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A MATSUMOTO-TYPE THEOREM FOR LINEAR GROUPS OVER SOME COMPLETED QUANTUM TORI

By

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0. Introduction

Let $K = F((X_1))$ be a field of formal power series in variable X_1 over an arbitrary field F. We fix an element $q \in F^{\times}$, and let $K_q = K[X_2, X_2^{-1}]$ be the ring of Laurent polynomials in variable X_2 over K with the relation $X_2 X_1 = q X_1 X_2$ (cf. Section 1). We call this K_q the completed quantum torus associated with $q \in F^{\times}$. For $\dot{\ell} \in \mathbf{Z}_{\geq 2}$, we let $A_{\dot{\ell}-1}$ be a Cartan matrix with simple roots $\Pi = \{\alpha_1, \dots, \alpha_{\dot{\ell}-1}\}$, and let $A_{\dot{\ell}-1}^{(1)}$ be an affine Cartan matrix of tier number 1 with affine simple roots $\Pi_1 = \{a_1 = (\alpha_1, 0), \dots, a_{\dot{\ell}-1} = (\alpha_{\dot{\ell}-1}, 0), a_0 = (-\alpha_0, 1)\}$, where α_0 is the highest root of the root system of type $A_{\dot{\ell}-1}$ with respect to Π. Let $M(\dot{\ell}, K_q)$ be the ring of $\dot{\ell} \times \dot{\ell}$ matrices with entries in K_q, and we let $GL(\dot{\ell}, K_q)$ be the multiplicative group of $M(\dot{\ell}, K_q)$. Then we can construct the elementary subgroup $E(A_{\dot{\ell}-1}^{(1)}, K)_q$ of $GL(\dot{\ell}, K_q)$, and the affine Steinberg group $St(A_{\dot{\ell}-1}^{(1)}, K)_q$ associated with $q \in F^{\times}$. Let $K_2(A_{\dot{\ell}-1}^{(1)}, K)_q$ be the kernel of the canonical homomorphism of $St(A_{\dot{\ell}-1}^{(1)}, K)_q$ onto $E(A_{\dot{\ell}-1}^{(1)}, K)_q$, and we have the fact that $K_2(A_{\dot{\ell}-1}^{(1)}, K)_q$ is central (cf. [17]). Using these notations, we obtain the main result below:

Theorem. *$K_2(A_{\dot{\ell}-1}^{(1)}, K)_q$ is isomorphic to the abelian group L generated by the symbols $c_a(u, v)$ and $d(w)$ for all $a \in \Pi_1$, $u, v \in K^{\times}$ and $w \in K_{q, X_2}^{\times} = \langle u \in K^{\times} \mid X_2 u X_2^{-1} = u \rangle$ with the following defining relations*:

(L1) $c_a(u, v) c_a(uv, t) = c_a(u, vt) c_a(v, t)$

(L2) $c_a(1, 1) = 1$

(L3) $c_a(u, v) = c_a(u^{-1}, v^{-1})$

(L4) $c_a(u, v) = c_a(u, (1 - u)v)$ with $u \neq 1$

(L5) $c_a(u, v(ab)) = c_b(u(ba), v)$

(L6) $c_{ab}(u, v)$ *is bimultiplicative*

(LD) $d(w) d(x) = d(wx) c_{a_1}(w, x) c_{a_0}(x, w) = d(wx) c_{a_1}(x, w) c_{a_0}(w, x)$

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for all $a = (\alpha, m)$, $b = (\beta, n) \in \Pi_1$, $u, v, t \in K^\times$ *and* $w, x \in K_{q,X_2}^\times$, *where* $c_{ab}(u, v) = c_a(u, v(ab)) = c_b(u(ba), v)$ *and the symbol* $u(ab)$ *is equal to* $u^{-1}X_2^{m-n}u^{-1}X_2^{n-m}$ *if* $\dot{\ell} = 2$ *and* $(a, b) = (a_0, a_1)$, *or* $X_2^{m-n}u^{-1}X_2^{n-m}$ *if* $\dot{\ell} \geq 3$ *and* $(a, b) = (a_0, a_1)$, *or* $u^{\langle \alpha, \beta \rangle}$ *otherwise.*

1. Completed Quantum Tori

Let F be a field (of any characteristic). We fix an element q of $F^\times$. Let $K = F((X_1))$ be the field of formal power series in X_1 over F, that is, $K = \{\sum_{j=m}^{\infty} a_j X_1^j \mid m \in \mathbf{Z}, a_j \in F\}$, and let $K_q = K[X_2, X_2^{-1}]$ be the (not necessarily commutative) ring of Laurent polynomials in X_2 over K with $X_2 X_1 = q X_1 X_2$, that is, $K_q = \{\sum_{i=k}^{\ell} a_i(X_1) X_2^i \mid k, \ell \in \mathbf{Z}, k \leq \ell, a_i(X_1) \in K\}$.

We call K_q the completed quantum torus associated with $q \in F^\times$. If $a(X_1) \in K$ and $i \in \mathbf{Z}$, then we have $X_2^i a(X_1) = a(q^i X_1) X_2^i$. In general, we obtain

$$\left(\sum_{i=k_1}^{\ell_1} a_i(X_1) X_2^i\right)\left(\sum_{j=k_2}^{\ell_2} b_j(X_1) X_2^j\right) = \sum_{i=k_1}^{\ell_1} \sum_{j=k_2}^{\ell_2} a_i(X_1) b_j(q^i X_1) X_2^{i+j}$$
$$= \sum_{m=k_1+k_2}^{\ell_1+\ell_2} \left(\sum_{i=k_1}^{m-k_2} a_i(X_1) b_{m-i}(q^i X_1)\right) X_2^m.$$

Using the spread of degrees in X_2, we find that K_q is a Euclidean ring and that K_q has no (nonzero) zero-divisor.

2. General Linear Groups

Let $M(\dot{\ell}, K_q)$ be the ring of $\dot{\ell} \times \dot{\ell}$ matrices whose entries are in K_q, and we set $GL(\dot{\ell}, K_q) = M(\dot{\ell}, K_q)^\times$, the multiplicative group of $M(\dot{\ell}, K_q)$.

Let $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq \dot{\ell}\}$ be a root system of type $A_{\dot{\ell}-1}$, where the ε_i give an orthonormal basis of a certain Euclidean space with an inner product $(\cdot, \cdot)$, and let $\Pi = \{\alpha_1, \ldots, \alpha_{\dot{\ell}-1}\}$ be a simple system of Φ, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. We put $\Phi^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 1 \leq i \leq j \leq \dot{\ell} - 1\}$, the set of positive roots, and $\Phi^- = -\Phi^+$, the set of negative roots, and hence $\Phi = \Phi^- \cup \Phi^+$. Then $\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_{\dot{\ell}-1}$ is the highest root of Φ with respect to Π. The associated abstract affine (real) root system is defined by $\Phi_1 = \Phi \times \mathbf{Z}$. As simple roots of Φ_1, we choose $a_1 = (\alpha_1, 0)$, $a_2 = (\alpha_2, 0), \ldots, a_{\dot{\ell}-1} = (\alpha_{\dot{\ell}-1}, 0)$, $a_0 = (-\alpha_0, 1)$, that is, $\Pi_1 = \{a_1, a_2 \ldots, a_{\dot{\ell}-1}, a_0\}$ is a simple system of Φ_1. Let $\Phi_1^+ = (\Phi^+ \times \mathbf{Z}_{\geq 0}) \cup (\Phi^- \times \mathbf{Z}_{>0})$ and $\Phi_1^- = (\Phi^+ \times \mathbf{Z}_{<0}) \cup (\Phi^- \times \mathbf{Z}_{\leq 0})$, which are called positive roots and negative roots of Φ_1 respectively. Sometimes we write $(\varepsilon_i - \varepsilon_j, m) = (ij, m)$ simply. For each $\alpha \in \Phi$, we define

$$e_\alpha = \begin{cases} E_{ij} & \text{if } \alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} = \varepsilon_i - \varepsilon_j \quad (i < j), \\ E_{ji} & \text{if } \alpha = -\alpha_i - \alpha_{i+1} - \cdots - \alpha_{j-1} = \varepsilon_j - \varepsilon_i \quad (i > j), \end{cases}$$

where E_{ij} is the matrix unit with 1 in the (i,j) position and 0 elsewhere. For $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$ and $r \in K_q$, we put $x_\alpha(r) = x_{ij}(r) = I_{\dot{\ell}} + re_\alpha$, where $I_{\dot{\ell}} = E_{11} + E_{22} + \cdots + E_{\dot{\ell}\dot{\ell}}$ is the identity matrix. Then the elementary subgroup $E(A_{\dot{\ell}-1}, K_q)$ is defined to be the subgroup of $GL(\dot{\ell}, K_q)$ generated by $x_\alpha(r)$ for all $\alpha \in \Phi$ and $r \in K_q$.

In a standard way, the Weyl group W of Φ is generated by σ_α for all $\alpha \in \Phi$, where σ_α is the reflection along with α. Then the associated affine Weyl group W_1 is generated by σ_a for all $a = (\alpha, m) \in \Phi_1$, where $\sigma_a(b) = \left(\sigma_\alpha \beta, n - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} m\right)$ for $a = (\alpha, m)$, $b = (\beta, n) \in \Phi_1$. We call W_1 the affine Weyl group of Φ. Usually Φ is identified with $\Phi \times \{0\}$ in Φ_1.

Now we define an automorphism f_n of K by $f_n(r) = \begin{cases} r & \text{if } n \geq 0 \\ X_2^n r X_2^{-n} & \text{otherwise} \end{cases}$ for all $r \in K$ and $n \in \mathbf{Z}$. For example, we have $f_n^{-1} \circ f_{-n}(r) = X_2^{-n} r X_2^n$ and $f_{-n}^{-1} \circ f_n(r) = X_2^n r X_2^{-n}$. And sometimes we write $r_n = X_2^n r X_2^{-n}$ for convenience.

Using these f_n, we will consider the elementary subgroup $E(A_{\dot{\ell}-1}^{(1)}, K)_q$, which is defined to be the subgroup of $GL(\dot{\ell}, K_q)$ generated by $x_a(r) = x_\alpha(f_m(r) X_2^m) = I_{\dot{\ell}} + f_m(r) X_2^m e_\alpha$ for all $a = (\alpha, m) \in \Phi_1$ and $r \in K$, and we have $E(A_{\dot{\ell}-1}^{(1)}, K)_q = E(A_{\dot{\ell}-1}, K_q)$.

For $a = (\alpha, m) \in \Phi_1$, $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$, $r \in K$ and $u \in K^\times$, we define the following symbols:

$$w_a(u) = x_a(u) x_{-a}(-u^{-1}) x_a(u) \ \ (= w_\alpha(f_m(u) X_2^m)),$$

$$h_a(u) = w_a(u) w_a(-1) \ \ (= \operatorname{diag}(1, \ldots, 1, \underbrace{f_m(u)}_{i\text{-th}}, 1, \ldots, 1, \underbrace{f_m(u_{-m}^{-1})}_{j\text{-th}}, 1, \ldots, 1)).$$

Then we put

$$E = E(A_{\dot{\ell}-1}^{(1)}, K)_q = E(A_{\dot{\ell}-1}, K_q),$$

$$U_a = \langle x_a(r) \mid r \in K \rangle \quad \text{for all } a \in \Phi_1,$$

$$U = \langle U_a \mid a \in \Phi_1^+ \rangle,$$

$$Y_a = \langle x_a(r) U_b x_a(-r) \mid r \in K, b \in \Phi_1^+ \backslash \{a\} \rangle \quad \text{for each } a \in \Pi_1,$$

$$N = \langle w_a(u) \mid a \in \Phi_1, u \in K^\times \rangle,$$

$$T = \langle h_a(u) \mid a \in \Phi_1, u \in K^\times \rangle,$$

$$B = \langle U, T \rangle,$$

$$S = \{ w_a(1) \bmod T \mid a \in \Pi_1 \}.$$

Sometimes we identify S with $\{w_a(1) \mid a \in \Pi_1\}$. Then, we have the following results as in [17].

LEMMA 2.1. *Notation is as above and let* $a \in \Pi_1$. *Then*:
(1) $B = U \rtimes T$.
(2) $T \triangleleft N$.
(3) $B \cap N = T$.
(4) $N/T \simeq W_1$.
(5) $(N/T, S)$ *is a Coxeter system.*
(6) $U = Y_a \rtimes U_a$.
(7) $w_a(u) Y_a w_a(-u) = Y_a$ for all $u \in K^\times$.

PROPOSITION 2.2. *Notation is as above. Then*, (E, B, N, S) *is a Tits system with the corresponding affine Weyl group* W_1. *In particular, we have* $E = \bigcup_{w \in W_1} BwB$ (*Bruhat decomposition*).

3. Affine Steinberg Groups

Let $St(A_{\dot{\ell}-1}, K_q)$ be the Steinberg group over K_q, which is defined by the generators $\hat{x}_{ij}(y)$ for all $1 \le i \ne j \le \dot{\ell}$ and $y \in K_q$ and the defining relations:

(RA) $\hat{x}_{ij}(y)\hat{x}_{ij}(z) = \hat{x}_{ij}(y+z)$

(RB) $$[\hat{x}_{ij}(y), \hat{x}_{kl}(z)] = \begin{cases} \hat{x}_{il}(yz) & \text{if } j = k, \\ \hat{x}_{kj}(-zy) & \text{if } i = l, \\ 1 & \text{otherwise} \end{cases}$$

for all $1 \le i \ne j \le \dot{\ell}$ and $1 \le k \ne l \le \dot{\ell}$ with $(i,j) \ne (k,l)$, and for all $y, z \in K_q$. Exactly this definition is valid for $\dot{\ell} \ge 3$. If $\dot{\ell} = 2$, then we should replace (RB) by the following (RB′):

(RB′) $\hat{w}_{ij}(t)\hat{x}_{ij}(y)\hat{w}_{ij}(-t) = \hat{x}_{ji}(-t^{-1}yt^{-1})$

for all i, j with $\{i,j\} = \{1,2\}$, that is $(i,j) = (1,2)$ or $(2,1)$, and for all $y \in K_q$ and $t \in K_q^\times$, where $\hat{w}_{ij}(t) = \hat{x}_{ij}(t)\hat{x}_{ji}(-t^{-1})\hat{x}_{ij}(t)$.

Next, we let $St(A^{(1)}_{\ell-1}, K)_q$ be the affine Steinberg group associated with $q \in F^\times$, which is defined by the generators $\hat{x}_a(r)$ for all $a \in \Phi_1$ and $r \in K$ with the defining relations:

(A) $\hat{x}_a(r)\hat{x}_a(s) = \hat{x}_a(r+s)$

(A′) $[\hat{x}_{(\alpha,m)}(r), \hat{x}_{(\alpha,n)}(s)] = 1$

(B) $$[\hat{x}_{(ij,m)}(r), \hat{x}_{(kl,n)}(s)] = \begin{cases} \hat{x}_{(il,n+m)}(f_{n+m}^{-1}(f_m(r)f_n(s_m))) & \text{if } j = k, \\ \hat{x}_{(kj,n+m)}(-f_{n+m}^{-1}(f_m(r_n)f_n(s))) & \text{if } i = l, \\ 1 & \text{otherwise} \end{cases}$$

for all $a \in \Phi_1$, $r, s \in K$, $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$, $m, n \in \mathbf{Z}$, $1 \le i \ne j \le \dot{\ell}$ and $1 \le k \ne l \le \dot{\ell}$ with $(i,j) \ne (k,l)$. Exactly this definition is valid for $\dot{\ell} \ge 3$. If $\dot{\ell} = 2$, then we should replace (B) by the following (B′):

(B′) $\hat{w}_{(\alpha,m)}(u)\hat{x}_{(\alpha,n)}(r)\hat{w}_{(\alpha,m)}(u)^{-1} = \hat{x}_{(-\alpha,n-2m)}(-f_{n-2m}^{-1}(f_{-m}(u^{-1}u_{n-m}^{-1})f_n(r_{-m})))$

for all $\alpha \in \Phi$, $r \in K$ and $u \in K^\times$, where $\hat{w}_a(u) = \hat{x}_a(u)\hat{x}_{-a}(-u^{-1})\hat{x}_a(u)$. We note that (B′) holds in case of $\dot{\ell} \ge 3$. In fact, if we choose an index k different from both i and j, such that $\hat{x}_{(ij,n)}(r) = [\hat{x}_{(ik,n)}(r), \hat{x}_{(kj,0)}(1)]$, then we have

$$\begin{aligned}
&\hat{w}_{(ij,m)}(u)\hat{x}_{(ij,n)}(r)\hat{w}_{(ij,m)}(u)^{-1}\\
&\quad= \hat{x}_{(ij,m)}(u)\hat{x}_{(ji,-m)}(-u^{-1})[\hat{x}_{(ik,n)}(r), \hat{x}_{(kj,0)}(1)]\hat{x}_{(ji,-m)}(u^{-1})\hat{x}_{(ij,m)}(-u)\\
&\quad= \hat{x}_{(ij,m)}(u)[\hat{x}_{(jk,n-m)}(-f_{n-m}^{-1}(f_{-m}(u^{-1})f_n(r_{-m})))\hat{x}_{(ik,n)}(r), \hat{x}_{(ki,-m)}(u^{-1})\hat{x}_{(kj,0)}(1)]\\
&\qquad\times \hat{x}_{(ij,m)}(-u)\\
&\quad= [\hat{x}_{(ik,n)}(-r)\hat{x}_{(jk,n-m)}(-f_{n-m}^{-1}(f_{-m}(u^{-1})f_n(r_{-m})))\hat{x}_{(ik,n)}(r),\\
&\qquad \hat{x}_{(kj,0)}(-1)\hat{x}_{(ki,-m)}(u^{-1})\hat{x}_{(kj,0)}(1)]\\
&\quad= [\hat{x}_{(jk,n-m)}(-f_{n-m}^{-1}(f_{-m}(u^{-1})f_n(r_{-m}))), \hat{x}_{(ki,-m)}(u^{-1})]\\
&\quad= \hat{x}_{(ji,n-2m)}(-f_{n-2m}^{-1}(f_{-m}(u^{-1}u_{n-m}^{-1})f_n(r_{-m}))).
\end{aligned}$$

For all $y \in K_q$, we can write $y = r_0X_2^k + \cdots + r_lX_2^{k+l}$ uniquely, where $k \in \mathbf{Z}$, $l \in \mathbf{Z}_{\ge 0}$ and $r_0, \ldots, r_l \in K$. Then, there is a natural homomorphism χ of $St(A_{\dot{\ell}-1}, K_q)$ onto $St(A_{\dot{\ell}-1}^{(1)}, K)_q$ with $\chi(\hat{x}_{ij}(r_0X_2^k + \cdots + r_lX_2^{k+l})) = \hat{x}_{(ij,k)}(f_k^{-1}(r_0)) \cdots \hat{x}_{(ij,k+l)}(f_{k+l}^{-1}(r_l))$ for all $1 \le i \ne j \le \dot{\ell}$, $k \in \mathbf{Z}$, $l \in \mathbf{Z}_{\ge 0}$ and $r_0, \ldots, r_l \in K$. Hence, we have the proposition below.

Proposition 3.1. *Notation is as above. Then, we have* $St(A_{\dot{\ell}-1}^{(1)}, K)_q \simeq St(A_{\dot{\ell}-1}, K_q)$.

Proof. We can define a homomorphism $\chi^{-1} : St(A_{\dot{\ell}-1}^{(1)}, K)_q \to St(A_{\dot{\ell}-1}, K_q)$ with $\chi^{-1}(\hat{x}_{(ij,m)}(r)) = \hat{x}_{ij}(f_m(r)X_2^m)$ for all $(ij,m) \in \Phi_1$ and $r \in K$. Then we should check the following:

① χ(left hand side of (RA) (resp. (RB), (RB′))) $= \chi$(right hand side of (RA) (resp. (RB), (RB′))),

② χ^{-1}(left hand side of (A) (resp. (A′), (B), (B′))) $= \chi^{-1}$(right hand side of (A) (resp. (A′), (B), (B′))).

By the definitions, it is easy to prove ② and the case of not (RB) in ①. Hence we should check χ(left hand side of (RB)) $= \chi$(right hand side of (RB)). For each $y, z \in K_q$, we have $y = y_0 X_2^m + \cdots + y_{\ell_1} X_2^{m+\ell_1}$, $z = z_0 X_2^n + \cdots + z_{\ell_2} X_2^{n+\ell_2}$ for $y_0, \ldots, y_{\ell_1}, z_0, \ldots, z_{\ell_2} \in K$, $m, n \in \mathbf{Z}$ and $\ell_1, \ell_2 \in \mathbf{Z}_{\geq 0}$. Then we have χ(left hand side of (RB)) $= \chi$(right hand side of (RB)) if $j = k$ as follows:

$$\begin{aligned}
\chi([\hat{x}_{ij}(y), \hat{x}_{jk}(z)]) &= [\hat{x}_{(ij,m)}(f_m^{-1}(y_0)) \cdots \hat{x}_{(ij,m+\ell_1)}(f_{m+\ell_1}^{-1}(y_{\ell_1})), \\
&\qquad \hat{x}_{(jk,n)}(f_n^{-1}(z_0)) \cdots \hat{x}_{(jk,n+\ell_2)}(f_{n+\ell_2}^{-1}(z_{\ell_2}))] \\
&= \hat{x}_{(ik,m+n)}(f_{m+n}^{-1}(y_0 X_2^m z_0 X_2^{-m})) \\
&\quad \times \hat{x}_{(ik,m+n+1)}(f_{m+n+1}^{-1}(y_0 X_2^m z_1 X_2^{-m} + y_1 X_2^{m+1} z_0 X_2^{-m-1})) \\
&\qquad \vdots \\
&\quad \times \hat{x}_{(ik,m+n+\ell_1+\ell_2)}(f_{m+n+\ell_1+\ell_2}^{-1}(y_{\ell_1} X_2^{m+\ell_1} z_{\ell_2} X_2^{-m-\ell_1})) \\
&= \chi(\hat{x}_{ik}(yz)).
\end{aligned}$$

Hence, a similar calculation yields our desired result in other case of (RB). q.e.d.

Similarly we put $\hat{h}_a(u) = \hat{w}_a(u)\hat{w}_a(-1)$ for all $a \in \Phi_1$ and $u \in K^\times$. Then, we obtain the lemma below by direct calculation.

LEMMA 3.2. *Let* $a \in \Phi_1$, $m, n \in \mathbf{Z}$, $r \in K$, $u, v \in K^\times$, $\alpha, \beta \in \Phi$, *where* α *and* β *are written as* $\alpha = \varepsilon_i - \varepsilon_j$ *and* $\beta = \varepsilon_k - \varepsilon_l$ *with* $(i, j) \neq (k, l)$, $1 \leq i \neq j \leq \dot{\ell}$ *and* $1 \leq k \neq l \leq \dot{\ell}$. *Then the following relations hold.*

(1) $\hat{w}_a(u)^{-1} = \hat{w}_a(-u), \quad \hat{w}_a(u) = \hat{w}_{-a}(-u^{-1})$.

(2) $\hat{w}_{(\alpha,m)}(u)\hat{x}_{(\beta,n)}(r)\hat{w}_{(\alpha,m)}(-u)$

$$= \begin{cases}
\hat{x}_{(\mp\alpha, n\mp 2m)}(-f_{n\mp 2m}^{-1}(f_{\mp m}(u^{\mp 1}u_{n\mp m}^{\mp 1})f_n(r_{\mp m}))) & \text{if } \beta = \pm\alpha, \\
\hat{x}_{(\beta,n)}(r) & \text{if } (\alpha,\beta) = 0, \\
\hat{x}_{(il,n+m)}(f_{n+m}^{-1}(f_m(u)f_n(r_m))) & \text{if } \alpha \pm \beta \neq 0,\ k = j, \\
\hat{x}_{(kj,n+m)}(-f_{n+m}^{-1}(f_m(u_n)f_n(r))) & \text{if } \alpha \pm \beta \neq 0,\ l = i, \\
\hat{x}_{(jl,n-m)}(-f_{n-m}^{-1}(f_{-m}(u^{-1})f_n(r_{-m}))) & \text{if } \alpha \pm \beta \neq 0,\ k = i, \\
\hat{x}_{(ki,n-m)}(f_{n-m}^{-1}(f_{-m}(u_n^{-1})f_n(r))) & \text{if } \alpha \pm \beta \neq 0,\ l = j.
\end{cases}$$

(3) $\hat{w}_{(\alpha,m)}(u)\hat{w}_{(\beta,n)}(v)\hat{w}_{(\alpha,m)}(-u)$

$$= \begin{cases} \hat{w}_{(\mp\alpha,n\mp 2m)}(-f_{n\mp 2m}^{-1}(f_{\mp m}(u^{\mp 1}u_{n\mp m}^{\mp 1})f_n(v_{\mp m}))) & \text{if } \beta = \pm\alpha, \\ \hat{w}_{(\beta,n)}(v) & \text{if } (\alpha,\beta) = 0, \\ \hat{w}_{(il,n+m)}(f_{n+m}^{-1}(f_m(u)f_n(v_m))) & \text{if } \alpha \pm \beta \neq 0,\ k = j, \\ \hat{w}_{(kj,n+m)}(-f_{n+m}^{-1}(f_m(u_n)f_n(v))) & \text{if } \alpha \pm \beta \neq 0,\ l = i, \\ \hat{w}_{(jl,n-m)}(-f_{n-m}^{-1}(f_{-m}(u^{-1})f_n(v_{-m}))) & \text{if } \alpha \pm \beta \neq 0,\ k = i, \\ \hat{w}_{(ki,n-m)}(f_{n-m}^{-1}(f_{-m}(u_n^{-1})f_n(v))) & \text{if } \alpha \pm \beta \neq 0,\ l = j. \end{cases}$$

(4) $\hat{h}_a(u) = \hat{h}_{-a}(u)^{-1}$.

(5) $\hat{w}_{(\alpha,m)}(u)\hat{h}_{(\beta,n)}(v)\hat{w}_{(\alpha,m)}(-u)$

$$= \begin{cases} \hat{h}_{(\mp\alpha,n\mp 2m)}(-f_{n\mp 2m}^{-1}(f_{\mp m}(u^{\mp 1}u_{n\mp m}^{\mp 1})f_n(v_{\mp m})))\hat{h}_{(\mp\alpha,n\mp 2m)} \\ \quad \times (-f_{n\mp 2m}^{-1}f_{\mp m}(u^{\mp 1}u_{n\mp m}^{\mp 1}))^{-1} \quad \text{if } \beta = \pm\alpha, \\ \hat{h}_{(\beta,n)}(v) \quad \text{if } (\alpha,\beta) = 0, \\ \hat{h}_{(il,n+m)}(f_{n+m}^{-1}(f_m(u)f_n(v_m)))\hat{h}_{(il,n+m)}(f_{n+m}^{-1}f_m(u))^{-1} \\ \quad \text{if } \alpha \pm \beta \neq 0,\ k = j, \\ \hat{h}_{(kj,n+m)}(-f_{n+m}^{-1}(f_m(u_n)f_n(v)))\hat{h}_{(kj,n+m)}(-f_{n+m}^{-1}f_m(u_n))^{-1} \\ \quad \text{if } \alpha \pm \beta \neq 0,\ l = i, \\ \hat{h}_{(jl,n-m)}(-f_{n-m}^{-1}(f_{-m}(u^{-1})f_n(v_{-m})))\hat{h}_{(jl,n-m)}(-f_{n-m}^{-1}f_{-m}(u^{-1}))^{-1} \\ \quad \text{if } \alpha \pm \beta \neq 0,\ k = i, \\ \hat{h}_{(ki,n-m)}(f_{n-m}^{-1}(f_{-m}(u_n^{-1})f_n(v)))\hat{h}_{(ki,n-m)}(f_{n-m}^{-1}f_{-m}(u_n^{-1}))^{-1} \\ \quad \text{if } \alpha \pm \beta \neq 0,\ l = j. \end{cases}$$

(6) $\hat{h}_{(\alpha,m)}(u)\hat{x}_{(\beta,n)}(r)\hat{h}_{(\alpha,m)}(u)^{-1}$

$$= \begin{cases} \hat{x}_{(\beta,n)}(rf_n^{-1}f_{\pm m}(u^{\pm 1}u_{n\mp m}^{\pm 1})) & \text{if } \beta = \pm\alpha, \\ \hat{x}_{(\beta,n)}(r) & \text{if } (\alpha,\beta) = 0, \\ \hat{x}_{(\beta,n)}(rf_n^{-1}f_{-m}(u^{-1})) & \text{if } \alpha \pm \beta \neq 0,\ k = j, \\ \hat{x}_{(\beta,n)}(rf_n^{-1}f_{-m}(u_{n+m}^{-1})) & \text{if } \alpha \pm \beta \neq 0,\ l = i, \\ \hat{x}_{(\beta,n)}(rf_n^{-1}f_m(u)) & \text{if } \alpha \pm \beta \neq 0,\ k = i, \\ \hat{x}_{(\beta,n)}(rf_n^{-1}f_m(u_{n-m})) & \text{if } \alpha \pm \beta \neq 0,\ l = j. \end{cases}$$

(7) $\hat{h}_{(\alpha,m)}(u)\hat{h}_{(\beta,n)}(v)\hat{h}_{(\alpha,m)}(u)^{-1}$

$$= \begin{cases} \hat{h}_{(\beta,n)}(vf_n^{-1}f_{\pm m}(u^{\pm 1}u_{n\mp m}^{\pm 1}))\hat{h}_{(\beta,n)}(f_n^{-1}f_{\pm m}(u^{\pm 1}u_{n\mp m}^{\pm 1}))^{-1} & \text{if } \beta = \pm\alpha, \\ \hat{h}_{(\beta,n)}(v) & \text{if } (\alpha,\beta) = 0, \\ \hat{h}_{(\beta,n)}(vf_n^{-1}f_{-m}(u^{-1}))\hat{h}_{(\beta,n)}(f_n^{-1}f_{-m}(u^{-1}))^{-1} & \text{if } \alpha \pm \beta \neq 0,\ k = j, \\ \hat{h}_{(\beta,n)}(vf_n^{-1}f_{-m}(u_{n+m}^{-1}))\hat{h}_{(\beta,n)}(f_n^{-1}f_{-m}(u_{n+m}^{-1}))^{-1} & \text{if } \alpha \pm \beta \neq 0,\ l = i, \\ \hat{h}_{(\beta,n)}(vf_n^{-1}f_m(u))\hat{h}_{(\beta,n)}(f_n^{-1}f_m(u))^{-1} & \text{if } \alpha \pm \beta \neq 0,\ k = i, \\ \hat{h}_{(\beta,n)}(vf_n^{-1}f_m(u_{n-m}))\hat{h}_{(\beta,n)}(f_n^{-1}f_m(u_{n-m}))^{-1} & \text{if } \alpha \pm \beta \neq 0,\ l = j. \end{cases}$$

Now we define the subgroups of $St(A_{\ell-1}^{(1)}, K)_q$:

$$\hat{U}_a = \langle \hat{x}_a(r) \mid r \in K \rangle \quad \text{for all } a \in \Phi_1,$$

$$\hat{U} = \langle \hat{U}_a \mid a \in \Phi_1^+ \rangle,$$

$$\hat{Y}_a = \langle x \hat{U}_b x^{-1} \mid x \in \hat{U}_a, b \in \Phi_1^+ \backslash \{a\} \rangle \quad \text{for each } a \in \Pi_1,$$

$$\hat{N} = \langle \hat{w}_a(u) \mid a \in \Phi_1, u \in K^\times \rangle,$$

$$\hat{T} = \langle \hat{h}_a(u) \mid a \in \Phi_1, u \in K^\times \rangle,$$

$$\hat{B} = \langle \hat{U}, \hat{T} \rangle,$$

$$\hat{S} = \{\hat{w}_a(1) \bmod \hat{T} \mid a \in \Pi_1\}.$$

Then, we have the following results (cf. [17]).

LEMMA 3.3. *Notation is as above and let* $a \in \Pi_1$. *Then*:
(1) $\hat{U} \triangleleft \hat{B} = \hat{U}\hat{T}$.
(2) $\hat{T} \triangleleft \hat{N}$.
(3) $\hat{B} \cap \hat{N} = \hat{T}$.
(4) $\hat{N}/\hat{T} \simeq W_1$.
(5) $(\hat{N}/\hat{T}, \hat{S})$ *is a Coxeter system.*
(6) $\hat{Y}_a \triangleleft \hat{U} = \hat{Y}_a \hat{U}_a$.
(7) $\hat{w}_a(u) \hat{Y}_a \hat{w}_a(-u) = \hat{Y}_a$ *for all* $u \in K^\times$.

PROPOSITION 3.4. *Notation is as above. Then,* $(St(A_{\ell-1}^{(1)}, K)_q, \hat{B}, \hat{N}, \hat{S})$ *is a Tits system with the corresponding affine Weyl group* W_1. *In particular, we have* $St(A_{\ell-1}^{(1)}, K)_q = \bigcup_{w \in W_1} \hat{B} w \hat{B}$ *(Bruhat decomposition).*

PROPOSITION 3.5. *Notation is as above. Then,* $St(A_{\ell-1}^{(1)}, K)_q$ *is a universal central extension of* $E(A_{\ell-1}^{(1)}, K)_q$.

4. Presentation of Elementary Subgroups

We put $K_2(A_{\ell-1}^{(1)}, K)_q = \operatorname{Ker} \phi$, where ϕ is the canonical homomorphism of $St(A_{\ell-1}^{(1)}, K)_q$ onto $E(A_{\ell-1}^{(1)}, K)_q$ with $\phi(\hat{x}_a(r)) = x_a(r)$ for all $a \in \Phi_1$ and $r \in K$.

First we suppose that $\ell = 2$, that is, the rank of Φ is 1 in the sense of root systems. Namely we assume $\Pi_1 = \{a_1 = (12, 0), a_0 = (21, 1)\}$. Let $\tilde{E}(A_1^{(1)}, K)_q$ be the group defined by generators $\tilde{x}_a(r)$ for all $a \in \Phi_1$ and $r \in K$ and the defining relations (A), (A′) and (B′) together with the following relation:

(C) $\tilde{h}_a(u)\tilde{h}_a(v) = \tilde{h}_a(uv)$ for all $a \in \Phi_1$ and $u, v \in K^\times$

(D) $\tilde{h}_{a_0}(w)\tilde{h}_{a_1}(w) = 1$ for all $w \in K^\times_{q,X_2}$

where $K^\times_{q,X_2} = \{u \in K^\times \mid [u, X_2] = 1\}$, and for all $a \in \Phi_1$ and $u, v \in K^\times$, we put

$$\tilde{w}_a(u) = \tilde{x}_a(u)\tilde{x}_{-a}(-u^{-1})\tilde{x}_a(u), \quad \tilde{h}_a(u) = \tilde{w}_a(u)\tilde{w}_a(-1)$$

and where $\hat{x}$, $\hat{w}$ in the relations (A), (A′) and (B′) should be changed into $\tilde{x}$, $\tilde{w}$ respectively. Using the above discussion, we obtain the following proposition.

Proposition 4.1. *Notation is as above. Then, we have* $\tilde{E}(A_1^{(1)}, K)_q \simeq E(A_1^{(1)}, K)_q$. *In particular,* $K_2(A_1^{(1)}, K)_q = \langle \{u, v\}_a, \hat{d}(w) \mid a \in \Pi_1, u, v \in K^\times, w \in K^\times_{q,X_2} \rangle$, *where* $\{u, v\}_a = \hat{h}_a(u)\hat{h}_a(v)\hat{h}_a(uv)^{-1}$ *and* $\hat{d}(w) = \hat{h}_{a_0}(w)\hat{h}_{a_1}(w)$.

Proof. The homomorphism $\phi : St(A_1^{(1)}, K)_q \to E(A_1^{(1)}, K)_q$ induces two canonical homomorphisms called $\hat{\phi}$ and $\tilde{\phi}$, that is, $\hat{\phi} : St(A_1^{(1)}, K)_q \to \tilde{E}(A_1^{(1)}, K)_q$, $\tilde{\phi} : \tilde{E}(A_1^{(1)}, K)_q \to E(A_1^{(1)}, K)_q$, which are defined by $\hat{\phi}(\hat{x}_a(r)) = \tilde{x}_a(r)$ and $\tilde{\phi}(\tilde{x}_a(r)) = x_a(r)$ with the following diagram:

$$\begin{array}{ccccc} & & \tilde{E}(A_1^{(1)}, K)_q & & \\ & \overset{\hat{\phi}}{\nearrow} & \phi & \overset{\tilde{\phi}}{\searrow} & \\ St(A_1^{(1)}, K)_q & & \longrightarrow & & E(A_1^{(1)}, K)_q \end{array}$$

We use the same notation for subgroups of $\tilde{E}(A_1^{(1)}, K)_q$ as in $St(A_1^{(1)}, K)_q$ changing $\hat{\ }$ into $\tilde{\ }$, namely $\hat{\phi}(\hat{\ }) = \tilde{\ }$. Then, we find two kinds of Bruhat decompositions:

$$\begin{array}{ccccc} \tilde{E}(A_1^{(1)}, K)_q = \bigsqcup_{w \in W_1} \tilde{B}w\tilde{B} & \supset & \tilde{B} & = & \tilde{U}\tilde{T} \\ \downarrow & & \downarrow & & \\ E(A_1^{(1)}, K)_q = \bigsqcup_{w \in W_1} BwB & \supset & B & = & U \rtimes T \end{array}$$

Therefore, by these decompositions, we can obtain $\operatorname{Ker} \tilde{\phi} \subset \tilde{B}$. We take an element $\tilde{x} \in \operatorname{Ker} \tilde{\phi}$. Then, we write $\tilde{x}$ as $\tilde{x} = \tilde{y}\tilde{z}$ for some $\tilde{y} \in \tilde{U}$ and $\tilde{z} \in \tilde{T}$. Put $y = \tilde{\phi}(\tilde{y})$ and $z = \tilde{\phi}(\tilde{z})$. Since $\tilde{x} \in \operatorname{Ker} \tilde{\phi}$, we have $1 = \tilde{\phi}(\tilde{x}) = \tilde{\phi}(\tilde{y})\tilde{\phi}(\tilde{z}) = yz \in U \rtimes T$ which implies $y = z = 1$. Hence, we obtain $\tilde{y}, \tilde{z} \in \operatorname{Ker} \tilde{\phi}$.

Claim 1. $\tilde{y} = 1$.

Using the degree map of $K[X_2]$ in X_2, we can establish that U is the free product of

$$\begin{pmatrix} 1 & K[X_2] \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ K[X_2]X_2 & 1 \end{pmatrix}.$$

Hence, $\tilde{U}$ is isomorphic to U, and $\tilde{y} = 1$.

Claim 2. $\tilde{z} = 1$.

By (B′) and (C), we have $\tilde{h}_a(u)\tilde{h}_b(v)\tilde{h}_a(u)^{-1} = \tilde{h}_b(v)$, so $\tilde{T}$ is commutative. And we have $\tilde{h}_a(u) \in \langle \tilde{T}_{a_1}, \tilde{T}_{a_0} \rangle$ by induction on the length of the shortest expression in W_1, where $\tilde{T}_{a_i} = \langle \tilde{h}_{a_i}(u) \mid u \in K^\times \rangle$. Hence we obtain $\tilde{T} = \tilde{T}_{a_0}\tilde{T}_{a_1}$, and one can write $\tilde{z} = \tilde{h}_{a_0}(u)\tilde{h}_{a_1}(v)$ for some $u, v \in K^\times$. Since $\tilde{\phi}(\tilde{z}) = 1$, we see that $u = v \in K^\times_{q,X_2}$ by:

$$h_{a_0}(u)h_{a_1}(v) = \begin{pmatrix} X_2^{-1}u^{-1}X_2 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} = \begin{pmatrix} X_2^{-1}u^{-1}X_2 v & 0 \\ 0 & uv^{-1} \end{pmatrix} = I_2.$$

Consequently, $\tilde{z} = 1$ by (D).

Therefore, we just reached $\tilde{x} = 1$, which implies that $\tilde{E}(A_1^{(1)}, K)_q \simeq E(A_1^{(1)}, K)_q$. q.e.d.

PROPOSITION 4.2.

$$K_2(A_1, K_q) \simeq K_2(A_1^{(1)}, K)_q = \langle \{u, v\}_a, \hat{d}(w) \mid a \in \Pi_1,\ u, v \in K^\times, w \in K^\times_{q,X_2} \rangle.$$

PROOF. Since $St(A_1^{(1)}, K)_q \simeq St(A_1, K_q)$ and $E(A_1^{(1)}, K)_q \simeq E(A_1, K_q)$, we have $K_2(A_1, K_q) \simeq K_2(A_1^{(1)}, K)_q$ from the following commutative diagram by the five lemma.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(A_1^{(1)}, K)_q & \longrightarrow & St(A_1^{(1)}, K)_q & \longrightarrow & E(A_1^{(1)}, K)_q & \longrightarrow & 1 \quad \text{(exact)} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_2(A_1, K_q) & \longrightarrow & St(A_1, K_q) & \longrightarrow & E(A_1, K_q) & \longrightarrow & 1 \quad \text{(exact)} \end{array}$$

And we have $K_2(A_1^{(1)}, K)_q = \langle \{u, v\}_a, \hat{d}(w) \mid a \in \Pi_1, u, v \in K^\times, w \in K^\times_{q,X_2} \rangle$ by Proposition 4.1. q.e.d.

We suppose $\ell \geq 3$. Then, in general, there exists a canonical homomorphism of $K_2(A_1^{(1)}, K)_q$ into $K_2(A_{\ell-1}^{(1)}, K)_q$, which is induced from the following diagram (cf. [13]):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(A_1^{(1)}, K)_q & \longrightarrow & St(A_1^{(1)}, K)_q & \longrightarrow & E(A_1^{(1)}, K)_q & \longrightarrow & 1 \quad \text{(exact)} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_2(A_{\ell-1}^{(1)}, K)_q & \longrightarrow & St(A_{\ell-1}^{(1)}, K)_q & \longrightarrow & E(A_{\ell-1}^{(1)}, K)_q & \longrightarrow & 1 \quad \text{(exact)} \end{array}$$

Since K_q is a Euclidean ring, the homomorphism of $K_2(A_1^{(1)}, K)_q$ into $K_2(A_{\ell-1}^{(1)}, K)_q$ is surjective by [5]. Hence, we have the following.

Theorem 4.3. *Suppose $\dot{\ell} \geq 3$. Let $\tilde{E}(A^{(1)}_{\dot{\ell}-1}, K)_q$ be the group generated by $\tilde{x}_a(r)$ for all $a \in \Phi_1$ and $r \in K$ with the defining relations* (A), (A′), (B), (C) *and* (D). *Then, $\tilde{E}(A^{(1)}_{\dot{\ell}-1}, K)_q$ is isomorphic to $E(A^{(1)}_{\dot{\ell}-1}, K)_q$.*

If $\dot{\ell} = 3$, we have the equations

$$\hat{w}_{(23,0)}(1)\hat{w}_{(12,0)}(1)\{u,v\}_{(23,0)}\hat{w}_{(12,0)}(-1)\hat{w}_{(23,0)}(-1) = \{u,v\}_{(12,0)} \quad \text{and}$$

$$\hat{w}_{(13,0)}(1)\hat{w}_{(21,1)}(1)\{u,v\}_{(31,1)}\hat{w}_{(21,1)}(-1)\hat{w}_{(13,0)}(-1) = \{u,v\}_{(12,0)},$$

then we have $\{u,v\}_0 = \{u,v\}_1 = \{u,v\}_2$. Similarly, by simply computation, we have $\{u,v\}_0 = \{u,v\}_1 = \cdots = \{u,v\}_{\dot{\ell}-1}$ for $\dot{\ell} \geq 3$. Hence we can write $\{u,v\}_i = \{u,v\}$ if $\dot{\ell} \geq 3$, for simple. Therefore, we have the following.

Proposition 4.4. *Suppose $\dot{\ell} \geq 3$. Then, we have*

$$K_2(A_{\dot{\ell}-1}, K_q) \simeq K_2(A^{(1)}_{\dot{\ell}-1}, K)_q = \langle \{u,v\}, \hat{d}(w) \mid u, v \in K^\times, w \in K^\times_{q,X_2} \rangle.$$

5. K_2-groups and Presentations

We put $\{u,v\}_{ab} = [\hat{h}_a(u), \hat{h}_b(v)]$ for any $u, v \in K^\times$ and $a = (\alpha, m)$, $b = (\beta, n) \in \Pi_1$. Then we have

$$\{u,v\}_{ab} = \begin{cases} \{u, v^{-1}v^{-1}_{m-n}\}_a = \{u^{-1}u^{-1}_{n-m}, v\}_b & \text{if } \dot{\ell} = 2 \text{ and the pair } (a,b) = (a_0, a_1), \\ \{u, v^{-1}_{m-n}\} = \{u_{n-m}, v^{-1}\} & \text{if } \dot{\ell} \geq 3 \text{ and the pair } (a,b) = (a_0, a_1), \\ \{u, v^{\langle \alpha, \beta \rangle}\}_a = \{u^{\langle \beta, \alpha \rangle}, v\}_b & \text{otherwise.} \end{cases}$$

Hence, using the notation

$$u(ab) = \begin{cases} u^{-1}u^{-1}_{m-n} & \text{if } \dot{\ell} = 2 \text{ and the pair } (a,b) = (a_0, a_1), \\ u^{-1}_{m-n} & \text{if } \dot{\ell} \geq 3 \text{ and the pair } (a,b) = (a_0, a_1), \\ u^{\langle \alpha, \beta \rangle} & \text{otherwise,} \end{cases}$$

we can write $\{u,v\}_{ab} = \{u, v(ab)\}_a = \{u(ba), v\}_b$ for convenience. In particular, we have $u(ab)(ba) = u$ if $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 1$ and $w(0a)w(1a)\cdots w(\dot{\ell}-1, a) = 1$, where $w \in K^\times_{q,X_2}$. Also, we have the following using the fact that $\{u,v\}_a$ is central:

$$\begin{aligned} \{u,v\}_{ab}\{u,w\}_{ab} &= \hat{h}_a(u)\hat{h}_b(v)\hat{h}_a(u)^{-1}[\hat{h}_a(u), \hat{h}_b(w)]\hat{h}_b(v)^{-1} \\ &= \hat{h}_a(u)\hat{h}_b(v)\hat{h}_b(w)\hat{h}_a(u)^{-1}\hat{h}_b(w)^{-1}\hat{h}_b(v)^{-1} \\ &= [\hat{h}_a(u), \hat{h}_b(vw)] \\ &= \{u, vw\}_{ab}. \end{aligned}$$

Also we can get $\{u,v\}_{ab}\{w,v\}_{ab} = \{uw, v\}_{ab}$ similarly, hence $\{u,v\}_{ab}$ is bimultiplicative.

For example, we can compute the following identities:

If $\dot{\ell} = 2$, we have $\{u,v\}_{10} = \{u, v^{-1}v_{-1}^{-1}\}_1 = \{u^{-1}u_1^{-1}, v\}_0$, $\{u,v\}_{aa} = \{u, v^2\}_a = \{u^2, v\}_a$, $\{u,-1\}_{ab} = \{-1,u\}_{ab} = 1$.

If $\dot{\ell} \geq 3$, we obtain $\{u,v\}_{12} = \cdots = \{u,v\}_{\dot{\ell}-2,\dot{\ell}-1} = \{u,v\}_{\dot{\ell}-1,0} = \{u, v^{-1}\}$ and $\{u,v\}_{01} = \{u_{-1}, v^{-1}\} = \{u, v_1^{-1}\}$.

LEMMA 5.1. *For any $u, v \in K^{\times}$, $a \in \Pi_1$, we have the following relations.*

(1) $\hat{h}_a(u)\hat{h}_a(v) = \hat{h}_a(u^2v)\hat{h}_a(u^{-1}) = \hat{h}_a(v^{-1})\hat{h}_a(uv^2)$.

(2) $\hat{h}_a(uv(1-v))\hat{h}_a(u(1-v))^{-1} = \hat{h}_a(uv)\hat{h}_a(u)^{-1}$.

(3) $\hat{h}_a(u^2) = \hat{h}_a(u)\hat{h}_a(u^{-1})^{-1} = \hat{h}_a(u^{-1})^{-1}\hat{h}_a(u) = \hat{h}_a(u^{-2})^{-1}$.

(4) $\{u,v\}_a = \hat{h}_a(u^{-1}v^{-1})^{-1}\hat{h}_a(u^{-1})\hat{h}_a(v^{-1})$.

Using the above Lemma 5.1 and the fundamental properties of $\{u,v\}_{ab}$, we have the following:

LEMMA 5.2. *Let $\dot{\ell} = 2$. Then, the following relations hold for all $t, u, v \in K^{\times}$, $w, x \in K_{q,X_2}^{\times}$ and $a = (\alpha, m)$, $b = (\beta, n) \in \Pi_1$:*

(L1) $\{u,v\}_a\{uv,t\}_a = \{u,vt\}_a\{v,t\}_a$,

(L2) $\{1,1\}_a = 1$,

(L3) $\{u,v\}_a = \{u^{-1}, v^{-1}\}_a$,

(L4) $\{u,v\}_a = \{u,(1-u)v\}_a$ with $u \neq 1$,

(L5) $\{u, v(ab)\}_a = \{u(ba), v\}_b$,

(L6) $\{u,v\}_{ab}$ *is bimultiplicative*,

(LD) $\hat{d}(w)\hat{d}(x) = \hat{d}(wx)\{w,x\}_1\{x,w\}_0 = \hat{d}(wx)\{x,w\}_1\{w,x\}_0$.

LEMMA 5.3. *Let $t, u, v \in K^{\times}$, $w, x \in K_{q,X_2}^{\times}$, $a \in \Pi_1$ and $\dot{\ell} = 2$. Then the relations* (L1)~(L6) *and* (LD) *in Lemma* 5.2 *yield the following relations.*

(1) $\{u,1\}_a = 1$,

(2) $\{u,v\}_a = \{u(1-v), v\}_a$,

(3) $\{u,v\}_a = \{u,-uv\}_a = \{-uv, v\}_a$,

(4) $\{u,u^2\}_a = 1$,

(5) $\{u,v\}_a = \{v^{-1}, u\}_a = \{v, u^{-1}\}_a = \{u, u^{-1}\}_a = \{v^{-1}, uv^2\}_a$,

(6) $\{-1, [u, X_2^{-1}]\}_a = 1$,

(7) $\hat{d}(w)\hat{d}(x^2) = \hat{d}(wx^2)$,

(8) $\hat{d}(wx)\hat{d}((1-w)x) = \hat{d}(x)\hat{d}(w(1-w)x)$.

PROOF.

(1) Put $v = t = 1$ and then apply (L1) and (L2) to obtain (1).

(2) We can obtain $\{u(1-v),v\}_a = \{1-v,uv\}_a\{u,v\}_a\{1-v,u\}_a^{-1} = \{u,v\}_a$ by (L1) and (L4).

(3) We have $\{u,v\}_a = \{u,(1-u)v\}_a = \{u,-uv(1-u^{-1})\}_a$
$= \{u^{-1},-u^{-1}v^{-1}(1-u^{-1})^{-1}\}_a$
$= \{u^{-1},-u^{-1}v^{-1}\}_a = \{u,-uv\}_a$
by the previous (2) and (L3), (L4).

(4) By (1), (3) of this Lemma 5.3, we can show (4).

(5) By (L1), (L3) and (3) of this Lemma 5.3, we see (5).

(6) We have $\{-1,uu_{-1}^{-1}\}_1 = \{u_{-1}^2,-u_{-1}^{-2}\}_1\{-1,uu_{-1}-1\}_1$
$= \{u_{-1}^2,-uu_{-1}\}_1\{-u_{-1}^{-2},uu_{-1}\}_1$ by (L1)
$= \{u_{-1},-uu_{-1}\}_{11}\{-u_{-1}^{-2},u^{-1}\}_{10}$
$= \{u_{-1},uu_{-1}\}_{11}\{u_{-1}^{-2},u^{-1}\}_{10}$ by (L5), (L6)
$= \{u_{-1}^2,uu_{-1}\}_1\{u_{-1}^{-2},uu_{-1}\}_1$
$= \{u_{-1}^2,u^{-1}\}_{10}\{u_{-1}^{-2},u^{-1}\}_{10} = 1$ by (L5), (L6)

Also we have $\{-1,uu_{-1}^{-1}\}_0 = 1$ similarly.

(7) By (4) of this Lemma 5.3, we have $1 = \{x,x^{\pm 2}\}_1 = \{x,x^{\mp 1}\}_{10}$, and this implies $\{x,w\}_{10} = \{x,wx\}_{10} = \{xw,w\}_{10}$ by (L6). Then we obtain $\{w,x^{-1}\}_{10} = \{w,x^{-1}w\}_{10} = \{x,x^{-1}w\}_{10} = \{x,w\}_{10}$.

(8) By (2) of this Lemma 5.3 and (L4). q.e.d.

Lemma 5.4. *Let $\dot{\ell} \geq 3$. Then, in $K_2(A_{\dot{\ell}-1}^{(1)},K)_q$, the relations corresponding to* (L1)$\sim$(L6) *and* (LD) *of Lemma* 5.2 *hold.*

Lemma 5.5. *Let $t,u,v \in K^\times$, $w,x \in K_{q,X_2}^\times$ and $\dot{\ell} \geq 3$. Then the relations* (L1)$\sim$(L6) *and* (LD) *yield the following relations.*

(1) $\{u,v\}\{u,t\} = \{u,vt\}$ *and* $\{u,v\}\{t,v\} = \{ut,v\}$.

(2) $\{u,v\} = \{v,u^{-1}\} = \{v^{-1},u\} = \{v,u\}^{-1}$.

(3) $\{u,v\} = \{u_m,v_m\}$ for all $m \in \mathbf{Z}$,

(4) $\hat{d}(w)\hat{d}(x) = \hat{d}(wx)$.

Let L be the abelian group generated by the symbols $c(u,v)_a$ and $d(w)$ for all $u,v \in K^\times$, $a \in \Pi_1$ and $w \in K_{q,X_2}^\times$, with the defining relations (L1)$\sim$(L6) and (LD) replacing $\{u,v\}_a$ and $\hat{d}(w)$ by $c_a(u,v)$ and $d(w)$ respectively. Hence there is one and only one homomorphism $\zeta : L \to K_2(A_{\dot{\ell}-1}^{(1)},K)_q$ which carries $c_a(u,v)$ to $\{u,v\}_a$ and $d(w)$ to $\hat{d}(w)$ respectively for all $u,v \in K^\times$, $w \in K_{q,X_2}^\times$ and $a \in \Pi_1$. Then we obtain the following, the proof of which will be given at the last part of this section.

Theorem 5.6. *Notation is as above. Then we have $L \simeq K_2(A_{\dot{\ell}-1}^{(1)},K)_q$.*

To prove this, we introduce the group $\tilde{H}$, which is generated by the symbols $\tilde{h}_a(u)$ and $z(l)$ for all $a \in \Pi_1$, $u \in K^\times$ and $l \in L$ with the following defining relations:

(H1) $\tilde{h}_a(u)\tilde{h}_a(v) = z(c_a(u,v))\tilde{h}_a(uv)$,

(H2) $\tilde{h}_a(u)\tilde{h}_b(v) = z(c_{ab}(u,v))\tilde{h}_b(v)\tilde{h}_a(u)$,

(H3) $z(l_1)z(l_2) = z(l_1 l_2)$,

(H4) $\tilde{h}_a(u)z(l) = z(l)\tilde{h}_a(u)$,

(H5) $\tilde{h}_0(w)\tilde{h}_1(w)\cdots\tilde{h}_{\ell-1}(w) = z(d(w))$

for all $a, b \in \Pi_1$, $u, v \in K^\times$, $w \in K^\times_{q,X_2}$ and $l, l_1, l_2 \in L$, where $c_{ab}(u,v)$ is the element of L corresponding to $\{u,v\}_{ab}$. We see that $\tilde{H}$ contains the subgroup consisting of $z(l)$ for all $l \in L$, which is isomorphic to L, hence we can identify $l \in L$ with $z(l)$. In particular, all the relations in Lemma 5.1 and Lemma 3.2 (7) hold in $\tilde{H}$.

Now we let $T_{a_i} = \langle h_{a_i}(u) \mid u \in K^\times \rangle \simeq K^\times$ for each $a_i \in \Pi_1$, as a subgroup of T. Then, using the fact $T = T_{a_0} \times T_{a_1} \times \cdots \times T_{a_{\ell-1}}$ we construct a central extension $(\tilde{H}, \pi)$

$$1 \to L \to \tilde{H} \xrightarrow{\pi} T \to 1$$

of T by L, where π denotes the associated homomorphism of $\tilde{H}$ onto T.

Next, we will construct some central extension of the monomial subgroup N by L which is compatible with the extension $(\tilde{H}, \pi)$ of T. To do so, we first obtain the presentation of N in a similar way as in Proposition 4.1, and then construct an action of N on $\tilde{H}$.

Lemma 5.7. *Notation is as above. Then N is the group generated by $w_a(u)$ for all $a \in \Pi_1$ and $u \in K^\times$ with the following defining relations*:

(N1) $w_a(u)^{-1} = w_a(-u)$,

(N2) $w_a(1)h_b(u)w_a(-1) = h_b(u)h_a(u(ab)^{-1})$,

(N3) $h_a(u)h_a(v) = h_a(uv)$,

(N4) $h_a(u)h_b(v) = h_b(v)h_a(v)$,

(N5) $h_0(w)h_1(w)\cdots h_{\ell-1}(w) = 1$,

(N6) $w_a(1)w_c(1)w_a(1) = w_c(1)w_a(1)w_c(1)$ if $\langle \alpha, \gamma \rangle \langle \gamma, \alpha \rangle = 1$,

(N7) $w_a(1)w_c(1) = w_c(1)w_a(1)$ if $\langle \alpha, \gamma \rangle \langle \gamma, \alpha \rangle = 0$

for all $u, v \in K^\times$, $w \in K^\times_{q,X_2}$, $a = (\alpha, m)$, $b = (\beta, n)$, $c = (\gamma, k) \in \Pi_1$ and $\alpha \neq \pm\gamma$.

Proposition 5.8. *We define the action of N on $\tilde{H}$ in the following way*:

$$w_a(u) \cdot \tilde{h}_b(v) = w_a(u)\tilde{h}_b(v)w_a(-u) = \tilde{h}_b(v)\tilde{h}_a(v(ab)^{-1})c_{ab}(u,v)^{-1}$$

for all $u, v \in K^\times$ and $a = (\alpha, m)$, $b = (\beta, n) \in \Pi_1$. Then $\tilde{H}$ becomes an N-group.

Proof. First we have $w_a(u)\cdot\tilde{h}_a(v)=\tilde{h}_a(v^{-1})c_{aa}(u,v)^{-1}$ and $h_a(u)\cdot\tilde{h}_b(v)=\tilde{h}_b(v)c_{ab}(u,v)$. We should check that the action of N preserves all the relations (H1)~(H5). We easily see that (H3) and (H4) are obvious, because of $w_a(u)\cdot l=l$ for all $l\in L$. We will comfirm the other relations.

(H1): $w_a(t)\cdot(\tilde{h}_b(u)\tilde{h}_b(v))$

$$\begin{aligned}
&=\tilde{h}_b(u)\tilde{h}_a(u(ab)^{-1})\tilde{h}_b(v)\tilde{h}_a(v(ab)^{-1})c_{ab}(t,uv)^{-1}\quad\text{by (L6)}\\
&=\tilde{h}_b(u)\tilde{h}_b(v)\tilde{h}_a(u(ab^{-1}))\tilde{h}_a(v(ab^{-1}))c_{ab}(t,uv)^{-1}c_{ab}(u(ab^{-1}),v)\quad\text{by (H2)}\\
&=\tilde{h}_b(uv)\tilde{h}_a((uv)(ab)^{-1})c_{ab}(t,uv)^{-1}c_{ab}(u(ab)^{-1},v)\\
&\quad\times c_a(u(ab)^{-1},v(ab)^{-1})c_b(u,v)\\
&=\tilde{h}_b(uv)\tilde{h}_a((uv)(ab)^{-1})c_{ab}(t,uv)^{-1}c_{ab}(u(ab)^{-1},v)c_{ab}(u(ab)^{-1},v^{-1})c_b(u,v)\\
&=\tilde{h}_b(uv)\tilde{h}_a((uv)(ab)^{-1})c_{ab}(t,uv)^{-1}c_b(u,v)\quad\text{by (L6)}\\
&=w_a(t)\cdot(c_b(u,v)\tilde{h}_b(uv)).
\end{aligned}$$

(H2): $w_a(t)\cdot(\tilde{h}_b(u)\tilde{h}_c(v))$

$$\begin{aligned}
&=\tilde{h}_b(u)\tilde{h}_a(u(ab)^{-1})\tilde{h}_c(v)\tilde{h}_a(v(ac)^{-1})c_{ab}(t,u)^{-1}c_{ac}(t,v)^{-1}\\
&=c_{bc}(u,v)c_{ab}(t,u)^{-1}c_{ac}(t,v)^{-1}\tilde{h}_c(v)\tilde{h}_b(u)\tilde{h}_a(u(ab)^{-1}v(ac)^{-1})\\
&\quad\times c_{ac}(u(ab)^{-1},v)c_a(u(ab)^{-1},v(ac)^{-1})\\
&=c_{bc}(u,v)c_{ab}(t,u)^{-1}c_{ac}(t,v)^{-1}\tilde{h}_c(v)\tilde{h}_b(u)\tilde{h}_a(u(ab)^{-1}v(ac)^{-1})\quad\text{by (L6)}\\
&=w_a(t)\cdot(c_{bc}(u,v)\tilde{h}_c(v)\tilde{h}_b(u)).
\end{aligned}$$

(H5): $w_a(t)\cdot(\tilde{h}_0(w)\cdots\tilde{h}_{\dot{\ell}-1}(w))$

$$\begin{aligned}
&=\tilde{h}_0(w)\tilde{h}_a(w(a0)^{-1})\cdots\tilde{h}_{\dot{\ell}-1}(w)\tilde{h}_a(w(a,\dot{\ell}-1)^{-1})c_{a0}(t,w)\cdots c_{a,\dot{\ell}-1}(t,w)\\
&=\tilde{h}_0(w)\cdots\tilde{h}_{\dot{\ell}-1}(w)\tilde{h}_a(w(a0)^{-1}\cdots w(a,\dot{\ell}-1)^{-1})\\
&\quad\times c_a(t,w(a0)\cdots w(a,\dot{\ell}-1))\quad\text{by (H2)}\\
&=\tilde{h}_0(w)\cdots\tilde{h}_{\dot{\ell}-1}(w)\quad\text{by } w(a0)\cdots w(a,\dot{\ell}-1)=1\\
&=w_a(t)\cdot(d(w)).
\end{aligned}$$

Therefore, $\tilde{w}_a$ gives an automorphism of $\tilde{H}$. Next we should check that both sides in the relations (N1)~(N7) give the same effect on $\tilde{H}$.

(N1):
$$\begin{aligned}
w_a(u)\cdot(w_a(-u)\cdot\tilde{h}_b(v))&=w_a(u)\cdot(\tilde{h}_b(v)\tilde{h}_a(v(ab)^{-1})c_{ab}(-u,v)^{-1})\\
&=\tilde{h}_b(v)\tilde{h}_a(v(ab)^{-1})c_{ab}(u,v)^{-1}\tilde{h}_a(v(ab))\\
&\quad\times c_{aa}(u,v(ab)^{-1})^{-1}c_{ab}(-u,v)^{-1}\\
&=\tilde{h}_b(v)c_a(-1,v(ab))c_{ab}(-1,v)^{-1}\\
&=\tilde{h}_b(v).
\end{aligned}$$

(N2): $(w_a(1)h_b(u)w_a(-1))\cdot\tilde{h}_c(v)$

$$\begin{aligned}
&=(w_a(1)h_b(u))\cdot(\tilde{h}_c(v)\tilde{h}_a(v(ac)^{-1})c_{ac}(-1,v)^{-1})\\
&=w_a(1)\cdot(\tilde{h}_c(v)\tilde{h}_a(v(ac)^{-1})c_{ac}(-1,v)^{-1}c_{bc}(u,v)c_{ba}(u,v(ac)^{-1})\\
&=\tilde{h}_c(v)c_a(-1,v(ac))c_{ac}(-1,v)^{-1}c_{bc}(u,v)c_{ba}(u,v(ac)^{-1})\\
&=\tilde{h}_c(v)c_{bc}(u,v)c_{ba}(u,v(ac)^{-1})\quad\text{by (L5)}\\
&=\tilde{h}_c(v)c_{bc}(u,v)c_{ac}(u(ab)^{-1},v)\quad\text{by (L3) and (L5)}\\
&=(h_b(u)h_a(u(ab)^{-1})\cdot\tilde{h}_c(v).
\end{aligned}$$

(N5): $(h_0(w)h_1(w)\cdots h_{\ell-1}(w))\cdot\tilde{h}_a(u)$
$= \tilde{h}_a(u)c_{0a}(w,u)c_{1a}(w,u)\cdots c_{\ell-1,a}(w,u)$
$= \tilde{h}_a(u)c_0(w(0a)w(1a)\cdots w(\ell-1,a),u)$ by (L5) and (L6)
$= \tilde{h}_a(u).$

(N6): $(w_a(1)w_c(1)w_a(1))\cdot\tilde{h}_b(u)$
$= \tilde{h}_b(u)\tilde{h}_a(u(ab)^{-1})\tilde{h}_c(u(cb)^{-1})\tilde{h}_a(u(cb)(ac))\tilde{h}_a(u(ab))$
$\times\tilde{h}_c(u(ab)(ca))\tilde{h}_a(u(ab)^{-1})$ by $u(ac)(ca) = u(ca)(ac) = u$
$= \tilde{h}_b(u)\tilde{h}_a(u(ab)^{-1})\tilde{h}_c(u(cb)^{-1})\tilde{h}_a(u(cb)(ac))\tilde{h}_c(u(ab)(ca)),$

$(w_c(1)w_a(1)w_c(1))\cdot\tilde{h}_b(u)$
$= \tilde{h}_b(u)\tilde{h}_c(u(cb)^{-1})\tilde{h}_a(u(ab)^{-1})\tilde{h}_c(u(ab)(ca))\tilde{h}_c(u(cb))$
$\times\tilde{h}_a(u(cb)(ac))\tilde{h}_c(u(cb)^{-1})$
$= \tilde{h}_b(u)\tilde{h}_a(u(ab)^{-1})\tilde{h}_c(u(cb)^{-1})\tilde{h}_a(u(cb)(ac))\tilde{h}_c(u(ab)(ca)).$

(N3), (N4) and (N7) are easy to be checked, hence $\tilde{H}$ is an N-group. q.e.d.

Let $\tilde{N}_0$ be the group generated by the symbols $\tilde{w}_a$ for all $a\in\Pi_1$ with the following defining relations:

(W1) $\tilde{h}_a\tilde{w}_b\tilde{h}_a^{-1} = \tilde{w}_b^d$

(W2) $\tilde{w}_a\tilde{w}_c\tilde{w}_a = \tilde{w}_c\tilde{w}_a\tilde{w}_c$ if $\langle\alpha,\gamma\rangle\langle\gamma,\alpha\rangle = 1$

(W3) $\tilde{w}_a\tilde{w}_c = \tilde{w}_c\tilde{w}_a$ if $\langle\alpha,\gamma\rangle\langle\gamma,\alpha\rangle = 0$

for all $a = (\alpha,m)$, $b = (\beta,n)$, $c = (\gamma,k)\in\Pi_1$ with $\alpha\neq\pm\gamma$ and $\tilde{h}_a = \tilde{w}_a^2$, where $d = (-1)^{\langle\alpha,\beta\rangle}$. Put $\tilde{T} = \langle\tilde{h}_a \mid a\in\Pi_1\rangle\subset\tilde{N}_0$, and $\tilde{N}^* = \tilde{H}\rtimes\tilde{N}_0$, where we note that $\tilde{N}_0$ acts on $\tilde{H}$ by $\tilde{w}_a\cdot\tilde{h} = w_a(-1)\cdot\tilde{h}$ for all $a\in\Pi_1$ and $\tilde{h}\in\tilde{H}$. Then $\tilde{T}$ is the group generated by $\tilde{h}_a$ for all $a\in\Pi_1$ with the following defining relation:

(T) $\tilde{h}_a\tilde{h}_b\tilde{h}_a^{-1} = \tilde{h}_b^d$ for all $a = (\alpha,m)$, $b = (\beta,n)\in\Pi_1$ with $d = (-1)^{\langle\alpha,\beta\rangle}$.

Hence, there is a canonical homomorphism ι of $\tilde{T}$ into $\tilde{H}$ with $\iota(\tilde{h}_a) = \tilde{h}_a(-1)$ for all $a\in\Pi_1$. Let J be the subgroup (which is normal in this case) of $\tilde{N}^*$ generated by $(t,\iota(t)^{-1})$ for all $t\in\tilde{T}$, and $\tilde{N} := \tilde{N}^*/J$ denote the quotient group of $\tilde{N}^*$ by J, and let $\tilde{w}_aJ$ be the canonical image of $\tilde{w}_a$ in $\tilde{N}$. Clearly $\tilde{H}$ can be embedded into $\tilde{N}$, we denote its image in $\tilde{N}$ by $\tilde{H}$ again. Similarly, we use the same notation of $\tilde{h}$ in $\tilde{N}$ as the original element $\tilde{h}\in\tilde{H}$. Then, putting $\tilde{w}_a(u) := \tilde{h}_a(u)\tilde{w}_a^{-1}J$, we have the following for all $a\in\Pi_1$ and $u\in K^\times$:

$$\tilde{w}_a(-1) = \tilde{h}_a(-1)\tilde{w}_a^{-1}J = \tilde{h}_a(-1)\tilde{h}_a(-1)^{-1}\tilde{w}_aJ = \tilde{w}_aJ,$$

$$\begin{aligned}\tilde{w}_a(u)\tilde{w}_a(-u) &= \tilde{h}_a(u)\tilde{w}_a^{-1}\tilde{h}_a(-u)\tilde{w}_a^{-1}J\\ &= \tilde{h}_a(u)(\tilde{w}_a^{-1}\cdot\tilde{h}_a(-u))\tilde{w}_a^{-2}J\\ &= \tilde{h}_a(u)\tilde{h}_a(-u^{-1})\tilde{h}_a(-1)^{-1}J = 1.\end{aligned}$$

Thus, we see that $\tilde{w}_a(-1) = \tilde{w}_a$, $\tilde{w}_a(u)^{-1} = \tilde{w}_a(-u)$ and $\tilde{h}_a(u) = \tilde{w}_a(u)\tilde{w}_a(-1)$ hold in $\tilde{N}$. Note that there is a canonical homomorphism ξ of N^* onto the monomial subgroup N such that $\xi(\tilde{w}_a) = w_a(-1)$ and $\xi(\tilde{h}_a(u)) = h_a(u)$ for all $a \in \Pi_a$ and $u \in K^\times$ and that $J \subset \operatorname{Ker} \xi$. Hence ξ induces a homomorphism, again called ξ, of $\tilde{N}$ onto N. Let ξ^* be the canonical homomorphism of N^* onto $\tilde{N}$, which implies $\tilde{w}_a(u) = \xi^*(\tilde{h}_a(u))\xi^*(\tilde{w}_a)^{-1}$. Since the restriction of ξ^* to $\tilde{H}$ is injective, we identify $\tilde{H}$ with $\xi^*(\tilde{H})$.

Proposition 5.9. *The pair* $(\tilde{N}, \xi)$ *is a central extension of* N *by* L:

$$1 \to L \to \tilde{N} \xrightarrow{\xi} N \to 1$$

In particular, the restriction of ξ *to* $\tilde{H}$ *coincides with* π.

Here, we will show the following lemma for later use.

Lemma 5.10. *Let* $\ell \geq 2$. *Then*:

(1) *Every matrix* $e \in E(A_{\ell-1}^{(1)}, K)_q$ *can be written as a product* $e = uwv$ *with* $u, v \in U$ *and* $w \in N$.

(2) *The monomial matrix part* w *is uniquely determined by* e. (*Thus a well defined retraction* $\tau : E(A_{\ell-1}^{(1)}, K)_q \to N$ *is defined by the formula* $\tau(uwv) = w$.)

Proof. (1) First, we have a Bruhat decomposition in $E(A_{\ell-1}^{(1)}, K)_q$ by Proposition 2.2:

$$E(A_{\ell-1}^{(1)}, K)_q = \bigsqcup_{s \in N/T} BsB \quad \text{(disjoint union)}$$

By this decomposition and $B = U \rtimes T$, we get $e = uwv$ for some $u, v \in U$ and $w \in N$.

(2) Next we show the uniqueness of w. If there are $u', v' \in U$ and $w' \in N$ such that $e = uwv = u'w'v'$, then we have $aw = u'^{-1}uw = w'v'v^{-1} = w'b$ for suitable elements $a, b \in U$. On the other hand, every matrix in U can be expressed as an element in

$$\begin{pmatrix} 1 + K[X_2]X_2 & & & \\ & \ddots & K[X_2] & \\ & & \ddots & \\ K[X_2]X_2 & & \ddots & \\ & & & 1 + K[X_2]X_2 \end{pmatrix}$$

since $e \in BsB$, we have $w' \equiv w \bmod T$. Thus, $a = (a_{ij})$, $b = (b_{ij})$, $w = (w_{ij})$, $w' = (w'_{ij})$ satisfy

$$a_{ij} = \begin{cases} 1 + f_{ii}X_2 & \text{if } i = j \\ f_{ij} & \text{if } i < j, \\ f_{ij}X_2 & \text{if } i > j \end{cases} \quad b_{ij} = \begin{cases} 1 + g_{ii}X_2 & \text{if } i = j \\ g_{ij} & \text{if } i < j, \\ g_{ij}X_2 & \text{if } i > j \end{cases}$$

$$w'_{ij} \equiv w_{ij} = \begin{cases} y_{k_j j} & \text{if } i = k_j \\ 0 & \text{otherwise} \end{cases}$$

with $f_{ij}, g_{ij} \in K[X_2]$, $y_{ij} \in K_q^\times$, and

$$aw = (c_{ij}), \quad c_{ij} = a_{i1}w_{1j} + \cdots + a_{i\dot{\ell}}w_{\dot{\ell}j} = a_{ik_j}w_{k_j j} = \begin{cases} y_{k_j j} + f_{ii}X_2 y_{k_j j} & \text{if } i = k_j, \\ f_{ik_j} y_{k_j j} & \text{if } i < k_j, \\ f_{ik_j}X_2 y_{k_j j} & \text{if } i > k_j. \end{cases}$$

Here, $w'b := (d_{ij})$ induces $(c_{ij}) = (d_{ij})$ and $d_{k_j j} = w'_{k_j 1}b_{1j} + \cdots + w'_{k_j\dot{\ell}}b_{\dot{\ell}j} = w'_{k_j j}b_{jj} = y'_{k_j j} + y'_{k_j j}g_{jj}X_2$ for $y'_{k_j j} \equiv y_{k_j j} \bmod T$, we get following.

$$c_{k_j j} = y_{k_j j} + f_{ii}X_2 y_{k_j j} = y'_{k_j j} + y'_{k_j j}g_{jj}X_2 = d_{k_j j}.$$

Therefore, $y_{k_j j} = y'_{k_j j}$ for all j, and then we have $w = w'$. q.e.d.

Now we proceed as in Matsumoto [12] (also see [13], [20]). For all $w \in N$, we can express $w = P_\varpi \operatorname{diag}(u_1, \ldots, u_{\dot{\ell}})$ using suitable elements $u_1, \ldots, u_{\dot{\ell}} \in K_q^\times$, where P_ϖ is the permutation matrix corresponding to some permutation ϖ of the numbers between 1 and $\dot{\ell}$. Then we have

$$wx_{(ij,m)}(r)w^{-1} = x_{(\varpi(ij),m+d)}(f_{m+d}^{-1}(u_i f_m(r)X_2^m u_j^{-1}X_2^{-m-d})),$$

$$w^{-1}x_{(ij,m)}(r)w = x_{(\varpi^{-1}(ij),m+d')}(f_{m+d'}^{-1}(u_{\varpi^{-1}(i)}^{-1} f_m(r)X_2^m u_{\varpi^{-1}(j)}X_2^{-m-d'}))$$

for all $(ij,m) \in \Pi_1$ and $r \in K$, where $d = \deg(u_j^{-1}u_i)$, $d' = \deg(u_{\varpi^{-1}(i)}^{-1}u_{\varpi^{-1}(j)})$. We often write $(\varpi(ij), m+d) = w(ij,m)$ and $(\varpi^{-1}(ij), m+d') = w^{-1}(ij,m)$ for convenience. Also, for all $e \in E(A_{\ell-1}^{(1)}, K)_q$ and $a, b \in \Pi_1$, we can assume $e = yx_a(r)wx_b(s)z$ for suitable $y \in Y_a$, $z \in Y_b$, $r, s \in K$ and $w \in N$. In particular, r (resp. s) is uniquely determined for a (resp. b). Thus, we have the following lemma below.

Lemma 5.11. *Let* $e = yx_a(r)wx_b(s)z \in E(A_{\ell-1}^{(1)}, K)_q$ *and* $w = P_\varpi$ *diag*$(u_1, \ldots, u_{\dot{\ell}}) \in N$ *be as above, where* $y \in Y_a$, $z \in Y_b$, $r, s \in K$, $u_1, \ldots, u_{\dot{\ell}} \in K_q^\times$ *and* $a = (ij,m)$, $b = (kl,n) \in \Pi_1$. *Then the following results hold.*

(1) $\tau(w_a(1)e)$ *is equal either to* $w_a(1)w$, *or to* $h_a(-r)^{-1}w$. *In detail, we have*

$$w_a(1)x_a(r)w$$
$$= \begin{cases} w_a(1)wx_{w^{-1}(a)}(f_{m+d'}^{-1}(u_{\varpi^{-1}(i)}^{-1}rX_2^m u_{\varpi^{-1}(j)}X_2^{-m-d'})) & \textit{if } r=0 \textit{ or } w^{-1}(a)\in\Phi_1^+, \\ x_a(-r^{-1})h_a(-r)^{-1}wx_{w^{-1}(-a)}(f_{-m-d'}^{-1}(u_{\varpi^{-1}(j)}^{-1}X_2^{-m}r^{-1}u_{\varpi^{-1}(i)}X_2^{m+d'})) & \\ \qquad \textit{if } w^{-1}(a)\notin\Phi_1^+ & \end{cases}$$

where $d' = deg(u_{\varpi^{-1}(i)}^{-1}u_{\varpi^{-1}(j)})$.

(2) $\tau(ew_b(-1))$ *is equal either to* $ww_b(-1)$, *or to* $wh_b(s)$. *In detail, we have*

$$wx_b(s)w_b(-1)$$
$$= \begin{cases} x_{w(b)}(f_{n+d}^{-1}(u_k sX_2^n u_l^{-1}X_2^{-n-d}))ww_b(-1) & \textit{if } s=0 \textit{ or } w(b)\in\Phi_1^+, \\ x_{w(-b)}(f_{-n-d}^{-1}(u_l X_2^{-n}s^{-1}u_k^{-1}X_2^{n+d}))wh_b(s)x_b(-s^{-1}) & \textit{if } w(b)\notin\Phi_1^+ \end{cases}$$

where $d = deg(u_l^{-1}u_k)$.

Proof. (1) We have

$$w_a(1)x_a(r)w = w_a(1)wx_{w^{-1}(a)}(f_{m+d'}^{-1}(u_{\varpi^{-1}(i)}^{-1}rX_2^m u_{\varpi^{-1}(j)}X_2^{-m-d'})) \quad \text{and}$$
$$x_{w^{-1}(a)}(f_{m+d'}^{-1}(u_{\varpi^{-1}(i)}^{-1}rX_2^m u_{\varpi^{-1}(j)}X_2^{-m-d'}))\in U \qquad \text{if } r=0 \text{ or } w^{-1}(a)\in\Phi_1^+.$$

Otherwise, we have

$$w_a(1)x_a(r)w = x_a(-r^{-1})h_a(-r)^{-1}wx_{w^{-1}(-a)}(f_{-m-d'}^{-1}(u_{\varpi^{-1}(j)}^{-1}X_2^{-m}r^{-1}u_{\varpi^{-1}(i)}X_2^{m+d'}))$$
$$\text{and} \qquad x_{w^{-1}(-a)}(f_{-m-d'}^{-1}(u_{\varpi^{-1}(j)}^{-1}X_2^{-m}r^{-1}u_{\varpi^{-1}(i)}X_2^{m+d'}))\in U,$$

using the equation $x_a(r) = x_{-a}(r^{-1})w_a(r)x_{-a}(r^{-1})$. (2) is proved in a similar way as above. q.e.d.

Now, we put $X := \{(e,\tilde{w})\in E(A_{\ell-1}^{(1)},K)_q\times\tilde{N} \mid \tau(e)=\xi(\tilde{w})\}$ and define several permutations $\gamma(h)$, $\gamma^*(h)$, $\mu(u)$, $\mu^*(u)$, η_λ, η_λ^* of X for $h\in\tilde{H}$, $u\in U$ and $\lambda\in\Pi_1$ as follows.

$$\gamma(h)(e,\tilde{w}) := (\xi(h)e, h\tilde{w})$$
$$(e,\tilde{w})\gamma^*(h) := (e\xi(h), \tilde{w}h)$$
$$\mu(u)(e,\tilde{w}) := (ue,\tilde{w})$$
$$(e,\tilde{w})\mu^*(u) := (eu,\tilde{w})$$
$$\eta_\lambda(e,\tilde{w}) := \begin{cases} (w_\lambda(1)e, \tilde{w}_\lambda(1)\tilde{w}) & \text{if } \tau(w_\lambda(1)e) = w_\lambda(1)w \\ (w_\lambda(1)e, \tilde{h}_\lambda(-r)^{-1}\tilde{w}) & \text{if } \tau(w_\lambda(1)e) = h_\lambda(-r)^{-1}w \end{cases}$$

$$(e,\tilde{w})\eta_\lambda^* := \begin{cases} (ew_\lambda(-1), \tilde{w}\tilde{w}_\lambda(-1)) & \text{if } \tau(ew_\lambda(-1)) = ww_\lambda(-1) \\ (ew_\lambda(-1), \tilde{w}\tilde{h}_\lambda(s)) & \text{if } \tau(ew_\lambda(-1)) = wh_\lambda(s) \end{cases}$$

Let G (resp. G^*) be the permutation group of X generated by $\gamma(h)$, $\mu(u)$, η_λ (resp. $\gamma^*(h)$, $\mu^*(u)$, η_λ^*) for all $h \in \tilde{H}$, $u \in U$ and $\lambda \in \Pi_1$.

Here, we will show some relations in $\tilde{N}$. For all $i \in \mathbf{Z}/\dot{\ell}\mathbf{Z}$, we put some notations in case of $\dot{\ell} \geq 3$:

$$\tilde{s}_i = \tilde{w}_{i+1}(1)\tilde{w}_i(1),$$

$$\tilde{t}_i = \tilde{s}_i\tilde{s}_{i+1}\cdots\tilde{s}_{i-1},$$

$$\tilde{u}_i = \tilde{w}_i(-1)\tilde{w}_{i+1}(-1)\cdots\tilde{w}_{i-2}(-1)\tilde{w}_{i-1}(-1)^{(-1)^{\dot{\ell}}}\tilde{w}_{i-2}(1)\cdots\tilde{w}_{i+1}(1).$$

In particular, we have $\xi(\tilde{t}_i) = \mathrm{diag}(X_2^{n_1}, \dots, X_2^{n_{\dot{\ell}}})$ with $n_1 \cdots n_{\dot{\ell}} = 1$ and $n_i = n_{i+1} = 1$, and $\xi(\tilde{u}_i) = \mathrm{diag}(1, \dots, 1, \underbrace{X_2}_{i\text{-th}}, \underbrace{X_2^{-1}}_{(i+1)\text{-th}}, 1, \dots, 1)$. Then we have the following:

$$\tilde{s}_{i-1} \cdot \tilde{h}_i(u) = \tilde{h}_{i-1}(u(i-1,i)^{-1}),$$

$$\tilde{s}_i^{-1} \cdot \tilde{h}_i(u) = \tilde{h}_{i+1}(u(i+1,i)^{-1}),$$

$$\tilde{t}_i \cdot \tilde{h}_i(u) = \tilde{h}_i(X_2uX_2^{-1}),$$

$$\tilde{u}_i \cdot \tilde{h}_j(u) = \tilde{h}_j(u) \quad \text{if } \langle\alpha,\beta\rangle = 0$$

$$\tilde{s}_{i-1} \cdot \tilde{w}_i(1) = \tilde{w}_{i-1}(1),$$

$$\tilde{s}_i^{-1} \cdot \tilde{w}_i(1) = \tilde{w}_{i+1}(1),$$

$$\tilde{t}_i \cdot \tilde{w}_i(1) = \tilde{w}_i(1),$$

$$\tilde{u}_i \cdot \tilde{w}_j(1) = \tilde{w}_j(1) \quad \text{if } \langle\alpha,\beta\rangle = 0$$

for all $i, j \in \mathbf{Z}/\dot{\ell}\mathbf{Z}$, $a_i = (\alpha, m)$, $a_j = (\beta, n) \in \Pi_1$ and $u \in K^\times$. Then, we obtain the following lemma.

LEMMA 5.12. *Let $u \in K^\times$, $a = (\alpha, m) \in \Pi_1$, $\alpha = \varepsilon_i - \varepsilon_j \in \Pi$ and $w \in \tilde{N}_0$ such that $\xi(w) = (b_{kl})$, where $1 \leq k \neq l \leq \dot{\ell}$, $b_{kl} \in K_q^\times$ with $b_{ii} = b_{jj} = 1$. Then we have $w \cdot \tilde{h}_a(u) = \tilde{h}_a(u)$ and $w \cdot \tilde{w}_a(1) = \tilde{w}_a(1)$.*

PROOF. By proposition 5.9, we have $\mathrm{Ker}\,\xi = L$. Then we can write $w = xyz$ for suitable $x \in L$, $y \in \langle \tilde{w}_c(1) \mid c = (\beta, n) \in \Pi_1, \langle\alpha,\beta\rangle = 0\rangle$ and

$z \in \langle \tilde{u}_{i'} \mid a_{i'} = (\beta, n) \in \Pi_1, \langle \alpha, \beta \rangle = 0 \rangle$. Hence, we obtain $w \cdot \tilde{h}_a(u) = \tilde{h}_a(u)$ and $w \cdot \tilde{w}_a(1) = \tilde{w}_a(1)$. q.e.d.

Lemma 5.13. *Let* $v_1, \dots, v_{\dot{\ell}} \in K^\times$, $v_1 \cdots v_{\dot{\ell}} \in [K^\times, K^\times]$, $n_1, \dots, n_{\dot{\ell}} \in \mathbf{Z}$, $a = (ij, m)$, $b \in \Pi_1$, $w \in \tilde{N}$ *such that* $w(a) = b$ *and* $\xi(w) = P_\varpi \, diag(c_1 X_2^{n_1}, \dots, c_{\dot{\ell}} X_2^{n_{\dot{\ell}}}) \, diag(v_1, \dots, v_{\dot{\ell}})$, *where* $c_1, \dots, c_{\dot{\ell}} \in \{1, -1\}$ *and* $c_i = c_j = 1$ *if* $\dot{\ell} \geq 3$. *Then we have the following relations.*

(1) $w\tilde{h}_a(u)w^{-1} = \tilde{h}_b(X_2^{n_i} v_i f_{-m}^{-1}(v_j^{-1}) u X_2^{-n_i}) \tilde{h}_b(X_2^{n_i} v_i f_{-m}^{-1}(v_j^{-1}) X_2^{-n_i})^{-1}$.

(2) $w\tilde{w}_a(-1)w^{-1} = \tilde{h}_b(-X_2^{n_i} v_i f_{-m}^{-1}(v_j^{-1}) X_2^{-n_i}) \tilde{w}_b(1)$.

Proof. (1) By definition of $\tilde{N}$, we can write $w = l\tilde{w}\tilde{h}$ for suitable $l \in L$, $\tilde{h} = \tilde{h}_0(r_0)\tilde{h}_1(r_1) \cdots \tilde{h}_{\dot{\ell}-1}(r_{\dot{\ell}-1})$ with $\xi(\tilde{h}) = \mathrm{diag}(v_1, \dots, v_{\dot{\ell}})$, $r_1, \dots, r_{\dot{\ell}-1} \in K^\times$ and $\tilde{w} \in \tilde{N}_0$ with $\xi(\tilde{w}) = P_\varpi \, \mathrm{diag}(c_1' X_2^{n_1}, \dots, c_{\dot{\ell}}' X_2^{n_{\dot{\ell}}})$, where $c_1', \dots, c_{\dot{\ell}}' \in \{1, -1\}$ and we can put $c_i' = c_j' = 1$ if $\dot{\ell} \geq 3$. Then, we easily obtain:

$$l\tilde{h}\tilde{h}_a(u)\tilde{h}^{-1}l^{-1} = \begin{cases} \tilde{h}_0(v_{\dot{\ell}} X_2 v_1^{-1} X_2^{-1} u) \tilde{h}_0(v_{\dot{\ell}} X_2 v_1^{-1} X_2^{-1})^{-1} & \text{if } a = a_0, \\ \tilde{h}_a(v_i v_j^{-1} u) \tilde{h}_a(v_i v_j^{-1})^{-1} & \text{otherwise.} \end{cases}$$

Next, we will calculate $\tilde{w}\tilde{h}_a(u)\tilde{w}^{-1}$.

Case 1. $a = (ij, m) = b = a_k$.

In this case, we have $n_i = n_j$. Then $\xi(\tilde{w}\tilde{t}_k^{-n_i})$ is a matrix with entry 1 at both positions ii and jj, which implies $\tilde{w}\tilde{h}_a(u)\tilde{w}^{-1} = \tilde{h}_b(u_{n_i})$ by Lemma 5.12.

Case 2. $a = a_i$ and $b = a_k$ with $1 \leq i \neq k \leq \dot{\ell} - 1$.

In this case, we have $n_i = n_{i+1}$ and $\tilde{h}_b(u) = \begin{cases} \tilde{s}_{k-1}^{-1} \cdots \tilde{s}_i^{-1} \cdot \tilde{h}_a(u) & \text{if } i < k \\ \tilde{s}_k \cdots \tilde{s}_{i-1} \cdot \tilde{h}_a(u) & \text{if } i > k \end{cases}$. Then we have

$$\tilde{w}\tilde{h}_a(u)\tilde{w}^{-1} = \tilde{s}_{k-1}^{-1} \cdots \tilde{s}_i^{-1} (\tilde{s}_i \cdots \tilde{s}_{k-1} \tilde{w} \tilde{t}_i^{-n_i}) \tilde{t}_i^{n_i} \cdot \tilde{h}_a(u) = \tilde{h}_b(u_{n_i})$$

if $i < k$. Similarly we have $\tilde{w}\tilde{h}_a(u)\tilde{w}^{-1} = \tilde{h}_b(u_{n_i})$ if $i > k$.

Case 3. $a = a_0$ and $b = a_k$ with $1 \leq k \leq \dot{\ell} - 1$.

In this case, we have $n_1 = n_{\dot{\ell}} + 1$ and $\tilde{s}_{k-1}^{-1} \cdots \tilde{s}_0^{-1} \cdot \tilde{h}_a(u) = \tilde{h}_b(u_{-1})$, then we have

$$\tilde{w}\tilde{h}_a(u)\tilde{w}^{-1} = \tilde{s}_{k-1}^{-1} \cdots \tilde{s}_0^{-1} (\tilde{s}_0 \cdots \tilde{s}_{k-1} \tilde{w} \tilde{t}_0^{-n_{\dot{\ell}}-1}) \tilde{t}_0^{n_{\dot{\ell}}+1} \cdot \tilde{h}_a(u) = \tilde{h}_b(u_{n_{\dot{\ell}}}) = \tilde{h}_b(u_{n_i})$$

since the diagonal matrix part of $\xi(\tilde{s}_{k-1}^{-1} \cdots \tilde{s}_0^{-1})$ is $\mathrm{diag}(z_1, \dots, z_{\dot{\ell}})$ with $z_{\dot{\ell}} = X_2^{-1}$, $z_1 = 1$.

Case 4. $a = a_i$ and $b = a_0$ with $1 \leq i \leq \dot{\ell} - 1$.

In this case, we have $n_{i+1} = n_i - 1$ and $\tilde{s}_0 \cdots \tilde{s}_{i-1} \cdot \tilde{h}_a(u) = \tilde{h}_b(u_1)$, then we have

$$\tilde{w}\tilde{h}_a(u)\tilde{w}^{-1} = \tilde{s}_0 \cdots \tilde{s}_{i-1}(\tilde{s}_{i-1}^{-1} \cdots \tilde{s}_0^{-1}\tilde{w}\tilde{t}_i^{-n_i+1})\tilde{t}_i^{n_i-1} \cdot \tilde{h}_a(u) = \tilde{h}_b(u_{n_i})$$

since the diagonal matrix part of $\xi(\tilde{s}_0 \cdots \tilde{s}_{i-1})$ is $\mathrm{diag}(z_1, \dots, z_\ell)$ with $z_i = X_2$, $z_{i+1} = 1$.

Therefore, we have

$$w\tilde{h}_a(u)w^{-1} = \begin{cases} \tilde{h}_b(X_2^{n_i}v_\ell X_2 v_1^{-1}X_2^{-1}uX_2^{-n_i})\tilde{h}_b(X_2^{n_i}v_\ell X_2 v_1^{-1}X_2^{-1}X_2^{-n_i})^{-1} & \text{if } a = a_0, \\ \tilde{h}_b(X_2^{n_i}v_i v_j^{-1}uX_2^{-n_i})\tilde{h}_b(X_2^{n_i}v_i v_j^{-1}X_2^{-n_i})^{-1} & \text{otherwise.} \end{cases}$$

(2) We assume $w = l\tilde{w}\tilde{h}$ as in (1) of this lemma. By Proposition 5.8, we have the equation

$$\tilde{h}_k(r) \cdot \tilde{w}_a(1) = \tilde{h}_a(r(ak))\tilde{w}_a(1)$$

which implies

$$l\tilde{h} \cdot \tilde{w}_a(1) = \tilde{h}_a(r_0(a0) \cdots r_{\ell-1}(a, \ell - 1))\tilde{w}_a(1) = \begin{cases} \tilde{h}_a(v_\ell X_2 v_1^{-1}X_2^{-1})\tilde{w}_a(1) & \text{if } a = a_0, \\ \tilde{h}_a(v_i v_j^{-1})\tilde{w}_a(1) & \text{otherwise} \end{cases}$$

$$\text{and} \qquad w\tilde{w}_a(1)w^{-1} = \begin{cases} \tilde{h}_b(X_2^{n_\ell}v_\ell X_2 v_1^{-1}X_2^{-n_\ell-1})\tilde{w}_b(1) & \text{if } a = a_0, \\ \tilde{h}_b(X_2^{n_i}v_i v_j^{-1}X_2^{-n_i})\tilde{w}_b(1) & \text{otherwise} \end{cases}$$

in a similar way as in (1) of this lemma. Then we have:

$$w\tilde{w}_a(-1)w^{-1} = \begin{cases} \tilde{h}_b(-X_2^{n_\ell}v_\ell X_2 v_1^{-1}X_2^{-n_\ell-1})\tilde{w}_b(1) & \text{if } a = a_0, \\ \tilde{h}_b(-X_2^{n_i}v_i v_j^{-1}X_2^{-n_i})\tilde{w}_b(1) & \text{otherwise.} \end{cases} \qquad \text{q.e.d.}$$

LEMMA 5.14. *Let* $(e, \tilde{w}) \in X$, $g \in G$ *and* $g^* \in G^*$. *Then the following equation* $(*)$ *holds.*

$$(*) \quad (g(e, \tilde{w}))g^* = g((e, \tilde{w})g^*).$$

PROOF. It suffices to show this for the generators of G and G^*. Then, the only nontrivial case is when $g = \eta_a$ and $g^* = \eta_b^*$, and one only has to compare the second components. Here, we let $\eta_a(\tilde{w}\eta_b^*)$ (resp. $(\eta_a\tilde{w})\eta_b^*$) be the second component of $\eta_a((e, \tilde{w})\eta_b^*)$ (resp. $(\eta_a(e, \tilde{w}))\eta_b^*$) for simplicity. We write $e = yx_a(r)wx_b(s)z \in E(A_{\ell-1}^{(1)}, K)_q$ for suitable $y \in Y_a$, $z \in Y_b$, $r, s \in K$, and $w = P_\varpi \,\mathrm{diag}(u_1, \dots, u_\ell) \in N$ with $u_1 = c_1 X_2^{n_1}v_1, \dots, u_\ell = c_\ell X_2^{n_\ell}v_\ell \in K_q^\times$ as in Lemma 5.13, and $a = (ij, m)$, $b = (kl, n) \in \Pi_1$.

Case 1 $w(b) \neq \pm a$.

In this case, we have the following by Lemma 5.11 and the general fact that $\Gamma(\sigma_\lambda) = \{\lambda\}$ for all $\lambda \in \Pi_1$, where $\Gamma(c) = \{d \in \Phi_1^+ \mid \sigma_c(d) \in \Phi_1^-\}$.

$$(\eta_a\tilde{w})\eta_b^*$$

$$= \left\{ \begin{array}{ll} \tilde{w}_a(1)\tilde{w}\tilde{w}_b(-1) & \text{if "}w^{-1}(a) \in \Phi_1^+ \text{ or } r = 0\text{" and "}w(b) \in \Phi_1^+ \text{ or } s = 0\text{"} \\ \tilde{w}_a(1)\tilde{w}\tilde{h}_b(s) & \text{if "}w^{-1}(a) \in \Phi_1^+ \text{ or } r = 0\text{" and } w(b) \notin \Phi_1^+ \\ \tilde{h}_a(-r)^{-1}\tilde{w}\tilde{w}_b(-1) & \text{if } w^{-1}(a) \notin \Phi_1^+ \text{ and "}w(b) \in \Phi_1^+ \text{ or } s = 0\text{"} \\ \tilde{h}_a(-r)^{-1}\tilde{w}\tilde{h}_b(s) & \text{if } w^{-1}(a) \notin \Phi_1^+ \text{ and } w(b) \notin \Phi_1^+ \end{array} \right\}$$

$$= \eta_a(\tilde{w}\eta_b^*).$$

Case 2 $w(b) = a$.

In this case, we have to show

$$\tilde{w}_a(1)\tilde{w}\tilde{h}_b(s + u_k^{-1}rX_2^m u_l X_2^{-m+n_k-n_l}) = \tilde{h}_a(-r - u_k s X_2^n u_l^{-1} X_2^{-n-n_k+n_l})^{-1}\tilde{w}\tilde{w}_b(-1)$$

where, $n_k = \deg(u_k)$. Here, we have $n - m = -n_k + n_l$ by $w(b) = a$, and put $t = -r - u_k s X_2^n u_l^{-1} X_2^{-m}$. Then we can write the equation above in the following way:

$$\tilde{w}_a(1)\tilde{w}\tilde{h}_b(-u_k^{-1}tX_2^m u_l X_2^{-n}) = \tilde{h}_a(t)^{-1}\tilde{w}\tilde{w}_b(-1).$$

If $t = 0$, then this is obvious. Otherwise, we obtain the above equation as follows.

$$\begin{aligned}
&\tilde{w}_a(1)\tilde{w}\tilde{h}_b(-u_k^{-1}tX_2^m u_l X_2^{-n}) \\
&\quad = \tilde{w}_a(1)\tilde{h}_a(-X_2^{n_k} v_k v_k^{-1} X_2^{-n_k} t X_2^{m+n_l} v_l X_2^{-n} f_{-n}^{-1}(v_l^{-1}) X_2^{n_k}) \\
&\qquad \times \tilde{h}_a(X_2^{n_k} v_k f_{-n}^{-1}(v_l^{-1}) X_2^{n_k})^{-1}\tilde{w} \quad \text{by Lemma 5.13 (1)} \\
&\quad = \tilde{w}_a(1)\tilde{h}_a(-t)\tilde{h}_a(X_2^{n_k} v_k f_{-n}^{-1}(v_l^{-1}) X_2^{n_k})^{-1}\tilde{w} \quad \text{by } X_2^{-n} f_{-n}^{-1}(v_l^{-1}) = v_l^{-1} X_2^{-n} \\
&\quad = \tilde{h}_a(-t^{-1})\tilde{h}_a(X_2^{n_k} v_k^{-1} f_{-n}^{-1}(v_l) X_2^{n_k})^{-1}\tilde{w}_a(1)\tilde{w} \\
&\quad = \tilde{h}_a(-t^{-1})\tilde{h}_a(X_2^{n_k} v_k^{-1} f_{-n}^{-1}(v_l) X_2^{n_k})^{-1}\tilde{h}_a(-X_2^{n_k} v_k f_{-n}^{-1}(v_l^{-1}) X_2^{n_k})^{-1} \\
&\qquad \times \tilde{w}\tilde{w}_b(-1) \quad \text{by Lemma 5.13 (2)} \\
&\quad = \tilde{h}_a(-t^{-1})\tilde{h}_a(-1)^{-1}\tilde{w}\tilde{w}_b(-1) \\
&\quad = \tilde{h}_a(t)^{-1}\tilde{w}\tilde{w}_b(-1).
\end{aligned}$$

Case 3 $w(b) = -a$.

First, if $r = s = 0$ then $(*)$ is obvious, and in the case when at least one of r and s is 0, $(*)$ holds by a simple computation (cf. [20]). Hence we assume that both r and s are not 0. Then, in this case, we have to show

$$\tilde{h}_a(-y)^{-1}\tilde{w}\tilde{h}_b(s) = \tilde{h}_a(-r)^{-1}\tilde{w}\tilde{h}_b(z)$$

where, $y = r + u_l X_2^{-n} s^{-1} u_k^{-1} X_2^{-m}$ and $z = s + u_k^{-1} X_2^{-m} r^{-1} u_l X_2^{-n}$. Then, by the fact that $w_a(-1)w(b) = a$ and

$$\xi(\tilde{w}_a(-1)\tilde{w}) = P'_{\varpi}\,\mathrm{diag}(u_1, \dots, u_{k-1}, -X_2^m u_k, u_{k+1}, \dots, u_{l-1}, X_2^{-m} u_l, \dots),$$

we have the following:

$$\begin{aligned}
&\tilde{h}_a(-r)\tilde{h}_a(-y)^{-1} \\
&= \tilde{w}\tilde{h}_b(z)\tilde{h}_b(s)^{-1}\tilde{w}^{-1} \\
&= \tilde{w}_a(1)(\tilde{w}_a(-1)\tilde{w})\tilde{h}_b(z)\tilde{h}_b(s)^{-1}(\tilde{w}_a(-1)\tilde{w})^{-1}\tilde{w}_a(-1) \\
&= \tilde{w}_a(1)\tilde{h}_a(-X_2^{n_k+m} v_k z f_{-n}^{-1}(v_l^{-1}) X_2^{-n_k-m})\tilde{h}_a(-X_2^{n_k+m} v_k s f_{-n}^{-1}(v_l^{-1}) X_2^{-n_k-m})^{-1}\tilde{w}_a(-1) \\
&= \tilde{h}_a(X_2^m u_k z X_2^n u_l^{-1})^{-1}\tilde{h}_a(X_2^m u_k s X_2^n u_l^{-1}).
\end{aligned}$$

This means that we have to show $\tilde{h}_a(X_2^m u_k s X_2^n u_l^{-1})\tilde{h}_a(-y) = \tilde{h}_a(X_2^m u_k z X_2^n u_l^{-1})\tilde{h}_a(-r)$. Setting $p = -r$ and $x = X_2^m u_k s X_2^n u_l^{-1}$, this is $\tilde{h}_a(x)\tilde{h}_a(p - x^{-1}) = \tilde{h}_a(x - p^{-1})\tilde{h}_a(p)$, so we have to establish $c_a(x, p - x^{-1}) = c_a(x - p^{-1}, p)$. However, we see the following by (L1)$\sim$(L4):

$$\begin{aligned}
c_a(x, p - x^{-1}) &= c_a(x, 1 - px) = c_a(x(1 - 1 + px)^{-1}, 1 - px) = c_a(p^{-1}, 1 - px) \\
&= c_a(1 - px, p) = c_a(x - p^{-1}, p).
\end{aligned}$$

Thus, the equation $(*)$ holds. q.e.d.

LEMMA 5.15. *The group G and G^* operates in a simply transitive manner on X.*

PROOF. Transitivity: $E(A_{\ell-1}^{(1)}, K_q)$ is generated by U and the elements $w_a(1)$. Therefore operating on $(e, \tilde{w}) \in X$ by some sequence of the permutations $\mu(u)$ and η_a, we can certainly transform the first component e of $(e, \tilde{w})$ to e'. That is, we can find a $g_0 \in G$ with $g_0(e, \tilde{w}) = (e', \tilde{w}^*)$. Since both $(e', \tilde{w}')$ and $(e', \tilde{w}^*)$ lie in X, we conclude $\tilde{w}' \equiv \tilde{w}^* \mod L$. Hence operating on $(e', \tilde{w}^*)$ by a suitable $\gamma(h)$ we obtain $(e', \tilde{w}')$. This proves the existence of g with $g(e, \tilde{w}) = (e', \tilde{w}')$ and we prove the transitivity of G. By a similar way, we prove the transitivity of G^*.

Uniqueness: For any $x \in X$, we choose $g_1, g_2 \in G$ such that $g_1 x = g_2 x$. Then, for any $g^* \in G^*$, we have $g_1(xg^*) = g_2(xg^*)$ by Lemma 5.14. This implies $g_1 x' = g_2 x'$ for every $x' \in X$ by the transitivity of G^*, which yields $g_1 = g_2$. Therefore, we prove the uniqueness of G, and uniqueness of G^* in a similar way. q.e.d.

Proposition 5.16. *The map* $\Psi : G \to E(A_{\ell-1}^{(1)}, K)_q$ *is an epimorphism, and the following exact sequence is a central extension of* $E(A_{\ell-1}^{(1)}, K)_q$.

$$1 \to L \to G \xrightarrow{\Psi} E(A_{\ell-1}^{(1)}, K)_q \to 1$$

Proof. First note that the action of an element $g \in G$ on the first component of a pair $(e, \tilde{w}) \in X$ is just the left multiplication by some element $\Psi(g) \in E(A_{\ell-1}^{(1)}, K)_q$. This fact is true for the generators of G, and hence is true for arbitrary elements of G. This defines a homomorphism $\Psi : G \to E(A_{\ell-1}^{(1)}, K)_q$. Since G acts transitively on X, it follows that Ψ is an epimorphism.

The kernel of Ψ can be computed as follows. If $g \in \operatorname{Ker} \Psi$ then $g(e, \tilde{w}) = (e, \tilde{w}')$. The equation $\tau(e) = \xi(\tilde{w}) = \xi(\tilde{w}')$ implies that $\tilde{w}' = l\tilde{w}$ for some $l \in L$. Thus, $g(e, \tilde{w}) = \gamma(l)(e, \tilde{w})$. Using the simply transitivity of the action of G, this proves that $g = \gamma(l)$. Therefore, we have $\operatorname{Ker} \Psi = \gamma(L) \simeq L \subset Z(G)$. q.e.d.

Hence, it is easy now to establish Theorem 5.6 in a standard way as follows (cf. [13]).

Proof of Theorem 5.6. We get the diagram below.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L & \longrightarrow & G & \xrightarrow{\Psi} & E(A_{\ell-1}^{(1)}, K)_q & \longrightarrow & 1 \quad \text{(c.e)} \\ & & \downarrow \zeta & & \uparrow \rho & & & & \\ 1 & \longrightarrow & K_2(A_{\ell-1}^{(1)}, K)_q & \longrightarrow & St(A_{\ell-1}^{(1)}, K)_q & \xrightarrow{\phi} & E(A_{\ell-1}^{(1)}, K)_q & \longrightarrow & 1 \quad \text{(u.c.e)} \end{array}$$

Then $\rho : St(A_{\ell-1}^{(1)}, K)_q \to G$ is unique. Hence we have $\rho(K_2(A_{\ell-1}^{(1)}, K)_q) \subseteq L$ and we see $\rho(\{u, v\}_a) = c_a(u, v)$ as well as $\rho(\hat{d}(w)) = d(w)$ for all $u, v \in K^\times$, $w \in K_{q, X_2}^\times$. But ζ carries $c_a(u, v)$ to $\{u, v\}_a$ and $d(w)$ to $\hat{d}(w)$ for all $u, v \in K^\times$, $w \in K_{q, X_2}^\times$. Since $K_2(A_{\ell-1}^{(1)}, K)_q$ is generated by $\{u, v\}_a$ and $\hat{d}(w)$, and L is generated by $c_a(u, v)$ and $d(w)$, one completes the proof of $L \simeq K_2(A_{\ell-1}, K)_q$. q.e.d.

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