ([3]). *Let* $A_j(z)$ $(j = 0, 1)$, *a, b, c satisfy the additional hypotheses of Theorem* A. *Let* d_0, d_1, d_2 *be complex constants that are not all equal to zero. If* $f(z) \not\equiv 0$ *is any meromorphic solution of equation* (1.2), *then*:

(i) f, f', f'' *all have infinitely many fixed points and satisfy*

$$\bar{\lambda}(f - z) = \bar{\lambda}(f' - z) = \bar{\lambda}(f'' - z) = \infty,$$

(ii) *the differential polynomial*

$$g(z) = d_2 f'' + d_1 f' + d_0 f$$

has infinitely many fixed points and satisfies $\bar{\lambda}(g - z) = \infty$.

Recently Theorem A has been generalized to higher order differential equations by the author as follows (see [1]):

Theorem C ([1]). *Let* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \dots, k-1)$ *be nonconstant polynomials where* $a_{0,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ *are complex numbers such that* $a_{n,j}a_{n,0} \neq 0$ $(j = 1, \dots, k-1)$, *let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0, \dots, k-1)$ *be meromorphic functions. Suppose that* $\arg a_{n,j} \neq \arg a_{n,0}$ *or* $a_{n,j} = c a_{n,0}$ $(0 < c < 1)$ $(j = 1, \dots, k-1)$, $\rho(A_j) < n$ $(j = 0, \dots, k-1)$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of the equation*

$$f^{(k)} + A_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)} + \cdots + A_1(z)e^{P_1(z)}f' + A_0(z)e^{P_0(z)}f = 0, \tag{1.3}$$

where $k \geq 2$, *is of infinite order.*

The main purpose of this paper is to study the relation between meromorphic functions of finite order and differential polynomials of second order linear differential equation (1.1). For some related results of linear differential equations with entire coefficients, we refer the reader to [2]. In fact we will prove the following results:

THEOREM 1.1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *and* $Q(z) = \sum_{i=0}^{n} b_i z^i$ *be nonconstant polynomials where* a_i, b_i $(i = 0, 1, \dots, n)$ *are complex numbers*, $a_n \neq 0$, $b_n \neq 0$ *such that* $\arg a_n \neq \arg b_n$ *or* $a_n = cb_n$ $(0 < c < 1)$ *and* $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ *be meromorphic functions with* $\rho(A_j) < n$ $(j = 0, 1)$. *Let* $d_0(z)$, $d_1(z)$, $d_2(z)$ *be polynomials that are not all equal to zero*, $\varphi(z) \not\equiv 0$ *is a meromorphic function with finite order. If* $f(z) \not\equiv 0$ *is a meromorphic solution of* (1.1) *with* $\lambda(1/f) < \infty$, *then the differential polynomial* $g(z) = d_2 f'' + d_1 f' + d_0 f$ *satisfies* $\overline{\lambda}(g - \varphi) = \infty$.

REMARK 1.1. In the following Theorem 1.2, we remove the condition $\lambda(1/f) < \infty$.

THEOREM 1.2. *Suppose that* $P(z)$, $Q(z)$, $A_1(z)$, $A_0(z)$ *satisfy the hypotheses of Theorem* 1.1. *If* $\varphi(z) \not\equiv 0$ *is a meromorphic function with finite order, then every meromorphic solution* f *of* (1.1) *satisfies* $\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty$.

2 Preliminary Lemmas

We need the following lemmas in the proofs of our theorems.

LEMMA 2.1 ([6]). *Let* f *be a transcendental meromorphic function of finite order* ρ, *let* $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geq 0$ *for* $i = 1, \dots, m$ *and let* $\varepsilon > 0$ *be a given constant. Then the following estimations hold*:

(i) *There exists a set* $E_1 \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\psi \in [0, 2\pi) - E_1$, *then there is a constant* $R_1 = R_1(\psi) > 1$ *such that for all* z *satisfying* $\arg z = \psi$ *and* $|z| \geq R_1$ *and for all* $(k, j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

(ii) *There exists a set* $E_2 \subset (1, \infty)$ *that has finite logarithmic measure* $lm(E_2) = \int_1^{+\infty} \frac{\chi_{E_2}(t)}{t}\,dt$, *where* χ_{E_2} *is the characteristic function of* E_2, *such that for all* z *satisfying* $|z| \notin E_2 \cup [0, 1]$ *and for all* $(k, j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.2}$$

Lemma 2.2 ([3]). *Let $f(z)$ be a transcendental meromorphic function of order $\rho(f) = \rho < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E_3 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_1 \in [0, 2\pi) \backslash E_3$, then there is a constant $R_2(\psi_1) > 1$ such that for all z satisfying $\arg z = \psi_1$ and $|z| = r \geq R_2$, we have*

$$\exp\{-r^{\rho+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\rho+\varepsilon}\}. \tag{2.3}$$

Lemma 2.3. *Let $P(z) = a_n z^n + \cdots + a_0$, $(a_n = \alpha + i\beta \neq 0)$ be a polynomial with degree $n \geq 1$ and $A(z)$ $(\not\equiv 0)$ be a meromorphic function with $\rho(A) < n$. Set $f(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $E_4 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) \backslash (E_4 \cup E_5)$, where $E_5 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ is a finite set, then for sufficiently large $|z| = r$, we have*

(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta)r^n\} \leq |f(z)| \leq \exp\{(1 + \varepsilon)\delta(P, \theta)r^n\}, \tag{2.4}$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta)r^n\} \leq |f(z)| \leq \exp\{(1 - \varepsilon)\delta(P, \theta)r^n\}. \tag{2.5}$$

Proof. Set $f(z) = h(z)e^{(\alpha+i\beta)z^n}$, where $h(z) = A(z)e^{P_{n-1}(z)}$, $P_{n-1}(z) = P(z) - (\alpha + i\beta)z^n$. Then $\rho(h) = \lambda < n$. By Lemma 2.2, for any given ε $(0 < \varepsilon < n - \lambda)$, there is $E_4 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) \backslash (E_4 \cup E_5)$, where $E_5 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$, then there is a constant $R_2 = R_2(\theta) > 1$, such that, for all z satisfying $\arg z = \theta$ and $r \geq R_2$, we have

$$\exp\{-r^{\lambda+\varepsilon}\} \leq |h(z)| \leq \exp\{r^{\lambda+\varepsilon}\}. \tag{2.6}$$

By $|e^{(\alpha+i\beta)(re^{i\theta})^n}| = e^{\delta(P,\theta)r^n}$ and (2.6), we have

$$\exp\{\delta(P, \theta)r^n - r^{\lambda+\varepsilon}\} \leq |f(z)| \leq \exp\{\delta(P, \theta)r^n + r^{\lambda+\varepsilon}\}. \tag{2.7}$$

By $\theta \notin E_5$ we see that:

(i) if $\delta(P, \theta) > 0$, then by $0 < \lambda + \varepsilon < n$ and (2.7), we know that (2.4) holds for a sufficiently large r;

(ii) if $\delta(P, \theta) < 0$, then by $0 < \lambda + \varepsilon < n$ and (2.7), we know that (2.5) holds for a sufficiently large r.

Lemma 2.4 ([5]). *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\rho(f) = \infty$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.8}$$

then $\bar{\lambda}(f) = \lambda(f) = \rho(f) = \infty$.

Lemma 2.5 [8, p. 344]. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function,* $\mu(r)$ *be the maximum term, i.e.,* $\mu(r) = \max\{|a_n|r^n; n = 0, 1, \dots\}$ *and let* $\nu_f(r)$ *be the central index of* f*, i.e.,* $\nu_f(r) = \max\{m, \mu(r) = |a_m|r^m\}$*. Then*

$$\nu_f(r) = r\frac{d}{dr}\log\mu(r) < [\log\mu(r)]^2 \le [\log M(r,f)]^2, \tag{2.9}$$

outside a set $E_6 \subset (1, +\infty)$ *of* r *of finite logarithmic measure.*

Remark 2.1 (see [10, pp. 33–35], [12, p. 51]). We have the following basic properties of $\mu(r)$ and $\nu_f(r)$:

(i) $\mu(r)$ is strictly increasing for all r sufficiently large, is continuous and tends to $+\infty$ as $r \to \infty$;

(ii) $\nu_f(r)$ is increasing, piecewise constant, right-continuous and also tends to $+\infty$ as $r \to \infty$.

Lemma 2.6 (see [10, pp. 36–37], [12, p. 51]). *If* $f(z)$ *is an entire of order* σ*, then*

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log \nu_f(r)}{\log r}. \tag{2.10}$$

Lemma 2.7 (Wiman-Valiron, [8], [14]). *Let* $f(z)$ *be a transcendental entire function, and let* z *be a point with* $|z| = r$ *at which* $|f(z)| = M(r,f)$*. Then the estimation*

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^k (1 + o(1)) \quad (k \ge 1 \text{ is an integer}), \tag{2.11}$$

holds for all $|z|$ *outside a set* E_7 *of* r *of finite logarithmic measure.*

Lemma 2.8 ([4]). *Suppose that* $f(z)$ *is a meromorphic function with* $\rho(f) = \beta < \infty$*. Then for any given* $\varepsilon > 0$*, there is a set* $E_8 \subset (1, +\infty)$ *of finite logarithmic measure, such that*

$$|f(z)| \le \exp\{r^{\beta+\varepsilon}\} \tag{2.12}$$

holds for $|z| = r \notin [0,1] \cup E_8$*,* $r \to +\infty$.

LEMMA 2.9. *Let $f(z)$ be a meromorphic function with $\rho(f) = \infty$ and the exponent $\lambda(1/f)$ of convergence of the poles of $f(z)$ is finite, $\lambda(1/f) < \infty$. Let $d_j(z)$ $(j = 0, 1, 2)$ be polynomials that are not all equal to zero. Then*

$$g(z) = d_2(z)f'' + d_1(z)f' + d_0(z)f \tag{2.13}$$

satisfies $\rho(g) = \infty$.

PROOF. We suppose that $\rho(g) = \rho < \infty$ and then we obtain a contradiction. First we suppose that $d_2(z) \not\equiv 0$. Set $f(z) = w(z)/h(z)$, where $h(z)$ is canonical product (or polynomial) formed with the non-zero poles of $f(z)$, $\lambda(h) = \rho(h) = \lambda(1/f) = \rho_1 < \infty$, $w(z)$ is an entire function with $\rho(w) = \rho(f) = \infty$. We have

$$f'(z) = \frac{w'h - h'w}{h^2} \quad \text{and} \quad f''(z) = \frac{w''}{h} - \frac{wh''}{h^2} - 2\frac{w'h'}{h^2} + 2\frac{(h')^2 w}{h^3}. \tag{2.14}$$

Hence

$$\frac{f''(z)}{f(z)} = \frac{w''}{w} - \frac{h''}{h} - 2\frac{w'h'}{wh} + 2\frac{(h')^2}{h^2}, \tag{2.15}$$

$$\frac{f'(z)}{f(z)} = \frac{w'}{w} - \frac{h'}{h}. \tag{2.16}$$

By Lemma 2.1 (ii), there exists a set $E_1 \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E_1 \cup [0, 1]$, we have

$$\left|\frac{h^{(j)}(z)}{h(z)}\right| \le |z|^{j(\rho_1 - 1 + \varepsilon)} \quad (j = 1, 2). \tag{2.17}$$

Substituting (2.17) into (2.15) and (2.16), we obtain for all z satisfying $|z| \notin E_1 \cup [0, 1]$

$$\frac{f''(z)}{f(z)} = \frac{w''}{w} + O(z^{\alpha})\frac{w'}{w} + O(z^{\alpha}), \tag{2.18}$$

$$\frac{f'(z)}{f(z)} = \frac{w'}{w} + O(z^{\alpha}), \tag{2.19}$$

where α $(0 < \alpha < \infty)$ is a constant and may be different at different places. Substituting (2.18) and (2.19) into (2.13), we have

$$d_2(z)\left(\frac{w''}{w} + O(z^{\alpha})\frac{w'}{w} + O(z^{\alpha})\right) + d_1(z)\left(\frac{w'}{w} + O(z^{\alpha})\right) + d_0(z) = \frac{g(z)h(z)}{w(z)}. \tag{2.20}$$

It follows that

$$d_2(z)\frac{w''}{w} + (O(z^{\alpha})d_2(z) + d_1(z))\frac{w'}{w} + O(z^{\alpha})d_2(z) + O(z^{\alpha})d_1(z) + d_0(z) = \frac{g(z)h(z)}{w(z)}. \tag{2.21}$$

Hence,

$$d_2(z)\frac{w''}{w} + O(z^m)\frac{w'}{w} + O(z^m) = \frac{g(z)h(z)}{w(z)}, \tag{2.22}$$

where m $(0 < m < \infty)$ is some constant. By Lemma 2.7, there exists a set $E_2 \subset (1, +\infty)$ with logarithmic measure $lm(E_2) < +\infty$ and we can choose z satisfying $|z| = r \notin [0, 1] \cup E_2$ and $|w(z)| = M(r, w)$, such that

$$\frac{w^{(j)}(z)}{w(z)} = \left(\frac{v_w(r)}{z}\right)^j (1 + o(1)) \quad (j = 1, 2). \tag{2.23}$$

Since $\rho(g) = \rho < \infty$ and $\rho(h) = \lambda(1/f) = \rho_1 < \infty$, by Lemma 2.8 there exists a set E_3 that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$, we have

$$|g(z)| \le \exp\{r^{\rho+1}\}, \quad |h(z)| \le \exp\{r^{\rho_1+1}\}. \tag{2.24}$$

By Lemma 2.5, there is a set $E_4 \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4$, we have

$$|v_w(r)| < (\log M(r, w))^2. \tag{2.25}$$

Since $\rho(w) = \overline{\lim}_{r\to+\infty} \frac{\log v_w(r)}{\log r} = +\infty$, there exists $\{r_n'\}$ $(r_n' \to +\infty)$ such that

$$\lim_{r_n'\to+\infty} \frac{\log v_w(r_n')}{\log r_n'} = +\infty. \tag{2.26}$$

Set the logarithmic measure of $E_1 \cup E_2 \cup E_3 \cup E_4$,

$$lm(E_1 \cup E_2 \cup E_3 \cup E_4) = \gamma < +\infty,$$

then there exists a point $r_n \in [r_n', (\gamma + 1)r_n'] - E_1 \cup E_2 \cup E_3 \cup E_4$. From

$$\frac{\log v_w(r_n)}{\log r_n} \ge \frac{\log v_w(r_n')}{\log((\gamma+1)r_n')} = \frac{\log v_w(r_n')}{\left[1 + \frac{\ln(\gamma+1)}{\log r_n'}\right] \log r_n'}, \tag{2.27}$$

it follows that

$$\lim_{r_n\to+\infty} \frac{\log v_w(r_n)}{\log r_n} = +\infty. \tag{2.28}$$

Then for a given arbitrary large $\beta > 2\ (\rho_1 + \rho + m + 3)$,

$$v_w(r_n) \geq r_n^{\beta} \tag{2.29}$$

holds for sufficiently large r_n. Now we take point z_n satisfying $|z_n| = r_n$ and $w(z_n) = M(r_n, w)$, by (2.22) and (2.23), we get

$$|d_2(z_n)|\left(\frac{v_w(r_n)}{r_n}\right)^2 |1 + o(1)| \leq 2Lr_n^m \left(\frac{v_w(r_n)}{r_n}\right)|1 + o(1)| + \left|\frac{g(z_n)h(z_n)}{w(z_n)}\right|, \tag{2.30}$$

where $L > 0$ is some constant. By Lemma 2.5 and (2.29), we get

$$M(r_n, w) > \exp(r_n^{\beta/2}). \tag{2.31}$$

Hence by (2.24), (2.31) as $r_n \to +\infty$

$$\frac{|g(z_n)h(z_n)|}{M(r_n, w)} \to 0 \tag{2.32}$$

holds. By (2.29), (2.30), (2.32), we get

$$|d_2(z_n)|r_n^{\beta} \leq |d_2(z_n)|v_w(r_n) \leq 2LKr_n^{m+1}, \tag{2.33}$$

where $K > 0$ is some constant. This is a contradiction by $\beta > 2\ (\rho_1 + \rho + m + 3)$. Hence $\rho(g) = \infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$. Using a similar reasoning as above we get a contradiction. Hence $\rho(g) = \infty$.

Finally, if $d_2 \equiv 0$, $d_1 \equiv 0$, $d_0 \not\equiv 0$, then we have $g(z) = d_0(z)f(z)$ and by d_0 is a polynomial, then we get $\rho(g) = \infty$.

LEMMA 2.10. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *and* $Q(z) = \sum_{i=0}^{n} b_i z^i$ *be nonconstant polynomials where* a_i, b_i $(i = 0, 1, \ldots, n)$ *are complex numbers,* $a_n \neq 0$, $b_n \neq 0$ *such that* $\arg a_n \neq \arg b_n$ *or* $a_n = cb_n$ $(0 < c < 1)$. *We denote index sets by*

$$\Lambda_1 = \{0, P\},$$

$$\Lambda_2 = \{0, P, Q, 2P, P + Q\}.$$

(i) *If* H_j $(j \in \Lambda_1)$ *and* $H_Q \not\equiv 0$ *are all meromorphic functions of orders that are less than* n, *setting* $\Psi_1(z) = \sum_{j \in \Lambda_1} H_j(z)e^j$, *then* $\Psi_1(z) + H_Q e^Q \not\equiv 0$.

(ii) *If* H_j $(j \in \Lambda_2)$ *and* $H_{2Q} \not\equiv 0$ *are all meromorphic functions of orders that are less than* n, *setting* $\Psi_2(z) = \sum_{j \in \Lambda_2} H_j(z)e^j$, *then* $\Psi_2(z) + H_{2Q}e^{2Q} \not\equiv 0$.

(iii) *Let* $\Psi_{20}(z)$, $\Psi_{21}(z)$, $\Psi_{22}(z)$, $\Psi_{23}(z)$, $\Psi_{24}(z)$ *have the form of* $\Psi_2(z)$ *which is defined as in* (ii), $H_{2Q} \not\equiv 0$ *are all meromorphic functions of orders that are less than* n, $\varphi(z) \not\equiv 0$ *and* $\phi(z) \not\equiv 0$ *are meromorphic functions with finite order. Then*

$$\frac{\varphi''(z)}{\varphi(z)}\Psi_{24}(z)+\frac{\varphi'(z)}{\varphi(z)}\Psi_{23}(z)+\frac{\varphi'(z)}{\varphi(z)}\frac{\phi'(z)}{\phi(z)}\Psi_{22}(z)+\frac{\phi'(z)}{\phi(z)}\Psi_{21}(z)$$
$$+\Psi_{20}(z)+H_{2Q}e^{2Q}\not\equiv 0. \tag{2.34}$$

Proof. The proof of (i) and (ii) are similar, we prove only (ii). We divide this into two cases to prove:

Case 1: Suppose first that $\arg a_n \neq \arg b_n$. Then $\arg a_n$, $\arg b_n$, $\arg(a_n+b_n)$ are three distinct arguments. Set $\rho(H_0)=\beta<n$. By Lemma 2.2, for any given ε $\left(0<\varepsilon<\min\left(\frac{1}{4}, n-\beta\right)\right)$, there is a set E_1 that has linear measure zero such that if $\arg z=\theta\in[0,2\pi)\backslash E_1$, then there is $R=R(\theta)>1$ such that for all z satisfying $\arg z=\theta$ and $|z|=r>R$, we have

$$|H_0(z)|\leq \exp\{r^{\beta+\varepsilon}\}. \tag{2.35}$$

By Lemma 2.3, there exists a ray $\arg z=\theta\in[0,2\pi)\backslash E_1\cup E_2\cup E_0$, $E_2, E_0\subset[0,2\pi)$ being defined as in Lemma 2.3, E_2 having linear measure zero, E_0 being a finite set, such that

$$\delta(2P,\theta)=2\delta(P,\theta)<0,\quad \delta(P+Q,\theta)<0,\quad \delta(2Q,\theta)=2\delta(Q,\theta)>0$$

and for the above ε, we have for sufficiently large $|z|=r$

$$|H_{2Q}e^{2Q}|\geq \exp\{(1-\varepsilon)2\delta(Q,\theta)r^n\}, \tag{2.36}$$

$$|H_Qe^Q|\leq \exp\{(1+\varepsilon)\delta(Q,\theta)r^n\}, \tag{2.37}$$

$$|H_{P+Q}e^{P+Q}|\leq \exp\{(1-\varepsilon)\delta(P+Q,\theta)r^n\}<1, \tag{2.38}$$

$$|H_{2P}e^{2P}|\leq \exp\{(1-\varepsilon)2\delta(P,\theta)r^n\}<1, \tag{2.39}$$

$$|H_Pe^P|\leq \exp\{(1-\varepsilon)\delta(P,\theta)r^n\}<1. \tag{2.40}$$

If $\Psi_2(z)+H_{2Q}e^{2Q}\equiv 0$, then by (2.35)–(2.40), we have

$$\exp\{(1-\varepsilon)2\delta(Q,\theta)r^n\}\leq |H_{2Q}e^{2Q}|\leq \exp\{r^{\beta+\varepsilon}\}+\exp\{(1+\varepsilon)\delta(Q,\theta)r^n\}+3$$
$$\leq 3\exp\{r^{\beta+\varepsilon}\}\exp\{(1+\varepsilon)\delta(Q,\theta)r^n\}. \tag{2.41}$$

By $2(1-\varepsilon)-(1+\varepsilon)=1-3\varepsilon>\frac{1}{4}$ and $\beta+\varepsilon<n$, we obtain from (2.41) a contradiction. Hence $\Psi_2(z)+H_{2Q}e^{2Q}\not\equiv 0$.

Case 2: Suppose now $a_n=cb_n$ $(0<c<1)$. Then for any ray $\arg z=\theta$, we have

$$\delta(P,\theta)=c\delta(Q,\theta),\quad \delta(2P,\theta)=2c\delta(Q,\theta),$$

$$\delta(P+Q,\theta)=(1+c)\delta(Q,\theta),\quad \delta(2Q,\theta)=2\delta(Q,\theta).$$

Then by Lemma 2.2 and Lemma 2.3, for any given ε $\left(0 < \varepsilon < \min\left(\frac{1-c}{4}, n-\beta\right)\right)$ there exist $E_j \subset [0, 2\pi)$ $(j = 0, 1, 2)$ that have linear measure zero, where E_0, E_1 and E_2 are defined as in the case 1 respectively. We take the ray $\arg z = \theta \in [0, 2\pi) \backslash E_1 \cup E_2 \cup E_0$ such that $\delta(Q, \theta) > 0$ and for sufficiently large $|z| = r$, we have (2.35)–(2.37) and

$$|H_P e^P| \le \exp\{(1+\varepsilon)c\delta(Q,\theta)r^n\}, \tag{2.42}$$

$$|H_{P+Q} e^{P+Q}| \le \exp\{(1+\varepsilon)(1+c)\delta(Q,\theta)r^n\}, \tag{2.43}$$

$$|H_{2P} e^{2P}| \le \exp\{(1+\varepsilon)2c\delta(Q,\theta)r^n\}. \tag{2.44}$$

If $\Psi_2(z) + H_{2Q}e^{2Q} \equiv 0$, then by (2.35)–(2.37) and (2.42)–(2.44) we have

$$\begin{aligned} &\exp\{(1-\varepsilon)2\delta(Q,\theta)r^n\} \\ &\quad \le |H_{2Q}e^{2Q}| \le \exp\{r^{\beta+\varepsilon}\} + 2\exp\{(1+\varepsilon)(1+c)\delta(Q,\theta)r^n\} \\ &\qquad + 2\exp\{(1+\varepsilon)2c\delta(Q,\theta)r^n\}. \end{aligned} \tag{2.45}$$

By $\beta + \varepsilon < n$ and $4\varepsilon < 1 - c$, we have as $r \to +\infty$

$$\frac{\exp\{r^{\beta+\varepsilon}\}}{\exp\{(1-\varepsilon)2\delta(Q,\theta)r^n\}} \to 0, \tag{2.46}$$

$$\frac{\exp\{(1+\varepsilon)(1+c)\delta(Q,\theta)r^n\}}{\exp\{(1-\varepsilon)2\delta(Q,\theta)r^n\}} \to 0, \tag{2.47}$$

$$\frac{\exp\{(1+\varepsilon)2c\delta(Q,\theta)r^n\}}{\exp\{(1-\varepsilon)2\delta(Q,\theta)r^n\}} \to 0. \tag{2.48}$$

By (2.45)–(2.48), we get $1 \le 0$. This is a contradiction, hence $\Psi_2(z) + H_{2Q}e^{2Q} \not\equiv 0$.

(iii) Set $\rho = \max\{\rho(\varphi), \rho(\phi)\} < \infty$. Then by Lemma 2.1, for any given $\varepsilon > 0$, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) - E$, then there is a constant $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \ge R$, we have

$$\left|\frac{\varphi^{(k)}(z)}{\varphi(z)}\right| \le |z|^{k(\rho-1+\varepsilon)}, \quad \left|\frac{\phi^{(k)}(z)}{\phi(z)}\right| \le |z|^{k(\rho-1+\varepsilon)} \quad (k = 1, 2). \tag{2.49}$$

It follows that on the ray $\arg z = \theta \in [0, 2\pi) - E$,

$$\frac{\varphi^{(k)}(z)}{\varphi(z)} H_j(z)e^j \quad (k = 1, 2;\, j \in \Lambda_2),$$

$$\frac{\varphi'(z)}{\varphi(z)}\frac{\phi'(z)}{\phi(z)} H_j(z)e^j \quad (j \in \Lambda_2), \quad \frac{\phi'(z)}{\phi(z)} H_j(z)e^j \quad (j \in \Lambda_2)$$

keep the properties of $H_j(z)e^j$ $(j \in \Lambda_2)$ which are defined as in (2.35), (2.37)–(2.40) or (2.35), (2.37), (2.42)–(2.44). By using similar reasoning to that in the proof of (ii), the proof of (iii) can be completed.

3 Proof of Theorem 1.1

Suppose that $f(z) \not\equiv 0$ is a meromorphic solution of equation (1.1) with $\lambda(1/f) < \infty$. Then by Theorem C we have $\rho(f) = \infty$. First we suppose that $d_2(z) \not\equiv 0$. Set $w = d_2 f'' + d_1 f' + d_0 f - \varphi$, then by Lemma 2.9 we have $\rho(w) = \rho(g) = \rho(f) = \infty$. In order to prove $\bar{\lambda}(g - \varphi) = \infty$, we need to prove $\bar{\lambda}(w) = \infty$. Substituting $f'' = -A_1 e^P f' - A_0 e^Q f$ into w, we get

$$w = (d_1 - d_2 A_1 e^P) f' + (d_0 - d_2 A_0 e^Q) f - \varphi. \tag{3.1}$$

Differentiating both sides of equation (3.1) and replacing f'' with $f'' = -A_1 e^P f' - A_0 e^Q f$, we obtain

$$\begin{aligned} w' = {} & [d_2 A_1^2 e^{2P} - ((d_2 A_1)' + P' d_2 A_1 + d_1 A_1) e^P - d_2 A_0 e^Q + d_0 + d_1'] f' \\ & + [d_2 A_0 A_1 e^{P+Q} - ((d_2 A_0)' + Q' d_2 A_0 + d_1 A_0) e^Q + d_0'] f - \varphi'. \end{aligned} \tag{3.2}$$

Set

$$\alpha_1 = d_1 - d_2 A_1 e^P, \quad \alpha_0 = d_0 - d_2 A_0 e^Q, \tag{3.3}$$

$$\begin{aligned} \beta_1 & = \alpha_1' + \alpha_0 - \alpha_1 A_1 e^P \\ & = d_2 A_1^2 e^{2P} - ((d_2 A_1)' + P' d_2 A_1 + d_1 A_1) e^P - d_2 A_0 e^Q + d_0 + d_1', \end{aligned} \tag{3.4}$$

$$\beta_0 = \alpha_0' - \alpha_1 A_0 e^Q = d_2 A_0 A_1 e^{P+Q} - ((d_2 A_0)' + Q' d_2 A_0 + d_1 A_0) e^Q + d_0'. \tag{3.5}$$

Then we have

$$\alpha_1 f' + \alpha_0 f = w + \varphi, \tag{3.6}$$

$$\beta_1 f' + \beta_0 f = w' + \varphi'. \tag{3.7}$$

Set

$$\begin{aligned} h & = \alpha_1 \beta_0 - \alpha_0 \beta_1 \\ & = (d_1 - d_2 A_1 e^P)[d_2 A_0 A_1 e^{P+Q} - ((d_2 A_0)' + Q' d_2 A_0 + d_1 A_0) e^Q + d_0'] \\ & \quad - (d_0 - d_2 A_0 e^Q)[d_2 A_1^2 e^{2P} - ((d_2 A_1)' + P' d_2 A_1 + d_1 A_1) e^P \\ & \quad - d_2 A_0 e^Q + d_0 + d_1']. \end{aligned} \tag{3.8}$$

Now check all the terms of h. Since the term $d_2^2 A_1^2 A_0 e^{2P+Q}$ is eliminated, by (3.8) we can write $h = \Psi_2(z) - d_2^2 A_0^2 e^{2Q}$, where $\Psi_2(z)$ is defined as in Lemma 2.10 (ii). By $d_2 \not\equiv 0$, $A_0 \not\equiv 0$ and Lemma 2.10 (ii) we see that $h \not\equiv 0$. By (3.6), (3.7) and (3.8), we obtain

$$hf = \alpha_1(w' + \varphi') - \beta_1(w + \varphi), \tag{3.9}$$

$$hf' = -\alpha_0(w' + \varphi') + \beta_0(w + \varphi). \tag{3.10}$$

Differentiating both sides of equation (3.10) we obtain

$$(hf')' = -\alpha_0(w'' + \varphi'') + (\beta_0 - \alpha_0')(w' + \varphi') + \beta_0'(w + \varphi). \tag{3.11}$$

On the other hand by (1.1), (3.9) and (3.10)

$$\begin{aligned}(hf')' &= h'f' + hf'' = (h' - hA_1e^P)f' - hA_0e^Qf \\ &= \left(\frac{h'}{h} - A_1e^P\right)(-\alpha_0(w' + \varphi') + \beta_0(w + \varphi)) \\ &\quad - A_0e^Q(\alpha_1(w' + \varphi') - \beta_1(w + \varphi)).\end{aligned} \tag{3.12}$$

By (3.11), (3.12) we get

$$\begin{aligned}&\alpha_0(w'' + \varphi'') - \alpha_0\left(\frac{h'}{h} - A_1e^P\right)(w' + \varphi') \\ &\quad + \left[\beta_0\left(\frac{h'}{h} - A_1e^P\right) + \beta_1A_0e^Q - \beta_0'\right](w + \varphi) = 0.\end{aligned} \tag{3.13}$$

Hence by (3.3), (3.4), (3.5), (3.13) we have

$$\begin{aligned}&\alpha_0w'' + \Phi_1w' + \Phi_0w \\ &\quad = -\left[\alpha_0\varphi'' - \alpha_0\left(\frac{h'}{h} - A_1e^P\right)\varphi' + \left(\beta_0\left(\frac{h'}{h} - A_1e^P\right) + \beta_1A_0e^Q - \beta_0'\right)\varphi\right] \\ &\quad = -\left[\alpha_0\varphi'' - \alpha_0\left(\frac{h'}{h} - A_1e^P\right)\varphi' \right. \\ &\qquad \left. - \left(\beta_0\left(\frac{h'}{h} - A_1e^P\right) + (\alpha_1' + \alpha_0 - \alpha_1A_1e^P)A_0e^Q - \beta_0'\right)\varphi\right] \\ &\quad = -\left[\alpha_0\varphi'' - \alpha_0\left(\frac{h'}{h} - A_1e^P\right)\varphi' - \left(\beta_0\frac{h'}{h} - \alpha_0'A_1e^P + (\alpha_1' + \alpha_0)A_0e^Q - \beta_0'\right)\varphi\right] \\ &\quad = F,\end{aligned} \tag{3.14}$$

where $\Phi_1(z)$ and $\Phi_0(z)$ are meromorphic functions with $\rho(\Phi_1) \le n$, $\rho(\Phi_0) \le n$. By (3.14) we can write

$$\frac{F}{\varphi(z)} = -\frac{\varphi''(z)}{\varphi(z)}\Psi_{24}(z) + \frac{h'(z)}{h(z)}\frac{\varphi'(z)}{\varphi(z)}\Psi_{24}(z) + \frac{\varphi'(z)}{\varphi(z)}\Psi_{23}(z) + \frac{h'(z)}{h(z)}\Psi_{22}(z) + \Psi_{21}(z) - d_2A_0^2e^{2Q},$$

where

$$\Psi_{24}(z) = \alpha_0 = d_0 - d_2A_0e^Q,$$

$$\Psi_{23}(z) = -A_1e^P\alpha_0 = d_2A_0A_1e^{P+Q} - d_0A_1e^P,$$

$$\Psi_{22}(z) = \beta_0 = d_2A_0A_1e^{P+Q} - ((d_2A_0)' + Q'd_2A_0 + d_1A_0)e^Q + d_0',$$

$$\begin{aligned}\Psi_{21}(z) &= A_0e^Q\alpha_1' - A_1e^P\alpha_0' - \beta_0' + d_0A_0e^Q \\ &= -d_0'A_1e^P + [A_0d_1' + (d_2A_0)'' + (Q'd_2A_0)' + (d_1A_0)' + Q'(d_2A_0)' \\ &\quad + (Q')^2d_2A_0 + d_1A_0Q' + d_0A_0]e^Q \\ &\quad + [(d_2A_0)'A_1 + Q'd_2A_1A_0 - A_0(d_2A_1)' - P'd_2A_1A_0 - (d_2A_1A_0)' \\ &\quad - (P' + Q')d_2A_0A_1]e^{P+Q} - d_0''.\end{aligned}$$

By $\rho(h) \le n$, $d_2 \not\equiv 0$, $A_0 \not\equiv 0$, $\rho(\varphi) < \infty$ and Lemma 2.10 (iii) we see that $F \not\equiv 0$ and by Lemma 2.4, we obtain $\overline{\lambda}(w) = \rho(w) = \infty$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$. Using a similar reasoning as above we get $\overline{\lambda}(w) = \rho(w) = \infty$.

Finally, if $d_2 \equiv 0$, $d_1 \equiv 0$, $d_0 \not\equiv 0$ then we have $w = d_0f - \varphi$, $\rho(w) = \infty$. By substituting

$$f = \frac{w}{d_0} + \frac{\varphi}{d_0}, \quad f' = \left(\frac{w}{d_0}\right)' + \left(\frac{\varphi}{d_0}\right)', \quad f'' = \left(\frac{w}{d_0}\right)'' + \left(\frac{\varphi}{d_0}\right)'' \tag{3.15}$$

into equation (1.1) we obtain

$$\left(\frac{w}{d_0}\right)'' + A_1e^P\left(\frac{w}{d_0}\right)' + A_0e^Q\frac{w}{d_0} = -\left(\left(\frac{\varphi}{d_0}\right)'' + A_1e^P\left(\frac{\varphi}{d_0}\right)' + A_0e^Q\frac{\varphi}{d_0}\right). \tag{3.16}$$

By $\varphi(z)$ being a meromorphic function of finite order and $d_0(z)$ is a polynomial, then $\frac{\varphi(z)}{d_0(z)}$ has finite order and by Theorem C we have

$$\left(\frac{\varphi}{d_0}\right)'' + A_1 e^P \left(\frac{\varphi}{d_0}\right)' + A_0 e^Q \frac{\varphi}{d_0} \not\equiv 0. \qquad (3.17)$$

Hence by Lemma 2.4, we have $\bar{\lambda}\left(\frac{w}{d_0}\right) = \rho\left(\frac{w}{d_0}\right) = \infty$ (d_0 is a polynomial). Then $\bar{\lambda}(w) = \infty$, i.e., $\bar{\lambda}(d_0 f - \varphi) = \infty$.

4 Proof of Theorem 1.2

Suppose that $f(z) \not\equiv 0$ is a meromorphic solution of equation (1.1). Then by Theorem C we have $\rho(f) = \rho(f') = \rho(f'') = \infty$. Since $\rho(\varphi) < \infty$, then $\rho(f - \varphi) = \rho(f' - \varphi) = \rho(f'' - \varphi) = \infty$. By using similar reasoning to that in the proof of Theorem 1.1, the proof of Theorem 1.2 can be completed.

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References

[1] B. Belaïdi, On the meromorphic solutions of linear differential equations, J. Syst. Sci. and Complexity (2007) 20: 41–46.
[2] Z. X. Chen and K. H. Shon, The relation between solutions of a class of second order differential equations with functions of small growth, Chinese Ann. Math. 2006, 27A (4): 431–442. (In Chinese)
[3] Z. X. Chen and K. H. Shon, On the growth and fixed points of solutions of second order differential equations with meromorphic coefficients, Acta Math. Sinica Engl. Ser., Vol. **21**, N°4 (2005), 753–764.
[4] Z. X. Chen and C. C. Yang, Some oscillation theorems for linear differential equations with meromorphic coefficients, Southeast Asian Bull. Math., **23** (1999), 409–417.
[5] Z. X. Chen, Zeros of meromorphic solutions of higher order linear differential equations, Analysis, **14** (1994), 425–438.
[6] G. G. Gundersen, Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates, J. London Math. Soc., (2) **37** (1988), 88–104.
[7] G. G. Gundersen, Finite order solutions of second order linear differential equations, Trans. Amer. Math. Soc. **305** (1988), 415–429.
[8] W. K. Hayman, The local growth of power series: a survey of the Wiman-Valiron method, Canad. Math. Bull., **17** (1974), 317–358.
[9] W. K. Hayman, Meromorphic functions, Clarendon Press, Oxford, 1964.
[10] G. Jank and L. Volkmann, Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, Birkhäuser, Basel-Boston, 1985.
[11] K. H. Kwon, Nonexistence of finite order solutions of certain second order linear differential equations, Kodai Math. J., **19** (1996), 378–387.
[12] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, New York, 1993.
[13] R. Nevanlinna, Eindeutige Analytische Funktionen, Zweite Auflage. Reprint. Die Grundlehren der mathematischen Wissenschaften, Band 46. Springer-Verlag, Berlin-New York, 1974.

[14] G. Valiron, Lectures on the General Theory of Integral Functions, translated by E. F. Collingwood, Chelsea, New York, 1949.

Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem
B. P. 227 Mostaganem-(Algeria)
belaidi@univ-mosta.dz
belaidi.benharrat@caramail.com