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CONSTRUCTION OF HARMONIC MAPS BETWEEN SEMI-RIEMANNIAN SPHERES

By

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Abstract. We describe a method of manufacturing harmonic maps between semi-Riemannian spheres out of those in Riemannian geometry. After normalization, the resulting maps give examples of helical geodesic immersions in semi-Riemannian geometry.

1. Introduction

Some harmonic maps between semi-Riemannian spheres were obtained by Konderak [5]. Unlike the Riemannian case, it is not so easy to construct harmonic maps between semi-Riemannian spheres, since the semiorthogonal group of the semi-Euclidean n-space $\boldsymbol{R}_t^n$ loses compactness and the Laplacian on $\boldsymbol{R}_t^n$ is not elliptic for $1 \le t \le n-1$. Therefore only finite many harmonic maps were constructed in [5]. Ding and Wang [2] proved that the d-homogeneous harmonic polynomials on the Lorentzian n-space $\boldsymbol{R}_1^n$ are given by a Wick rotation of those on the Euclidean n-space $\boldsymbol{R}^n = \boldsymbol{R}_0^n$. Using this result, they constructed all harmonic maps of the Lorentzian 2-sphere (resp. hyperbolic 2-space) all of whose components form a basis of the space of d-homogeneous harmonic polynomials on $\boldsymbol{R}_1^3$.

In this paper, we shall construct all harmonic maps between semi-Riemannian spheres all of whose components form a basis of the space of d-homogeneous harmonic polynomials on a semi-Euclidean space. Using Weyl algebras, we first generalize the Ding and Wang's result on the harmonic polynomials, that is, the d-homogeneous harmonic polynomials on $\boldsymbol{R}_t^n$ $(0 \le t \le n)$ are given by Wick rotations of those on $\boldsymbol{R}^n$. Applying this result to the canonical basis of the space of d-homogeneous harmonic polynomials on $\boldsymbol{R}^{n+1}$ (Vilenkin [12]), we obtain the required harmonic maps in the explicit form. By multiplying a suitable constant

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factor, the resulting maps are isometric immersions and corresponding to the standard minimal immersions of Riemannian spheres. In Riemannian geometry, it is well-known that the standard minimal immersions are helical geodesic immersions. We show that our isometric immersions are helical geodesic immersions in semi-Riemannian geometry.

2. Harmonic Polynomials on Semi-Euclidean Spaces

Let $\boldsymbol{F}[x] = \boldsymbol{F}[x_1, \dots, x_n]$ be the polynomial algebra in n-variables $x_1, \dots, x_n$, where $\boldsymbol{F}$ is the complex numbers $\boldsymbol{C}$ or the real numbers $\boldsymbol{R}$. The natural decomposition $\boldsymbol{C} = \boldsymbol{R} \oplus \sqrt{-1}\boldsymbol{R}$ induces

$$\boldsymbol{C}[x] = \boldsymbol{R}[x] \oplus \sqrt{-1}\boldsymbol{R}[x]. \tag{1}$$

So, for any polynomial f of $\boldsymbol{C}[x]$, there exist two polynomials $\Re f$ and $\Im f$ of $\boldsymbol{R}[x]$ such that $f = \Re f + \sqrt{-1}\Im f$. Then we denote $\Re f - \sqrt{-1}\Im f$ by $\bar{f}$. We shall denote by $\boldsymbol{F}_d[x]$ the space of d-homogeneous polynomials in $\boldsymbol{F}[x]$ $(d \in \boldsymbol{N}_0)$.

Let $\mathrm{End}_{\boldsymbol{F}}(\boldsymbol{F}[x])$ be the set of all $\boldsymbol{F}$-linear mappings of $\boldsymbol{F}[x]$. We define addition in $\mathrm{End}_{\boldsymbol{F}}(\boldsymbol{F}[x])$ to be the addition of $\boldsymbol{F}$-linear mappings, and multiplication to be the composition. For any $\xi, \eta \in \mathrm{End}_{\boldsymbol{F}}(\boldsymbol{F}[x])$, we will write the multiplication $\xi \circ \eta$ simply $\xi\eta$ when no confusion can arise. We can consider any $f \in \boldsymbol{F}[x]$ as an element of $\mathrm{End}_{\boldsymbol{F}}(\boldsymbol{F}[x])$ by $g \mapsto fg$ for any $g \in \boldsymbol{F}[x]$. Thus $\boldsymbol{F}[x] \subset \mathrm{End}_{\boldsymbol{F}}(\boldsymbol{F}[x])$. Moreover, we put $\mathrm{Der}_{\boldsymbol{F}}(\boldsymbol{F}[x]) = \{\theta \in \mathrm{End}_{\boldsymbol{F}}(\boldsymbol{F}[x]) \mid \theta(fg) = \theta(f)g + f\theta(g) \text{ for } f, g \in \boldsymbol{F}[x]\}$, whose element is called a *derivation* of $\boldsymbol{F}[x]$. There exists $\partial_i \in \mathrm{Der}_{\boldsymbol{F}}(\boldsymbol{F}[x])$ for $1 \le i \le n$ such that, $\partial_i(x_j) = \delta_{ij}$. We can see $\mathrm{Der}_{\boldsymbol{F}}(\boldsymbol{F}[x]) = \bigoplus_{i=1}^n \boldsymbol{F}[x]\partial_i$. Denote the subalgebra of $\mathrm{End}_{\boldsymbol{F}}(\boldsymbol{F}[x])$ which $\boldsymbol{F}[x]$ and $\mathrm{Der}_{\boldsymbol{F}}(\boldsymbol{F}[x])$ generate by $W_n(\boldsymbol{F})$. We will use the symbol $\boldsymbol{N}_0$ to denote the set of all non-negative integers. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{N}_0^n$, we put $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For arbitrary $D \in W_n(\boldsymbol{F})$, we have the unique expression:

$$D = \sum_{|\alpha| \le k} f_\alpha \partial^\alpha,$$

where $k \in \boldsymbol{N}_0$ and $f_\alpha \in \boldsymbol{F}[x]$ $(|\alpha| \le k)$.

Let $GL(n, \boldsymbol{F})$ be the general linear group of degree n over $\boldsymbol{F}$. To each element $g = (g_{ij})$ of $GL(n, \boldsymbol{F})$ corresponds a ring homomorphism $L(g)$ in the space $\boldsymbol{F}[x]$, which transforms the generators x_i into the polynomial of degree one:

$$L(g)(x_i) = \sum_{j=1}^n \hat{g}_{ij} x_j, \quad g^{-1} = (\hat{g}_{ij}),$$

and $L(g)(1) = 1$. Evidently, $L(g_1 g_2) = L(g_1)L(g_2)$ and so L is a representation of $GL(n, \boldsymbol{F})$. We note that $L(g)$ is degree-preserving (i.e., $L(g)(\boldsymbol{F}_d[x]) \subset \boldsymbol{F}_d[x]$ for any $d \in \boldsymbol{N}_0$), and that $L(I) = \mathrm{id}_{\boldsymbol{F}[x]}$, where $I = (\delta_{ij})$ is the unit matrix of degree n.

Hereafter, for any integer t satisfying $0 \le t \le n$, we set

$$\varepsilon_{t,i}^n = \begin{cases} -1 & \text{for } 1 \le i \le t, \\ +1 & \text{for } t < i \le n. \end{cases}$$

Then we put $I_t^n = (\varepsilon_{t,i}^n \delta_{ij}) \in GL(n, \boldsymbol{R})$. Let $O_t(n)$ be a semiorthogonal group with signature $(t, n-t)$, that is,

$$O_t(n) = \{g \in GL(n, \boldsymbol{R}) \mid I_t^n {}^t g I_t^n = g^{-1}\},$$

where ${}^t g$ denotes the transpose of g. We define $\Delta_t^n = \sum_{i=1}^n \varepsilon_{t,i}^n \partial_i^2 \in W_n(\boldsymbol{R})$ and $\mathscr{H}(\boldsymbol{R}_t^n) = \ker \Delta_t^n \subset \boldsymbol{R}[x]$. From a straightforward calculation, we can see that Δ_t^n is an $O_t(n)$-invariant operator, that is,

$$\Delta_t^n \circ L(g) = L(g) \circ \Delta_t^n \quad \text{for any } g \in O_t(n).$$

Hence the kernel $\mathscr{H}(\boldsymbol{R}_t^n)$ of Δ_t^n is an $O_t(n)$-invariant vector subspace of $\boldsymbol{R}[x]$. A polynomial f is *harmonic with respect to* Δ_t^n, if $f \in \mathscr{H}(\boldsymbol{R}_t^n)$. Especially we put

$$\mathscr{H}_d(\boldsymbol{R}_t^n) = \boldsymbol{R}_d[x] \cap \mathscr{H}(\boldsymbol{R}_t^n),$$

that is, the space of d-homogeneous harmonic polynomials with respect to Δ_t^n. It is also an $O_t(n)$-invariant space of $\boldsymbol{R}[x]$ since $L(g)$ $(g \in O_t(n))$ is degree-preserving. In a similar way to discussions of Vilenkin [12, pp. 444–445] (see Liu [6, p. 7] also), we can see

$$\dim \mathscr{H}_d(\boldsymbol{R}_t^n) = (2d + n - 2)\frac{(n + d - 3)!}{(n-2)!d!}.$$

Thus the dimension of $\mathscr{H}_d(\boldsymbol{R}_t^n)$ is independent of the index t. Put $\Delta_t^{n\boldsymbol{C}} = \sum_{i=1}^n \varepsilon_{t,i}^n (\partial_i)^2 \in W_n(\boldsymbol{C})$, $\mathscr{H}(\boldsymbol{C}_t^n) = \ker \Delta_t^{n\boldsymbol{C}} \subset \boldsymbol{C}[x]$ and $\mathscr{H}_d(\boldsymbol{C}_t^n) = \boldsymbol{C}_d[x] \cap \mathscr{H}(\boldsymbol{C}_t^n)$. Then, from (1), $f \in \mathscr{H}_d(\boldsymbol{C}_t^n)$ if and only if both $\Re f$ and $\Im f$ are in $\mathscr{H}_d(\boldsymbol{R}_t^n)$, that is,

(2) $$\mathscr{H}_d(\boldsymbol{C}_t^n) = \mathscr{H}_d(\boldsymbol{R}_t^n) \oplus \sqrt{-1}\mathscr{H}_d(\boldsymbol{R}_t^n).$$

We put $\rho_t = L(\sqrt{I_t^n}^{-1})$, where $\sqrt{I_t^n} = (\sqrt{\varepsilon_{t,i}^n}\delta_{ij}) \in GL(n, \boldsymbol{C})$. There exists the ring homomorphism $\widetilde{\rho_t} : W_n(\boldsymbol{R}) \to W_n(\boldsymbol{C})$ satisfying $\widetilde{\rho_t} = \rho_t$ on $\boldsymbol{R}[x]$. By the definition of $\widetilde{\rho_t}$, we have for the generators ∂_i of $\mathrm{Der}_{\boldsymbol{R}}(\boldsymbol{R}[x])$,

$$\widetilde{\rho_t}(\partial_i) = \frac{1}{\sqrt{\varepsilon_{t,i}^n}}\partial_i = \begin{cases} -\sqrt{-1}\partial_i & \text{for } 1 \le i \le t, \\ \partial_i & \text{for } t < i \le n. \end{cases}$$

Therefore, for the ordinary Laplacian $\Delta_0^n = \Delta$, we have

(3) $$\widetilde{\rho_t}(\Delta_0^n) = \Delta_t^{n\boldsymbol{C}} \in W_n(\boldsymbol{C}).$$

Moreover the following diagram is commutative for any $D \in W_n(\boldsymbol{R})$:

(4) $$\begin{array}{ccc} \boldsymbol{R}[x] & \xrightarrow{\rho_t} & \boldsymbol{C}[x] \\ {\scriptstyle D}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\rho_t}(D)} \\ \boldsymbol{R}[x] & \xrightarrow[\rho_t]{} & \boldsymbol{C}[x]. \end{array}$$

We define $\sigma_t = \rho_t \circ \rho_t = L(I_t^n)$ for $0 \le t \le n$. Since σ_t is involutive (i.e., $\sigma_t \circ \sigma_t = \mathrm{id}_{\boldsymbol{R}[x]}$), we obtain for $0 \le t \le n$,

$$\boldsymbol{R}[x] = P_t^+ \oplus P_t^-,$$

where $P_t^\pm$ is the eigenspace of σ_t corresponding to the eigenvalue ± 1. It is easily seen that $\sigma_t(x^\alpha) = (-1)^{\alpha_1 + \cdots + \alpha_t} x^\alpha$. Hence $\sigma_t \circ \partial_i^2 = \partial_i^2 \circ \sigma_t$ for $1 \le i \le n$ and $0 \le t \le n$. This implies for $0 \le s, t \le n$,

(5) $$\sigma_t \circ \Delta_s^n = \Delta_s^n \circ \sigma_t.$$

Putting $\mathscr{H}_{d,t}^\pm(\boldsymbol{R}_s^n) = P_t^\pm \cap \mathscr{H}_d(\boldsymbol{R}_s^n)$ for $0 \le s, t \le n$, by virtue of (5), we have the following direct decomposition:

$$\mathscr{H}_d(\boldsymbol{R}_s^n) = \mathscr{H}_{d,t}^+(\boldsymbol{R}_s^n) \oplus \mathscr{H}_{d,t}^-(\boldsymbol{R}_s^n).$$

By the definition, P_t^+ (resp. P_t^-) is the subspace which consists of polynomials whose terms are of even (resp. odd) degree with respect to $x_1, \dots, x_t$. Thus ρ_t maps any polynomials in $\mathscr{H}_{d,t}^+(\boldsymbol{R}_s^n)$ (resp. $\mathscr{H}_{d,t}^-(\boldsymbol{R}_s^n)$) to those in $\boldsymbol{C}_d[x]$ which have purely real (resp. imaginary) coefficients. So, because of the injectivity of ρ_t, we have

(6) $$\rho_t(\mathscr{H}_d(\boldsymbol{R}_s^n)) = \rho_t(\mathscr{H}_{d,t}^+(\boldsymbol{R}_s^n)) \oplus \sqrt{-1}\,\Im(\rho_t(\mathscr{H}_{d,t}^-(\boldsymbol{R}_s^n))).$$

LEMMA 2.1. *For any $n, d, t \in \boldsymbol{N}_0$ satisfying $n \ge 1$ and $0 \le t \le n$, we obtain*

$$\rho_t(\mathscr{H}_{d,t}^+(\boldsymbol{R}_0^n)) = \mathscr{H}_{d,t}^+(\boldsymbol{R}_t^n), \quad \Im(\rho_t(\mathscr{H}_{d,t}^-(\boldsymbol{R}_0^n))) = \mathscr{H}_{d,t}^-(\boldsymbol{R}_t^n).$$

Hence we obtain $\rho_t(\mathscr{H}_d(\boldsymbol{R}_0^n)) = \mathscr{H}_{d,t}^+(\boldsymbol{R}_t^n) \oplus \sqrt{-1}\mathscr{H}_{d,t}^-(\boldsymbol{R}_t^n)$.

PROOF. By the commutative diagram (4), the decomposition (6) and Equation (3), we have for any $f^\pm \in \mathscr{H}_{d,t}^\pm(\boldsymbol{R}_0^n)$,

$$0 = \rho_t(\Delta_0^n f^+) = (\widetilde{\rho_t}(\Delta_0^n))(\rho_t(f^+)) = \Delta_t^{n\boldsymbol{C}} \rho_t(f^+) = \Delta_t^n \rho_t(f^+),$$

$$0 = \rho_t(\Delta_0^n f^-) = (\widetilde{\rho_t}(\Delta_0^n))(\rho_t(f^-)) = \Delta_t^{n\boldsymbol{C}} \sqrt{-1}\Im\rho_t(f^-) = \sqrt{-1}\Delta_t^n \Im\rho_t(f^-).$$

Hence $\rho_t(\mathscr{H}_{d,t}^+(\boldsymbol{R}_0^n)) \subset \mathscr{H}_{d,t}^+(\boldsymbol{R}_t^n)$ and $\Im(\rho_t(\mathscr{H}_{d,t}^-(\boldsymbol{R}_0^n))) \subset \mathscr{H}_{d,t}^-(\boldsymbol{R}_t^n)$. Since ρ_t is injective, the real dimension of $\rho_t(\mathscr{H}_d(\boldsymbol{R}_0^n))$ is equal to the one of $\mathscr{H}_d(\boldsymbol{R}_t^n)$ (or $\mathscr{H}_d(\boldsymbol{R}_0^n)$). Therefore we obtain this lemma. □

Remark 2.2. It is known that the following identity on $W_n(\boldsymbol{R})$:

$$r_{0,n}^2 \Delta_0^n - E(E+n-2) = \sum_{i<j} (x_i \partial_j - x_j \partial_i)^2 \tag{7}$$

holds, where $E = \sum_{i=1}^n x_i \partial_i$ is Euler's degree operator. This identity is known as Capelli's identity for $O_0(n)$ ([13]). Applying $\widetilde{\rho_t}$ to (7), we can immediately get Capelli's identity for $O_t(n)$ $(1 \le t \le n)$ as follows:

$$r_{t,n}^2 \Delta_t^n - E(E+n-2) = \sum_{i<j} \varepsilon_{t,i}^n \varepsilon_{t,j}^n (x_i \partial_j - x_j \partial_i)^2.$$

3. Harmonic Maps between Semi-Riemannian Spheres

In this section, we construct harmonic maps of semi-Riemannian unit spheres. We denote the semi-Riemannian n-sphere with constant sectional curvature k and index t by $S_t^n(k) \subset \boldsymbol{R}_t^{n+1}$, that is, $S_t^n(k) = \{p \in \boldsymbol{R}_t^{n+1} \mid -x_1^2(p) - \cdots - x_t^2(p) + x_{t+1}^2(p) + \cdots + x_{n+1}^2(p) = k^{-1}\}$ and the unit n-sphere $S_t^n(1)$ by S_t^n. So, from now on, $\boldsymbol{R}[x]$ stands for $\boldsymbol{R}[x_1, \ldots, x_n, x_{n+1}]$ as the space of all polynomials on $\boldsymbol{R}^{n+1}$. We denote the semi-Riemannian n-sphere with constant sectional curvature k and index t by $S_t^n(k) \subset \boldsymbol{R}_t^{n+1}$, and the unit n-sphere $S_t^n(1)$ by S_t^n.

Now we recall the standard λ-eigenmaps and the standard minimal immersions of the ordinary n-sphere $S^n = S_0^n$ (see [3], [4] and [11] for examples). Let Δ^{S^n} be the Laplacian on S^n. It is well-known that all eigenvalues are given by $\lambda_d = d(d+n-1)$ $(d \in \boldsymbol{N}_0)$ and the eigenspace V_d of Δ^{S^n} corresponding to the eigenvalue λ_d is

$$V_d = \{f|_{S^n} \mid f \in \mathscr{H}_d(\boldsymbol{R}_0^{n+1})\}.$$

(See [4, Theorem (1.9), p. 132] for more details.) Then V_d is an orthogonal $O_0(n+1)$-module. The $O_0(n+1)$-module structure on V_d is given by L and we choose as an $O_0(n+1)$-invariant scalar product $\langle\,,\rangle$ on V_d the L^2-scalar product:

$$\langle f_1, f_2 \rangle = \int_{S^n} f_1 f_2 \, dv_{S^n}, \quad f_1, f_2 \in V_d,$$

where dv_{S^n} is proportional to the volume element of S^n and normalized in such a way that $\int_{S^n} dv_{S^n} = \dim V_d$. For simplicity, we put

$$k(d) = \frac{n}{d(d+n-1)}, \quad m(d) = (2d+n-1)\frac{(d+n-2)!}{d!(n-1)!} - 1.$$

Then we have $\dim V_d = m(d) + 1 = \dim \mathscr{H}_d(\boldsymbol{R}_0^{n+1})$. Let $\{f_i\}_{i=1}^{m(d)+1}$ be an orthonormal basis of V_d, which, at the same time, identifies V_d with $\boldsymbol{R}^{m(d)+1}$. We obtain

$$\sum_{i=1}^{m(d)+1} (f_i)^2 = 1 \quad \text{on } S^n. \tag{8}$$

Then we have the *standard λ_d-eigenmaps* (resp. the *standard minimal immersions of order d*):

$$\phi_{n,d} = \phi_{n,d,0} = (f_1, \dots, f_{m(d)+1}) : S^n \to S^{m(d)},$$

$$(\text{resp. } \psi_{n,d} = \psi_{n,d,0} = \phi_{n,d} \circ \chi_{n,d} : S^n(k(d)) \to S^{m(d)}),$$

where $\chi_{n,d}$ is the homothetic transformation such that $\chi_{n,d}(p) = k(d)^{1/2} \cdot p$ for $p \in \boldsymbol{R}^{n+1}$. These are uniquely determined up to congruence on the range. Let $T^{n+1,d} = T_0^{n+1,d} : O_0(n+1) \to O_0(m(d)+1)$ denote the homomorphism associated to the $O_0(n+1)$-module structure of V_d under the identification $V_d \cong \boldsymbol{R}^{m(d)+1}$. It is obvious that $\phi_{n,d}$ and $\psi_{n,d}$ are equivariant with respect to $T^{n+1,d}$. We note that $\mathscr{H}_d(\boldsymbol{R}_0^{n+1})$ has an $O_0(n+1)$-invariant scalar product induced from the one of V_d since every d-homogeneous polynomials are uniquely determined by its values on S^n.

Hereafter we put

$$\mathscr{K}_d = \{(k_1, \dots, k_{n-2}, k_{n-1}) \in \boldsymbol{Z}^{n-1} \mid d \ge k_1 \ge \cdots \ge k_{n-2} \ge |k_{n-1}|\}.$$

For convenience' sake, we may put $k_0 = d$ and $k_n = 0$. It is easy to check $\#\mathscr{K}_d = \dim \mathscr{H}_d(\boldsymbol{R}_t^{n+1})$. We put $r_{t,n+1}^2 = \sum_{i=1}^{n+1} \varepsilon_{t,i}^{n+1} x_i^2 \in \boldsymbol{R}_2[x]$, which is an $O_0(n+1)$-invariant polynomial in $\boldsymbol{R}[x]$. For the later use, we summarize [12, pp. 466–467] as follows.

LEMMA 3.1. *For any $K = (k_1, \dots, k_{n-2}, k_{n-1}) \in \mathscr{K}_d$, we put*

$$\Xi_K^d(x) = A_K^d \prod_{j=0}^{n-2} \left(r_{0,n+1-j}^{k_j - |k_{j+1}|} C_{k_j - |k_{j+1}|}^{(n-j-1)/2 + |k_{j+1}|} \left(\frac{x_{n+1-j}}{r_{0,n+1-j}} \right) \right)$$

$$\times (x_2 + \operatorname{sgn}(k_{n-1}) \sqrt{-1} x_1)^{|k_{n-1}|},$$

where $\mathrm{sgn}(k_{n-1})$ *is the signature of* k_{n-1}, $C_m^p(t)$ *are Gegenbauer polynomials*

$$C_m^p(t) = \frac{2^m\Gamma(p+m)}{\Gamma(p)}\left(\sum_{k=0}^{[m/2]}\frac{(-1)^k(p+m-k-1)!}{2^{2k}k!(m-2k)!(p+m-1)!}t^{m-2k}\right),$$

and A_K^d *is the normalizing factor. Then* Ξ_K^d *are in* $\mathscr{H}_d(\boldsymbol{C}_0^{n+1})$, *and any harmonic polynomial of* $\mathscr{H}_d(\boldsymbol{R}_0^{n+1})$ *can be uniquely represented as a linear combination of* Ξ_K^d. *Moreover we have for any* $K, M \in \mathscr{K}_d$,

$$\int_{S^n} \Xi_K^d \overline{\Xi_M^d}\, dv_{S^n} = \delta_{KM},$$

where the measure dv_{S^n} *is normalized by* $\int_{S^n} dv_{S^n} = \dim \mathscr{H}_d(\boldsymbol{R}_0^{n+1})$, *and* $\delta_{KM} = 1$ *if* $K = M$, $\delta_{KM} = 0$ if $K \neq M$.

In this paper, for $1 \leq i_1 < \cdots < i_t \leq n+1$, we denote by $\deg_{x_{i_1},\dots,x_{i_t}} f$ the degree of a polynomial $f \in \boldsymbol{R}[x]$ with respect to variables $x_{i_1}, \dots, x_{i_t}$. By the definition of Ξ_K^d, we can show that

$$\text{(9)} \qquad \deg_{x_{n+1-i}} \Xi_K^d = 2\sum_{j=0}^{i-1}\left[\frac{k_j - |k_{j+1}|}{2}\right] + |k_i| - |k_{i+1}| \quad \text{for } 0 \leq i \leq n-1.$$

We note that $k_{n-1} = 0$ if and only if Ξ_K^d is real. So $\{\Xi_K^d\}_{K\in\mathscr{K}_d}$ is not a basis of $\mathscr{H}_d(\boldsymbol{R}_0^{n+1})$. However, from the decomposition (2), it is a simple matter to obtain an orthonormal basis of $\mathscr{H}_d(\boldsymbol{R}_0^{n+1})$. In fact, for $K = (k_1, \dots, k_{n-2}, k_{n-1}) \in \mathscr{K}_d$, we put

$$U_K^d = \begin{cases} \Xi_K^d & \text{for } k_{n-1} = 0, \\ \sqrt{2}\Re\Xi_K^d & \text{for } k_{n-1} > 0, \\ \sqrt{2}\Im\Xi_K^d & \text{for } k_{n-1} < 0. \end{cases}$$

Then $\{U_K^d\}_{K\in\mathscr{K}_d}$ is an orthonormal basis of $\mathscr{H}_d(\boldsymbol{R}_0^{n+1})$. The following two polynomials in $\boldsymbol{R}_l[x_1, x_2]$:

$$U_+^l := \Re(x_2 + \sqrt{-1}x_1)^l = \sum_{i=0}^{[l/2]}(-1)^i\binom{l}{2i}x_1^{2i}x_2^{l-2i},$$

$$U_-^l := \Im(x_2 + \sqrt{-1}x_1)^l = \sum_{i=0}^{[(l-1)/2]}(-1)^i\binom{l}{2i+1}x_1^{2i+1}x_2^{l-(2i+1)}$$

satisfy

$$\mathscr{H}_l(\boldsymbol{R}_0^2) = \mathrm{Span}\{U_+^l, U_-^l\}, \quad (U_+^l)^2 + (U_-^l)^2 = r_{0,2}^{2l},$$

and U_+^l (resp. U_-^l) is of even (resp. odd) degree with respect to x_1. For $2 \le t \le n+1$, by virtue of (9) and the definitions of U_K^d and Ξ_K^d, we can get

$$\deg_{x_1,\dots,x_t} U_K^d \equiv k_{n+1-t} \pmod 2, \tag{10}$$

and see that if $\deg_{x_1,\dots,x_t} U_K^d$ is even (resp. odd), then each terms of U_K^d are also of even (resp. odd) degree with respect to $x_1,\dots,x_t$. So we put $\mathscr{K}_{d,0}^+ := \mathscr{K}_d$, $\mathscr{K}_{d,0}^- := \varnothing$,

$$\mathscr{K}_{d,1}^+ := \{(k_1,\dots,k_{n-2},k_{n-1}) \in \mathscr{K}_d \,|\, k_{n-1} \ge 0\},$$

$$\mathscr{K}_{d,1}^- := \{(k_1,\dots,k_{n-2},k_{n-1}) \in \mathscr{K}_d \,|\, k_{n-1} < 0\},$$

from (10), for $2 \le t \le n+1$,

$$\mathscr{K}_{d,t}^+ := \{K \in \mathscr{K}_d \,|\, k_{n+1-t} : \text{even}\}, \quad \mathscr{K}_{d,t}^- := \{K \in \mathscr{K}_d \,|\, k_{n+1-t} : \text{odd}\}.$$

Then we have for $0 \le t \le n+1$,

$$\mathscr{H}_{d,t}^{\pm}(\boldsymbol{R}_0^{n+1}) = \mathrm{Span}\{U_K^d \,|\, K \in \mathscr{K}_{d,t}^{\pm}\}. \tag{11}$$

Then, since $\{U_K^d\}_{K \in \mathscr{K}_d}$ is an orthonormal basis of $\mathscr{H}_d(\boldsymbol{R}_0^{n+1})$, and $r_{0,n+1}^2 = 1$ on S_0^n, from Equation (8), we have in $\boldsymbol{R}_{2d}[x]$,

$$\sum_{K \in \mathscr{K}_d} (U_K^d)^2 = r_{0,n+1}^{2d}. \tag{12}$$

We put $U_{K,t}^d = \rho_t(U_K^d)$ for $K \in \mathscr{K}_{d,t}^+$ and $U_{K,t}^d = \Im\rho_t(U_K^d)$ for $K \in \mathscr{K}_{d,t}^-$, hence $U_{K,0}^d = U_K^d$ for $K \in \mathscr{K}_d$. Using (11) and Lemma 2.1 and applying ρ_t to (12), we have

Lemma 3.2. *The polynomials $\{U_{K,t}^d\}_{K \in \mathscr{K}_d}$ form a basis of $\mathscr{H}_d(\boldsymbol{R}_t^{n+1})$. Especially we have*

$$\mathrm{Span}\{U_{K,t}^d \,|\, K \in \mathscr{K}_t^+\} = \mathscr{H}_{d,t}^+(\boldsymbol{R}_t^{n+1}),$$

$$\mathrm{Span}\{U_{K,t}^d \,|\, K \in \mathscr{K}_t^-\} = \sqrt{-1}\mathscr{H}_{d,t}^-(\boldsymbol{R}_t^{n+1}).$$

Moreover we obtain

$$-\sum_{K \in \mathscr{K}_{d,t}^-} (U_{K,t}^d)^2 + \sum_{K \in \mathscr{K}_{d,t}^+} (U_{K,t}^d)^2 = r_{t,n+1}^{2d}. \tag{13}$$

We call $\{U_{K,t}^d\}_{K \in \mathscr{K}_d}$ the *canonical basis* of $\mathscr{H}_d(\boldsymbol{R}_t^{n+1})$. Hereafter we put $l(d,t) = \#\mathscr{K}_{d,t}^-$.

PROPOSITION 3.3. *$\mathscr{H}_d(\boldsymbol{R}_t^{n+1})$ has the $O_t(n+1)$-invariant scalar product of index $l(d,t)$, for which the canonical basis $\{U_{K,t}^d\}_{K\in\mathscr{K}_d}$ is orthonormal. With respect to the scalar product, $\mathscr{H}_d(\boldsymbol{R}_t^{n+1})$ is the orthogonal $O_t(n+1)$-module by $T_t^{n+1,d}$ which is the representation given by L in $\mathscr{H}_d(\boldsymbol{R}_t^{n+1})$.*

PROOF. Let $\langle\,,\rangle$ be the scalar product in $\mathscr{H}_d(\boldsymbol{R}_t^n)$ for which $\{U_{K,t}^d\}_{K\in\mathscr{K}_d}$ is an orthonormal basis such that $\langle U_K^d, U_K^d\rangle = +1$ (resp. -1) when $K\in\mathscr{K}_{d,t}^+$ (resp. $K\in\mathscr{K}_{d,t}^-$). We identify $\mathscr{H}_d(\boldsymbol{R}_t^{n+1})$ with $\boldsymbol{R}_{l(d,t)}^{m(d)+1}$ by $\{U_K^d\}_{K\in\mathscr{K}_d}$. For any $g\in O_t(n+1)$, we can write $L(g)(U_K^d) = \sum_{M\in\mathscr{K}_d} c_{MK}U_M^d$ since $\mathscr{H}_d(\boldsymbol{R}_t^{n+1})$ is an $O_t(n+1)$-invariant subspace of $\boldsymbol{R}[x]$. Using Equation (13), we can see that the scalar product is $O_t(n+1)$-invariant and $(c_{MK})\in O_{l(d,t)}(m(d)+1)$ under the identification $\mathscr{H}_d(\boldsymbol{R}_t^{n+1})\cong\boldsymbol{R}_{l(d,t)}^{m(d)+1}$, since $r_{t,n+1}^2$ is an $O_t(n+1)$-invariant polynomial. Therefore the proof is complete. □

For any $n,d,t\in\boldsymbol{N}_0$ satisfying $n,d\geq 1$ and $0\leq t\leq n$, we define $\phi_{n,d,t}: S_t^n\to\boldsymbol{R}_{l(d,t)}^{m(d)+1}$ by

$$\phi_{n,d,t} = (U_{K_1,t}^d,\dots,U_{K_{l(d,t)},t}^d, U_{K_{l(d,t)+1},t}^d,\dots,U_{K_{m(d)+1},t}^d)$$

where $K_1,\dots,K_{l(d,t)}\in\mathscr{K}_{d,t}^-$ and $K_{l(d,t)+1},\dots,K_{m(d)+1}\in\mathscr{K}_{d,t}^+$, and $K_i\neq K_j$ when $i\neq j$. From Proposition 3.3, we can see that these are uniquely determined up to congruence on the range. To prove Theorem 3.5, we quote a special case of a result in [5, Corollary I.3.7]:

LEMMA 3.4. *If $w:\boldsymbol{R}_t^{n+1}\to\boldsymbol{R}_s^{m+1}$ consists of d-homogeneous harmonic polynomials and $w(S_t^n)\subset S_s^m$, then $w|_{S_t^n}: S_t^n\to S_s^m$ is a harmonic map.*

Moreover we note that if $f\in\mathscr{H}_d(\boldsymbol{R}_t^{n+1})$, then $f|_{S_t^n}$ is an eigenfunction of the Laplacian on S_t^n with eigenvalue $\lambda_d = d(d+n-1)$ ([5, Corollary I.3.5] and [6, Theorem 2]).

THEOREM 3.5. *For any $n,d,t\in\boldsymbol{N}_0$ satisfying $n,d\geq 1$ and $0\leq t\leq n$, $\phi_{n,d,t}$ are harmonic maps $S_t^n\to S_{l(d,t)}^{m(d)}$, which is equivariant with respect to the homomorphism $T_t^{n+1,d}$.*

PROOF. By (13) in Lemma 3.2, the image of $\phi_{n,d,t}$ is contained in the unit sphere $S_{l(d,t)}^{m(d)}\subset\boldsymbol{R}_{l(d,t)}^{m(d)+1}\cong\mathscr{H}_d(\boldsymbol{R}_t^{n+1})$. From Proposition 3.3, it is obvious that $\phi_{n,d,t}$ is $O(\boldsymbol{R}_t^{n+1})$-equivariant. So, according to Lemma 3.4, we have the required harmonic map. □

We note that the map $A_t^{n+1} : \boldsymbol{R}_t^{n+1} \to \boldsymbol{R}_{n+1-t}^{n+1}$ given by

$$A_t^{n+1}(x_1, \dots, x_{n+1}) = (x_{t+1}, \dots, x_{n+1}, x_1, \dots, x_t)$$

is an anti-isometry that carries $S_t^n(k)$ anti-isometrically onto $H_{n-t}^n(k)$ ([10, Lemma 24, p. 110]), and if $f : X \to S_t^n(k)$ and $g : S_t^n(k) \to Y$ are maps between semi-Riemannian manifolds, then f is harmonic if and only if $A_t^{n+1} \circ f$ is harmonic; the same equivalence we have for g and $g \circ A_t^{n+1}$ ([5, Remark I.3.2, p. 471]). Thus we obtain

COROLLARY 3.6. *For any $n, d, t \in \boldsymbol{N}_0$ satisfying $n, d \geq 1$ and $0 \leq t \leq n$, there exist the following harmonic maps*:

$$\phi_{n,d,t} : S_t^n \to S_{l(d,t)}^{m(d)},$$

$$\phi_{n,d,t}^{+-} = A_{l(d,t)}^{m(d)+1} \circ \phi_{n,d,t} : S_t^n \to H_{m(d)-l(d,t)}^{m(d)},$$

$$\phi_{n,d,n-t}^{-+} = \phi_{n,d,t} \circ A_t^{n+1} : H_{n-t}^n \to S_{l(d,t)}^{m(d)},$$

$$\phi_{n,d,n-t}^{H} = A_{l(d,t)}^{m(d)+1} \circ \phi_{n,d,t} \circ A_t^{n+1} : H_{n-t}^n \to H_{m(d)-l(d,t)}^{m(d)}.$$

Furthermore we put $\psi_{n,d,t} = \phi_{n,d,t} \circ \chi_{n,d} : S_t^n(k(d)) \to S_{l(d,t)}^{m(d)}$. Of course, $\psi_{n,d,0} = \psi_{n,d}$ is the standard minimal immersion of order d of the ordinary n-sphere $S_0^n = S^n$. It is well known that $\psi_{n,d,0} : S_0^n(k(d)) \to S_0^{m(d)}$ is a helical geodesic immersion of order d (see [8]).

Here we recall the definition of helices and helical geodesic immersions in semi-Riemannian geometry. (For details, see [9].) Let N be a semi-Riemannian manifold. Let c be a unit speed curve in N. The curve c is said to be a *helix of order d in N*, if it has the orthonormal frame field $c_1 = c', c_2, \dots, c_d$ and the following Frenet formulas along c are satisfied for all $1 \leq i \leq d (\leq \dim N)$:

$$\begin{cases} \langle c_i, c_i \rangle = \varepsilon_i, \\ \nabla_{c'} c_i = -\varepsilon_{i-1}\varepsilon_i \lambda_{i-1} c_{i-1} + \lambda_i c_{i+1}, \end{cases}$$

where ∇ denotes the Levi-Civita connection of $N, d \in \boldsymbol{N}$, $\lambda_0 = \lambda_d = \varepsilon_0 = 0$, $c_0 = c_{d+1} = 0$, λ_i $(1 \leq i \leq d-1)$ is a positive constant and $\varepsilon_i \in \{-1, +1\}$ $(1 \leq i \leq d)$. In this paper, we may call such a curve a helix of type $\Lambda = (d; \lambda_1, \dots, \lambda_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$. Let $f : M \to \tilde{M}$ be an isometric immersion between semi-Riemannian manifolds. Suppose that there exist space-like geodesics on M, let γ be any unit speed space-like geodesic of M. If the curve $f \circ \gamma$ in $\tilde{M}$ is a helix of type Λ which are independent of the choice of γ, then f is called a *helical space-like geodesic immersion of type* Λ (or of order d simply). We also define that f is a *helical time-like geodesic immersion* in the same way. To prove the following proposition, we quote [1, Lemma 1.1].

Lemma 3.7. *Let V be an n-dimensional real vector space equipped with a non-degenerate scalar product g of index t. For any r-linear map T on V to a real vector space W and $\varepsilon = -1$ or $+1$ $(-t \le \varepsilon \le t)$, the following conditions are equivalent*:

(a) $T(u, \dots, u) = 0$ *for any vector u of V such that $g(u, u) = \varepsilon$,*
(b) $T(v, \dots, v) = 0$ *for any vector v* of V.

Since $\psi_{n,d,0}$ is a helical geodesic immersion of order d between Riemannian spheres, we can put its type $\Lambda_0 = (d; \lambda_1, \dots, \lambda_{d-1}; +1, \dots, +1)$.

Proposition 3.8. *For any $n, d, t \in \boldsymbol{N}_0$ such that $n, d \ge 1$ and $0 \le t \le n$, $\psi_{n,d,t}$ is an isometric immersion with vanishing mean curvature. Moreover, for $1 \le t \le n-1$, $\psi_{n,d,t}$ is a helical space-like geodesic immersion of type Λ_0.*

Proof. It suffices to prove that the assertion follows for the maps $\tilde{\chi}_{n,d} \circ \phi_{n,d,t} : S^n_t \to S^{m(d)}_{l(d,t)}(k(d)^{-1})$, where $\tilde{\chi}_{n,d}$ is the homothetic transformation such that $\tilde{\chi}_{n,d}(p) = k(d)^{1/2} \cdot p$ for $p \in \boldsymbol{R}^{m(d)+1}$. We use the same latter $\psi_{n,d,t}$ for $\tilde{\chi}_{n,d} \circ \phi_{n,d,t}$.

When $x_1 = \cdots = x_t = 0$, we have

$$U^d_{K,0} = U^d_{K,t} \qquad \text{for any } K \in \mathscr{K}^+_{d,t},$$

$$U^d_{K,0} = U^d_{K,t} = 0 \quad \text{for any } K \in \mathscr{K}^-_{d,t}.$$

At first, we deal with the case of $1 \le t \le n-1$. Let γ be a unit speed space-like geodesic $(0, \dots, 0, \cos s, \sin s)$ of S^n_0 (resp. S^n_t), which is on $S^n_0 \cap S^n_t$ since $1 \le t \le n-1$. When $K \in \mathscr{K}^-_{d,t}$, the components of $\psi_{n,d,t} \circ \gamma$ and $\psi_{n,d,0} \circ \gamma$ are vanishing. Hence $\psi_{n,d,t} \circ \gamma$ is in a positive definite subspace properly. Noting that the Levi-Civita connection of $\mathscr{H}_d(\boldsymbol{R}^{n+1}_0)$ coincides with the one of $\mathscr{H}_d(\boldsymbol{R}^{n+1}_t)$, we can see that $\psi_{n,d,t} \circ \gamma$ satisfies the same Frenet equation of $\psi_{n,d,0} \circ \gamma$. Therefore $\psi_{n,d,t} \circ \gamma$ is a helix of type Λ_0. Since $\psi_{n,d,t}$ is $O_t(n+1)$-equivariant, $\psi_{n,d,t}$ maps any space-like geodesic c of S^n_t to a helix $\psi_{n,d,t} \circ c$ of type Λ_0 in $S^{m(d)}_{l(d,t)}$. Especially $\psi_{n,d,t} \circ c$ is unit speed. So we have $g(x, x) = \psi_{n,d,t}{}^*\tilde{g}(x, x)$ for x is any unit space-like vector of S^n_t, where g (resp. $\psi_{n,d,t}{}^*\tilde{g}$) is the metric of S^n_t (resp. the pull-back of the metric $\tilde{g}$ of $\mathscr{H}_d(\boldsymbol{R}^{n+1}_t)$). Using Lemma 3.7, we see that $g = \psi_{n,d,t}{}^*\tilde{g}$ on

S_t^n. By a semi-Riemannian version of Takahashi's theorem ([7, Theorem 1] for example), the mean curvature of $\psi_{n,d,t}$ is vanishing. Therefore we have this proposition in this case.

Next, we deal with the case of $t = n$. By the definition, we have

$$U_{K,n}^d(x_1, 0, \dots, 0, x_{n+1}) = U_{K,1}^d(x_1, 0, \dots, 0, x_{n+1}) \quad \text{for } K \in \mathscr{K}_d.$$

Moreover, for any $K \in (\mathscr{K}_{d,n}^+ \cap \mathscr{K}_{d,1}^-) \cup (\mathscr{K}_{d,n}^- \cap \mathscr{K}_{d,1}^+) =: \mathscr{L}$, each terms of $U_{K,n}^d$ and $U_{K,1}^d$ are of odd degree with respect to variables $x_2, \dots, x_n$. So $\deg_{x_2,\dots,x_n} U_{K,n}^d = \deg_{x_2,\dots,x_n} U_{K,1}^d \geq 1$. Thus, for any $K \in \mathscr{L}$, we have

$$U_{K,n}^d(x_1, 0, \dots, 0, x_{n+1}) = U_{K,1}^d(x_1, 0, \dots, 0, x_{n+1}) = 0.$$

We note that, for $K \in (\mathscr{K}_d \backslash \mathscr{L})$, the components $U_{K,n}^d$ of $\psi_{n,d,n}$ and $U_{K,1}^d$ of $\psi_{n,d,1}$ are the same causal character each other. Let γ be a unit speed time-like geodesic $(\sinh s, 0, \dots, 0, \cosh s)$ of S_1^n (resp. S_n^n), which is on $S_1^n \cap S_n^n$. Since we had seen that $\psi_{n,d,1}$ is isometric, $\psi_{n,d,1} \circ \gamma$ is a unit speed time-like curve in $S_{l(d,1)}^{m(d)}$. On account of the above arguments, we can see that $\psi_{n,d,n} \circ \gamma$ satisfies the same equation of $\psi_{n,d,1} \circ \gamma$, hence it is a unit speed time-like curve in $S_{l(d,n)}^{m(d)}$. Therefore the same arguments as in the case of $0 \leq t \leq n-1$ imply that $\psi_{n,d,n}$ is isometric. We accomplished the proof. □

By the same reason to get Corollary 3.6, we have

COROLLARY 3.9. *For any $n, d, t \in \boldsymbol{N}_0$ such that $n, d \geq 1$ and $0 \leq t \leq n$, $\psi_{n,d,n-t}^H = A_{l(d,t)}^{m(d)+1} \circ \psi_{n,d,t} \circ A_t^{n+1} : H_{n-t}^n(k(d)) \to H_{m(d)-l(d,t)}^{m(d)}$ is an isometric immersion with vanishing mean curvature, where A_t^{n+1} and $A_{l(d,t)}^{m(d)+1}$ are the anti-isometries in respective vector spaces. Moreover, for $1 \leq t \leq n-1$, $\psi_{n,d,n-t}^H$ is a helical time-like geodesic immersion of type $(d; \lambda_1, \dots, \lambda_{d-1}; -1, \dots, -1)$.*

REMARK 3.10. In [9], the author showed the following result. *Let $f : M \to \tilde{M}$ be an isometric immersion between semi-Riemannian manifolds and M indefinite. If f is a helical space-like geodesic immersion of type $\Lambda = (d; \lambda_1, \dots, \lambda_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$, then f is a helical time-like geodesic immersion of type $\overline{\Lambda} = (d; \lambda_1, \dots, \lambda_{d-1}; (-1)^1\varepsilon_1, \dots, (-1)^d\varepsilon_d)$.* Using this result, we can see that $\psi_{n,d,t}$ $(1 \leq t \leq n-1)$ is a helical time-like geodesic immersion of type $\overline{\Lambda_0} = (d; \lambda_1, \dots, \lambda_{d-1}, (-1)^1, \dots, (-1)^d)$. From the same arguments as in the case of $t = n$ in the proof of Proposition 3.8, we can prove that $\psi_{n,d,n} \circ \gamma$ satisfies the same Frenet equation of $\psi_{n,d,1} \circ \gamma$, hence it is a helix of type $\overline{\Lambda_0}$ in $S_{l(d,n)}^{m(d)}$.

Consequently, $\psi_{n,d,n}$ is a helical time-like geodesic immersion of type $\overline{\Lambda_0}$ since $\psi_{n,d,n}$ is $O_n(n+1)$-equivariant.

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