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ON SCATTERING FOR WAVE EQUATIONS WITH TIME DEPENDENT COEFFICIENTS

By

Kiyoshi MOCHIZUKI

Abstract. We consider the wave equations with perturbation of the first order differential operators, the coefficients of which depend on both the space and time variables. Under suitable conditions on the coefficients, we show the existence of the scattering operator. The main tool used is space-time energy estimates of solutions.

1. Introduction and Results

Let Ω be an exterior domain in $\mathbf{R}^n$ $(n \geq 3)$ with smooth boundary $\partial\Omega$ which is star-shaped with respect to the origin 0. We consider in Ω the wave equation

$$\partial_t^2 w - \Delta w + b_0(x,t)\partial_t w + \sum_{j=1}^n b_j(x,t)\partial_j w + c(x,t)w = 0, \quad (x,t) \in \Omega \times \mathbf{R} \tag{1}$$

with Dirichlet boundary condition

$$w(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbf{R}, \tag{2}$$

where $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, Δ is the n-dimensional Laplacian, and $b_j(x,t)$ $(j = 0,1,\dots,n)$ and $c(x,t)$ are real-valued bounded continuous functions.

Throughout this paper solutions are assumed to be real-valued. Moreover, we restrict ourselves to solutions with finite energy.

The energy at time t of solution $w(x,t)$ is defined by

$$\|\vec{w}(t)\|_E^2 = \frac{1}{2}\int_\Omega \{|\nabla w(x,t)|^2 + w_t(x,t)^2\}\, dx,$$

where ∇w is the gradient of w, $w_t = \partial_t w$, and we mean by $\vec{w}$ the pair of functions $\{w, w_t\}$.

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We compare the solutions of problem (1), (2) with those of the free problem

$$\partial_t^2 w_0 - \Delta w_0 = 0, \quad (x,t) \in \Omega \times \mathbf{R} \tag{3}$$

$$w_0(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbf{R}, \tag{4}$$

More precisely, under suitable smallness conditions on $b_j(x,t)$ and $c(x,t)$, we shall develop a scattering theory between these two problems.

When the coefficients $b_j(x,t)$ and $c(x,t)$ are independent of t, the scattering theory has been studied by Mochizuki [5], [7] under the conditions

$$|b_0(x)| \le \varepsilon\xi(r), \quad b_j(x) \equiv 0 \ (j = 1, \dots, n), \quad |c(x)| \le \varepsilon\xi(r)\frac{n-2}{2r}$$

$(r = |x|)$, where ε is a small positive constant, and $\xi(r)$ is a positive L^1-function of $r > 0$. On the other hand, when they are independent of the space variable, the scattering operator is constructed by Wirth [10] under the conditions

$$b_0(t) \in L^1(\mathbf{R}), \quad b_j(t) \equiv 0 \ (j = 1, \dots, n), \quad c(t) \equiv 0.$$

In this paper, we shall develop a theory which generalizes both of these two results. Our results will cover the coefficients which satisfy

$$\begin{cases} |b_j(x,t)| \le b_{j0}(1+r)^{-\alpha_j}(1+|t|)^{-\beta_j} & (j = 0, 1, \dots, n), \\ |c(x,t)| \le c_0 r^{-1}(1+r)^{-\tilde{\alpha}}(1+|t|)^{-\tilde{\beta}}, \end{cases} \tag{5}$$

where b_{j0}, c_0 are positive constants, and $\alpha_j, \beta_j, \tilde{\alpha}, \tilde{\beta} \ge 0$ satisfy

$$\alpha_j + \beta_j > 1, \quad \tilde{\alpha} + \tilde{\beta} > 1.$$

b_{j0} (or c_0) should be chosen sufficiently small if β_j (or $\tilde{\beta}$) $= 0$.

For the Schrödinger equation

$$i\partial_t u - \Delta u + V(x,t)u = 0, \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}$$

with time dependent complex potential, the scattering operator is constructed in Mochizuki-Motai [8] under similar smallness conditions on the potential. Note that time dependent real potentials have been treated in Howland [1], Yafaev [11], Yajima [12] and Kitada-Yajima [4] without requiring smallness conditions. For time independent complex potentials, the so called smooth perturbation theory has been developed in Kato's classical paper [2] (see also [3]). His theory based on the weighted resolvent estimate is not available in our prolem. In this paper, we directly obtain the necessary space-time weighted energy estimate for problem (1), (2), and use it to obtain asymptotics of solutions (cf., [5], [7]).

Now, let H_D^1 be the closure in the Dirichlet norm

$$\|u\|_D = \left(\int |\nabla u|^2 \, dx \right)^{1/2}$$

of scalar functions u with compact support in Ω, and let L^2 be the usual L^2 space with norm

$$\|u\| = \left(\int |u|^2 \, dx \right)^{1/2}.$$

Here and in the following we denote by $\int$ the integration over the domain Ω. We define $\mathscr{H}_E = H_D^1 \times L^2$. Then $\mathscr{H}_E$ forms a Hilbert space with energy norm

$$\|f\|_E = \frac{1}{\sqrt{2}} \{ \|f_1\|_D^2 + \|f_2\|^2 \}^{1/2}, \quad f = \{f_1, f_2\}.$$

As an evolution equation in $\mathscr{H}_E$, the free problem (3), (4) is rewritten in the matrix form

$$i\partial_t \vec{w}_0 = \Lambda_0 \vec{w}_0, \quad \vec{w}_0 = \{w_0, w_{0t}\} \tag{6}$$

where $i = \sqrt{-1}$ and $\Lambda_0 = i \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$. The operator Λ_0 becomes selfadjoint in $\mathscr{H}_E$ if the domain is defined by $\mathscr{D}(\Lambda_0) = H_D^2 \times \{H_D^1 \cap L^2\}$, where H_D^2 is the set of functions $f_1 \in H_D^1$ such that $\Delta f_1 \in L^2$. Thus, Λ_0 generates a unitary group $\{U_0(t) = e^{-it\Lambda_0}; t \in \mathbf{R}\}$ in $\mathscr{H}_E$, and for given initial data $\vec{w}_0(0) = f \in \mathscr{H}_E$, the solution of (6) is represented as

$$\vec{w}_0(t) = U_0(t) f.$$

The perturbed problem (1), (2) is similarly rewritten in the form

$$i\partial_t \vec{w} = \Lambda_0 \vec{w} + V(t)\vec{w}, \quad \vec{w} = \{w, w_t\}. \tag{7}$$

where

$$V(t) = -i \begin{pmatrix} 0 & 0 \\ b(x,t) \cdot \nabla + c(x,t) & b_0(x,t) \end{pmatrix} \tag{8}$$

with $b(x,t) \cdot \nabla = \sum_{j=1}^{n} b_j(x,t) \partial_j$.

By use of the free unitary group $U_0(t)$, problem (7) is reduced to the integral equation

$$\vec{w}(t) = U_0(t)f + \int_0^t U_0(t-s)V(s)\vec{w}(s)\,ds. \tag{9}$$

As a bounded opertor in $\mathscr{H}_E$, $V(t)$ depends continously on $t \in \mathbf{R}$. So, for given $f \in \mathscr{H}_E$, this equation has a unique solution $\vec{w}(t) \in C(\mathbf{R}; \mathscr{H}_E)$. We denote by $U(t,s) \in \mathscr{B}(\mathscr{H}_E)$ the evolution operator which maps solutions at time s to those at time t. The unique existence of solutions of (9) implies that for each fixed s and t, $U(t,s)$ defines a bijection on $\mathscr{H}_E$.

Let $\eta = \eta(t)$ be a positive L^1-function of $t \in \mathbf{R}$, and let $\xi = \xi(r)$ be a smooth positive L^1-function of $r > 0$ which also satisfies

$$\xi'(r) \le 0, \quad \xi'(r)^2 \le 2\xi(r)\xi''(r). \tag{10}$$

With these functions, we require the following conditions on the coefficients of the perturbed problem (1), (2).

$$(A1) \qquad \begin{cases} |b_j(x,t)| \le \varepsilon\xi(r) + \eta(t) \quad (j = 0, 1, \dots, n) \\ |c(x,t)| \le \{\varepsilon\xi(r) + \eta(t)\} \dfrac{n-2}{2r} \end{cases}$$

in $(x,t) \in \mathbf{R}^n \times \mathbf{R}$, where ε is a small positive constant.

Note that the following functions satisfy condition (10) (cf., Mochizuki-Nakazawa [9]).

$$\xi(r) = (1+r)^{-1-\delta},$$

$$\xi(r) = (e+r)^{-1}[\log(e+r)]^{-1-\delta},$$

$$\xi(r) = (e_k+r)^{-1}[\log(e_k+r)]^{-1} \cdots [\log^{[k]}(e_k+r)]^{-1-\delta},$$

where $\delta > 0$ and e_k, $\log^{[k]}$ are defined by

$$e_0 = 1, e_1 = e, \dots, e_k = e^{e_{k-1}},$$

$$\log^{[0]} a = a, \log^{[1]} a = \log a, \dots, \log^{[k]} a = \log \log^{[k-1]} a.$$

The functions (5) satisfy $(A1)$. To verify this we consider the function

$$a(r,t) = a_0(1+r)^{-\alpha}(1+|t|)^{-\beta},$$

where $a_0 > 0$, $\alpha, \beta \ge 0$ and $\alpha + \beta > 1$. The case $\beta = 0$ is obvious since we can choose $\varepsilon = a_0$ and $\xi = (1+r)^{-\alpha}$. In case $\beta > 0$ we use the Young inequality to obtain

$$a(r,t) \le a_0 \varepsilon \frac{\alpha}{\alpha+\beta}(1+r)^{-\alpha-\beta} + a_0 \varepsilon^{-\alpha/\beta} \frac{\beta}{\alpha+\beta}(1+|t|)^{-\alpha-\beta},$$

where ε is any positive number. Thus, each function of (5) is estimated like

$$\begin{cases} |b_j(x,t)| \le \varepsilon_j \xi_j(r) + \eta_j(t) \quad (j = 0, 1, \dots, n) \\ |c(x,t)| \le \{\tilde{\varepsilon}\tilde{\xi}(r) + \tilde{\eta}(t)\} \dfrac{n-2}{2r} \end{cases}$$

and $(A1)$ holds if we choose

$$\varepsilon = \max\{\varepsilon_j, \tilde{\varepsilon}\}, \quad \xi(r) = \max\{\xi_j(r), \tilde{\xi}(r)\}, \quad \eta(t) = \max\{\eta_j(t), \tilde{\eta}(t)\}.$$

We can now state the main theorem of this paper.

Theorem 1. *Assume $(A1)$ with small $\varepsilon > 0$. Then*

(i) *Every solution of* (1) *is asymptotically free, that is, for every $f \in \mathscr{H}_E$ there exists $f_0^\pm \in \mathscr{H}_E$ such that*

$$\|U(t,0)f - U_0(t)f_0^\pm\|_E \to 0 \quad as \ t \to \pm\infty.$$

(ii) *We put*

$$Z^\pm = s - \lim_{t\to\pm\infty} U_0(-t)U(t,0).$$

Then $Z^\pm$ defines a nontrivial bounded operator on $\mathscr{H}_E$.

(iii) *If ε is chosen to be small enough, then $Z^\pm$ gives a bijection on $\mathscr{H}_E$. Thus, the scattering operator*

$$S = Z^+(Z^-)^{-1} : f_0^- \to f_0^+$$

is well defined and also gives a bijection on $\mathscr{H}_E$.

Remark. The smallness of ε is estimated as

$$\varepsilon \frac{9(2n-3)}{n-2}\|\xi\|_{L^1} < 1 \quad \text{for (i) and (ii)},$$

$$\varepsilon \frac{9(2n-3)}{n-2}\|\xi\|_{L^1}\left\{1 + \varepsilon \frac{20(2n-3)}{n-2}\|\xi\|_{L^1} e^{9\|\eta\|_{L^1}}(1 + 3\|\eta\|_{L^1})\right\} < 1$$

for (iii).

This theorem will be proved in §4 based on the space-time weighted energy estimates of solutions. The estimates for free solutions are treated in the next §2, and the results will be applied in §3 to obtain similar estimates for perturbed solutions.

2. Space-time Weighted Energy Estimates of Free Solutions

The basic identities used in this and next sections are originated by Morawetz [10] (cf., also Strauss [11]), and they are summarized in the present forms in Mochizuki [6].

In this section we consider solutions to the free problem (3), (4).

First we multiply by w_{0t} on both sides of (3). Then

$$\frac{1}{2}\partial_t(w_{0t}^2) - \nabla\cdot(\nabla w_0 w_{0t}) + \frac{1}{2}\partial_t(|\nabla w_0|^2) = 0.$$

Integrating this over $\Omega \times [0, t]$ and taking account of the boundary condition $w_{0t}|_{\partial\Omega} = 0$, we obtain

$$\|\vec{w}_0(t)\|_E^2 = \|\vec{w}_0(0)\|_E^2 \quad \text{for any } t \in \mathbf{R}, \tag{11}$$

which verifies the conservation of the energy.

Next, we multiply by $\psi\left(w_{0r} + \frac{n-1}{2r}w_0\right)$ on both sides of (3), where $\psi = \psi(r)$ is a positive, bounded smooth function of $r = |x| > 0$, and $w_{0r} = \tilde{x}\cdot\nabla w_0$ with $\tilde{x} = x/|x|$. We have

$$\begin{aligned} w_{0tt}\psi&\left(w_{0r} + \frac{n-1}{2r}w_0\right) \\ &= \partial_t\left[w_{0t}\psi\left(w_{0r} + \frac{n-1}{2r}w_0\right)\right] - \frac{1}{2}\nabla\cdot(\tilde{x}\psi w_{0t}^2) + \frac{1}{2}\psi' w_{0t}^2, \end{aligned}$$

$$\begin{aligned} -\Delta w_0\psi\left(w_{0r} + \frac{n-1}{2r}w_0\right) &= -\nabla\cdot\left[\psi\nabla w_0\left(w_{0r} + \frac{n-1}{2r}w_0\right)\right] \\ &\quad + \psi\nabla w_0\cdot(\tilde{x}\cdot\nabla)\nabla w_0 + \psi\frac{1}{r}\{|\nabla w_0|^2 - w_{0r}^2\} + \psi' w_{0r}^2 \\ &\quad + \psi\frac{n-1}{2r}|\nabla w_0|^2 + \left(\psi\frac{n-1}{2r}\right)'(\tilde{x}\cdot\nabla w_0)w_0, \end{aligned}$$

and also

$$\psi\nabla w_0\cdot(\tilde{x}\cdot\nabla)\nabla w_0 = \frac{1}{2}\nabla\cdot[\tilde{x}\psi|\nabla w_0|^2] - \frac{n-1}{2r}\psi|\nabla w_0|^2 - \frac{1}{2}\psi'|\nabla w_0|^2,$$

$$\left(\psi \frac{n-1}{2r}\right)'(\tilde{x} \cdot \nabla w_0) w_0 = \frac{1}{2} \nabla \cdot \left[\tilde{x}\left(\psi \frac{n-1}{2r}\right)' w_0^2\right] + \left(\frac{1}{r}\psi - \psi'\right) \frac{(n-1)(n-3)}{4r^2} w_0^2 - \psi'' \frac{n-1}{2r} w_0^2.$$

Thus, it follows that

$$\partial_t X_0 + \frac{1}{2} \nabla \cdot Y_0 + Z_0 = 0, \tag{12}$$

where

$$X_0 = X_0(x,t) = \psi\left(w_{0t} w_{0r} + \frac{n-1}{2r} w_{0t} w_0\right),$$

$$Y_0 = Y_0(x,t) = \tilde{x}\psi(-w_{0t}^2 + |\nabla w_0|^2) - 2\psi \nabla w_0 \left(w_{0r} + \frac{n-1}{2r} w_0\right) + \tilde{x}\left(\psi \frac{n-1}{2r}\right)' w_0^2,$$

$$Z_0 = Z_0(x,t) = \left(\frac{1}{r}\psi - \psi'\right)\left\{|\nabla w_0|^2 - w_{0r}^2 + \frac{(n-1)(n-3)}{4r^2} w_0^2\right\} + \frac{1}{2}\psi'(w_{0t}^2 + |\nabla w_0|^2) - \psi'' \frac{n-1}{4r} w_0^2.$$

Integrating (12) over $\Omega \times [s,t]$ $(s<t)$, we have

$$\int X_0(t)\, dx - \int X_0(s)\, dx + \frac{1}{2}\int_s^t \int_{\partial\Omega} \nu \cdot Y_0 \, dS d\tau + \int_s^t \int Z_0 \, dx d\tau = 0 \tag{13}$$

With this equation, we can prove the following space-time weighted energy estimate of solutions $w_0(x,t)$.

Theorem 2. *Let $\xi(r)$, $r>0$, be a positive L^1-function satisfying* (10). *Then for any $s<t$ we have*

$$\frac{1}{2}\int_s^t \int \left\{\xi(|\nabla w_0|^2 + w_{0t}^2) - \xi' \frac{n-1}{2r} w_0^2\right\} dx d\tau \le C_1 \|\vec{w}_0(0)\|_E^2,$$

where $C_1 = C_1(\xi) = \dfrac{4n-6}{n-2}\|\xi\|_{L^1}$.

Proof. Let $\psi(r) = \int_0^r \xi(\sigma)\, d\sigma$ in (13). Then since

$$\psi(r) \ge \xi(r) r = \psi'(r) r, \quad |\nabla w_0|^2 - w_{0r}^2 \ge 0 \quad \text{and} \quad n \ge 3,$$

it follows that

$$\int_s^t \int Z_0 \, dxd\tau \geq \frac{1}{2} \int_s^t \int \left\{ \xi(|\nabla w_0|^2 + w_{0t}^2) - \xi' \frac{n-1}{2r} w_0^2 \right\} dxd\tau.$$

Since $v \cdot \tilde{x} \leq 0$ by the starshapedness assumption of $\partial\Omega$, the boundary condition

$$w_0|_{\partial\Omega} = w_{0t}|_{\partial\Omega} = \tilde{\tau} \cdot \nabla w_0|_{\partial\Omega} = 0,$$

where $\tilde{\tau}$ is any tangential vector to the boundary, shows that

$$\begin{aligned} \int_s^t \int_{\partial\Omega} v \cdot Y_0 \, dSd\tau &= \int_s^t \int_{\partial\Omega} \psi\{(v \cdot \tilde{x})|\nabla w_0|^2 - 2(v \cdot \nabla w_0)(\tilde{x} \cdot \nabla w_0)\} \, dSd\tau \\ &= -\int_s^t \int_{\partial\Omega} (v \cdot \tilde{x})\psi|v \cdot \nabla w_0|^2 \, dSd\tau \geq 0. \end{aligned}$$

On the other hand, the well known inequality

$$\int \left(\frac{n-2}{2r}\right)^2 |u(x)|^2 \, dx \leq \int |\nabla u(x)|^2 \, dx \tag{14}$$

and the Schwarz inequality imply that

$$\begin{aligned} \left| \int X_0(\tau) \, dx \right| &\leq \frac{2n-3}{2(n-2)} \|\xi\|_{L^1} \int (|\nabla w_0|^2 + w_{0t}^2) \, dx \\ &\leq \frac{1}{2} C_1 \|\vec{w}_0(\tau)\|_E^2. \end{aligned}$$

These inequalities applied to (13) and the energy identity (11) show the theorem. □

3. Space-time Weighted Energy Estimates of Perturbed Solutions

In this section we consider solutions of the perturbed problem (1), (2).

First we multiply (1) by w_t and integrate by parts over $\Omega \times [s,t]$. Then corresponding to (11) we have

$$\|\vec{w}(t)\|_E^2 - \|\vec{w}(s)\|_E^2 + \int_s^t \int P_1 \, dxd\tau = 0, \tag{15}$$

where

$$P_1 = P_1(x,t) = \{b_0 w_t + b \cdot \nabla w + cw\} w_t.$$

Next, we multiply (1) by $\psi\left(w_r + \dfrac{n-1}{2r} w\right)$ with $\psi(r) = \int_0^r \xi(\sigma) \, d\sigma$. Integration by parts then gives us

$$\int X(t)\,dx - \int X(s)\,dx + \int_s^t \int \{Z + P_2\}\,dsd\tau \le 0 \tag{16}$$

since $\int_s^t \int_{\partial\Omega} \nu \cdot Y\,dSd\tau \ge 0$. Here $X,\ Y,\ Z$ are respectively the functions $X_0,\ Y_0,\ Z_0$ with w_0 replaced by w, and

$$P_2 = P_2(x,t) = \psi\{b_0 w_t + b \cdot \nabla w + cw\}\left(w_r + \frac{n-1}{2r}w\right).$$

LEMMA 1. *We have*

$$\left[\pm C_1 \|\vec{w}(\tau)\|_E^2 + \int X(\tau)\,dx\right]_s^t + \frac{1}{2}\int_s^t \int \left\{\xi(|\nabla w|^2 + w_t^2) - \xi' \frac{n-1}{2r} w^2\right\} dxd\tau$$
$$- \int_s^t \int \{C_1|P_1| + |P_2|\}\,dxd\tau \le 0,$$

where

$$C_1\|\vec{w}(\tau)\|_E^2 - \left|\int X(\tau)\,dx\right| \ge \frac{1}{2}C_1\|\vec{w}(\tau)\|_E^2,$$

$$C_1\|\vec{w}(\tau)\|_E^2 + \left|\int X(\tau)\,dx\right| \le \frac{3}{2}C_1\|\vec{w}(\tau)\|_E^2.$$

PROOF. The above inequalities follow from (15) multiplied by C_1 and (16) if we note

$$\int_s^t \int Z\,dxd\tau \ge \frac{1}{2}\int_s^t \int \left\{\xi(|\nabla w|^2 + w_t^2) - \xi' \frac{n-1}{2r} w^2\right\} dxd\tau$$

and

$$\left|\int X(t)\,dx\right| \le \frac{1}{2}C_1\|\vec{w}(t)\|_E^2, \qquad \square$$

LEMMA 2. *We have*

$$\int_s^t \int \{C_1|P_1| + |P_2|\}\,dxd\tau \le \frac{9}{4}C_1\varepsilon \int_s^t \int \left\{\xi(w_t^2 + |\nabla w|^2) - \xi' \frac{n-1}{2r} w^2\right\} dxdt$$
$$+ \frac{9}{2}C_1 \int_s^t \eta(\tau)\|\vec{w}(\tau)\|_E^2\,d\tau.$$

PROOF. Since $\psi(r) \leq \|\xi\|_{L^1} = \dfrac{n-2}{4n-6} C_1$, as is easily seen

$$\left(|w_t| + |\nabla w| + \frac{n-2}{2r}|w|\right)\left\{C_1|w_t| + \psi\left(|w_r| + \frac{n-1}{2r}|w|\right)\right\}$$

$$\leq C_1\left\{w_t^2 + \left(|\nabla w| + \left|\frac{n-2}{2r}w\right|\right)\left(\frac{\alpha}{2}|\nabla w| + \frac{\alpha'}{2}\left|\frac{n-2}{2r}w\right|\right)\right.$$

$$\left. + \frac{5}{2}|w_t|\left(\beta|\nabla w| + \beta'\left|\frac{n-2}{2r}w\right|\right)\right\}$$

$$\leq C_1\left\{\left(1+\frac{5}{4}\right)w_t^2 + \frac{1}{2}\left(\alpha + \frac{1}{2} + \frac{5}{2}\beta\right)|\nabla w|^2 + \frac{1}{2}\left(\alpha' + \frac{1}{2} + \frac{5}{2}\beta'\right)\left(\frac{n-2}{2r}w\right)^2\right\}$$

$$= \frac{9}{4}C_1\left\{w_t^2 + \gamma|\nabla w|^2 + \gamma'\left(\frac{n-2}{2r}w\right)^2\right\}.$$

Here α, α', β, β', γ and γ' are positive numbers given by

$$\alpha + \alpha' = 1, \quad \beta + \beta' = 1, \quad \gamma + \gamma' = 1,$$

$$\alpha = \frac{n-2}{2n-3}, \quad \beta = \frac{5n-8}{10n-15}, \quad \gamma = \frac{2}{9}\left(\alpha + \frac{1}{2} + \frac{5}{2}\beta\right).$$

So, by use of $(A1)$ we have

$$\int_s^t \int \{C_1|P_1| + |P_2|\}\, dx d\tau$$

$$\leq \frac{9}{4}C_1 \int_s^t \int \{\varepsilon\xi(r) + \eta(\tau)\}\left\{w_t^2 + \gamma|\nabla w|^2 + \gamma'\left(\frac{n-2}{2r}w\right)^2\right\} dx d\tau. \tag{17}$$

Note here

$$|\nabla(\sqrt{\xi}w)|^2 = \left|\sqrt{\xi}\left(\nabla w + \frac{\xi'}{2\xi}\tilde{x}w\right)\right|^2$$

$$= \xi|\nabla w|^2 + \frac{1}{2}\nabla\cdot(\tilde{x}\xi' w^2) - \xi'\frac{n-1}{2r}w^2 - \frac{2\xi''\xi - \xi'^2}{4\xi}w^2.$$

Then since $2\xi''\xi - \xi'^2 \geq 0$ by (10), integrating this, we have

$$\int_s^t \int \xi \left(\frac{n-2}{2r} w \right)^2 dx d\tau \le \int_s^t \| \sqrt{\xi} w \|_D^2 \, d\tau$$

$$\le \int_s^t \int \left\{ \xi |\nabla w|^2 - \xi' \frac{n-1}{2r} |w|^2 \right\} dx d\tau, \tag{18}$$

This and (17) show with (14) the inequality of the lemma. □

Now we put together the inequalities of the above two lemmas. Then

$$\left[\pm C_1 \| \vec{w}(\tau) \|_E^2 + \int X(\tau) \, dx \right]_s^t$$
$$+ \left(\frac{1}{2} - \frac{9}{4} C_1 \varepsilon \right) \int_s^t \int \left\{ \xi (w_t^2 + |\nabla w|^2) - \xi' \frac{n-1}{2r} w^2 \right\} dx d\tau$$
$$- \frac{9}{2} C_1 \int_s^t \eta(\tau) \| \vec{w}(\tau) \|_E^2 \, d\tau \le 0. \tag*{$(19)_\pm$}$$

Based on this inequality we can prove the following theorem.

Theorem 3. *Assume* $(A1)$ *with* $\varepsilon < \dfrac{2}{9C_1}$. *Then*

(i) *There exists* $C_2 = C_2(\eta) > 0$ *such that for any* $t \in \mathbf{R}$, *we have*

$$\| \vec{w}(t) \|_E^2 \le C_2 \| \vec{w}(0) \|_E^2.$$

(ii) *There exists* $C_3 = C_3(\varepsilon, \xi, \eta) > 0$ *such that for any* $s < t$ *we have*

$$\frac{1}{2} \int_s^t \int_\Omega \left\{ \xi (|\nabla w|^2 + w_t^2) - \xi' \frac{n-1}{2r} w^2 \right\} dx d\tau \le C_3 \| \vec{w}(0) \|_E^2.$$

Proof. (i) If $t > 0$, we put $s = 0$ in $(19)_+$. Since $\frac{1}{2} - \frac{9}{4} C_1 \varepsilon > 0$, by means of the second and third inequalities of Lemma 1,

$$\| \vec{w}(t) \|_E^2 \le 3 \| \vec{w}(0) \|_E^2 + 9 \int_0^t \eta(\tau) \| \vec{w}(\tau) \|_E^2 \, d\tau.$$

The Gronwall inequality then shows

$$\| \vec{w}(t) \| \le 3 \| \vec{w}(0) \|_E^2 e^{9 \int_0^t \eta(\tau) \, d\tau}.$$

If $t < 0$, we put $t = 0$ and $s = t$ in $(19)_-$. Then we similarly have

$$\|\vec{w}(t)\|_E^2 \le 3\|\vec{w}(0)\|_E^2 + 9\int_t^0 \eta(\tau)\|\vec{w}(\tau)\|_E^2\, d\tau,$$

and

$$\|\vec{w}(t)\|_E^2 \le 3\|\vec{w}(0)\|_E^2 e^{9\int_t^0 \eta(\tau)\,d\tau}.$$

Thus, the desired inequality holds with $C_2 = 3e^{9\|\eta\|_{L^1}}$.

(ii) Once the boundedness of the energy $\|\vec{w}(t)\|_E^2$ is assured to hold, the remainder of the theorem is obvious from $(19)_+$. In fact

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{9}{4}C_1\varepsilon\right)\int_s^t \int \left\{\xi(w_t^2 + |\nabla w|^2 - \xi' \frac{n-1}{2r} w^2\right\} dx d\tau \\
&\le \frac{3}{2}C_1\|\vec{w}(s)\|_E^2 + \frac{9}{2}C_1\int_s^t \eta(\tau)\|\vec{w}(\tau)\|_E^2\, d\tau \\
&\le \frac{3}{2}C_1C_2(1 + 3\|\eta\|_{L^1})\|\vec{w}(0)\|_E^2,
\end{aligned}$$

and hence (ii) is concluded with

$$C_3 = \frac{3C_1C_2(1 + 3\|\eta\|_{L^1})}{2 - 9C_1\varepsilon}. \qquad \square$$

4. Proof of Theorem 1

Theorem 3 will play a key role to prove Theorem 1. So, ε in $(A1)$ is restricted less than $\dfrac{2}{9C_1}$ from the beginning of this section.

We put $\vec{w}_0(t) = U_0(t)f_0$ and $\vec{w}(t) = U(t,0)f$, and consider the innerproduct

$$(\vec{w}(t), \vec{w}_0(t))_E \quad \text{in } \mathscr{H}_E.$$

Differentiate this. Then by means of (6) and (7) we obtain

$$\frac{d}{dt}(\vec{w}(t), \vec{w}_0(t))_E = -\frac{1}{2}(\{b_0 w_t + b\cdot\nabla w + cw\}(t), w_{0t}(t)), \tag{20}$$

where $(\cdot\,,\cdot)$ is the usual innerproduct of L^2.

PROOF OF THEOREM 1 (i). Since $U(t,0)f = U_0(t)U_0(-t)U(t,0)f$, to verify the assertion we have only to show that $U_0(-t)U(t,0)$ strongly converges in $\mathscr{H}_E$ as $t \to \pm\infty$. We shall show this when $t \to \infty$. A similar argument can be applied when $t \to -\infty$.

Integrating (20) on (s, t) $(s < t)$ gives us

$$(U_0(-t)U(t,0)f - U_0(-s)U(s,0)f, f_0)_E$$
$$= -\frac{1}{2}\int_s^t \int \{b_0 w_t + b \cdot \nabla w + cw\} w_{0t}\, dxd\tau.$$

Here by $(A1)$

$$|b_0 w_t + b \cdot \nabla w + cw| \leq \sqrt{5}\{\varepsilon\xi(r) + \eta(t)\}\left\{w_t^2 + \frac{1}{2}|\nabla w|^2 + \frac{1}{2}\left(\frac{n-2}{2r}w\right)^2\right\}^{1/2}.$$

So, after using the Schwarz inequality, we can apply (18), Theorems 3 (i) and (11) to get

$$|(U_0(-t)U(t,0)f - U_0(-s)U(s,0)f, f_0)_E|$$
$$\leq \frac{1}{2}\sqrt{5}\varepsilon \times \left(\int_s^t \int \left\{\xi(w_t^2 + |\nabla w|^2) - \xi' \frac{n-1}{2r} w^2\right\} dxd\tau\right)^{1/2} \left(\int_s^t \int \xi w_{0t}^2\, dxd\tau\right)^{1/2}$$
$$+ \sqrt{5}C_2 \int_s^t \eta(\tau)\, d\tau \|\vec{w}(0)\|_E \|\vec{w}_0(0)\|_E. \tag{21}$$

Note that

$$\int_s^\infty \int \xi w_{0t}^2\, dxd\tau \leq 2C_1 \|\vec{w}_0(0)\|_E^2 = 2C_1 \|f_0\|_E^2$$

by Theorem 2. Then since

$$\int_s^\infty \int \left\{\xi(w_t^2 + |\nabla w|^2 - \xi' \frac{n-1}{2r} w^2\right\} dxd\tau \to 0 \quad \text{as } s \to \infty$$

by Theorem 3 (ii), it follows from (21) that

$$|(U_0(-t)U(t,0)f - U_0(-s)U(s,0)f, f_0)_E|$$
$$\leq \left[\frac{1}{2}\sqrt{10C_1}\varepsilon \times \left(\int_s^\infty \int \left\{\xi(w_t^2 + |\nabla w|^2) - \xi' \frac{n-1}{2r} w^2\right\} dxd\tau\right)^{1/2}\right.$$
$$\left. + \sqrt{5}C_2 \int_s^\infty \eta(\tau)\, d\tau \|f\|_E\right] \|f_0\|_E \to 0 \quad \text{as } s \to \infty.$$

This shows that $U_0(-t)U(t,0)$ converges strongly in $\mathscr{H}_E$ as $t \to \infty$.

To conclude the assertion we put

$$f_0^\pm = s - \lim_{t \to \pm\infty} U_0(-t)U(t,0)f.$$

Then as is expected

$$\|U(t,0)f - U_0(t)f_0^{\pm}\|_E = \|U_0(t)\{U_0(-t)U(t,0)f - f_0^{\pm}\}\|_E \to 0$$

when $t \to \pm\infty$. □

PROOF OF THEOREM 1 (ii). We apply the above argument to the operator $U_0(s-t)U(t,s)$ with fixed $s \in \mathbf{R}$, and put

$$Z^{\pm}(s) = s - \lim_{t \to \pm\infty} U_0(s-t)U(t,s).$$

$Z^{\pm}(s)$ defines a bounded operator in $\mathscr{H}_E$. We shall show that for each $0 \neq f \in \mathscr{H}_E$ there exists $s > 0$ sufficiently large such that $Z^+(s)U_0(s)f \neq 0$. (Similarly, we can show the existence of $s < 0$ sufficiently small such that $Z^-(s)U_0(s)f \neq 0$.) Then since

$$Z^{\pm}U(0,s)U_0(s)f = U_0(-s)Z^{\pm}(s)U_0(s)f,$$

$Z^{\pm}$ is verified to be a nontrivial bounded operator.

Let $\vec{w}(t) = U(t,s)f$ and $\vec{w}_0(t) = U_0(t-s)f_0$ in (2). Then the argument of the proof of (1) yields

$$\begin{aligned} &|(Z^+(s)f, f_0)_E - (f, f_0)_E| \\ &\quad \leq \frac{1}{2}\sqrt{5}\varepsilon \times \left(\int_s^{\infty}\int\left\{\xi(w_t^2 + |\nabla w|^2) - \xi'\frac{n-1}{2r}w^2\right\} dxd\tau\right)^{1/2}\left(\int_s^{\infty}\int \xi w_{0t}^2\, dxd\tau\right)^{1/2} \\ &\qquad + \sqrt{5}C_2\int_s^{\infty}\eta(\tau)\, d\tau\|\vec{w}(s)\|_E\|\vec{w}_0(s)\|_E, \end{aligned} \tag{22}$$

We choose here $f = f_0 = U_0(s)g$, where $g \neq 0$, and assume that $Z^+(s)U_0(s)g = 0$ for any $s > 0$. Then since

$$\int_s^{\infty}\int \xi w_{0t}^2\, dxd\tau \leq \int_s^{\infty}\int \xi|U_0(\tau - s)U_0(s)g|^2\, dxd\tau,$$

it follows from Theorem 3 (ii) that

$$\begin{aligned} \|U_0(s)g\|_E^2 \leq \frac{1}{2}\sqrt{10C_3}\varepsilon\|U_0(s)g\|_E\left(\int_s^{\infty}\int \xi|U_0(\tau)g|^2\, dxd\tau\right)^{1/2} \\ + \sqrt{5}C_2\int_s^{\infty}\eta(\tau)\, d\tau\|U_0(s)g\|_E^2. \end{aligned}$$

$\|U_0(s)g\|_E = \|g\|_E > 0$ is independent of s, whereas the right side goes to 0 as $s \to \infty$. These cause a contradiction and the assertion (ii) is proved. $\square$

Proof of Theorem 1 (iii). We shall show that if ε is chosen to be small enough, then there exists $s \geq 0$ (or $s \leq 0$) such that $Z^+(s)$ (or $Z^-(s)$) defines a bijection on $\mathscr{H}_E$. Then since

$$Z^{\pm} = U_0(-s)Z^{\pm}(s)U(s,0),$$

$Z^{\pm}$ is verified to be a bijection on $\mathscr{H}_E$, and the scattering operator $S = (Z^+)^{-1}Z^-$ is well defined also as a bijection on $\mathscr{H}_E$.

Assume that $g \in \mathscr{H}_E$ satisfies $Z^+(s)g = 0$ or $g \perp Z^+(s)\mathscr{H}_E$. Then putting $f = f_0 = g$ in (22), we obtain

$$\|g\|_E^2 \leq \sqrt{5C_3C_1}\varepsilon\|g\|_E^2 + \sqrt{5}C_2\int_s^\infty \eta(\tau)\,d\tau\|g\|_E^2.$$

So, if ε is small enough to satisfy $\sqrt{5C_3C_1}\varepsilon < 1$, then we have

$$\sqrt{5C_3C_1}\varepsilon + \sqrt{5}C_2\int_s^\infty \eta(\tau)\,d\tau < 1$$

choosing $s > 0$ sufficiently large. These conclude $g = 0$, and hence $Z^+(s)$ becomes a bijection.

The same conclusion is valid also for $Z^-(s)$ with sufficiently small $s < 0$.

The assertion (iii) is proved. $\square$

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Faculty of Science and Engineering
Chuo University
Kasuga
Bunkyo
Tokyo 112-8551, Japan