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A FURTHER NOTE ON THE GENERALIZED JOSEPHUS PROBLEM

By

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1. In our previous papers [1] and [2] we have observed several interesting and significant aspects of the generalized Josephus problem. In the present article we shall again concern ourselves with this problem. Thus, given a total number $n \geq 1$ and certain n objects numbered from 1 to n, and another integer $m \geq 1$, called the reduction coefficient, we arrange these n objects in a circle and, starting with the object numbered 1, and counting each object in turn around the circle, we eliminate every mth object until all of them are removed. By $a_m(k,n)$ $(1 \leq k \leq n)$ we denote as before the kth Josephus number, that is, the object number to be removed in the kth step of elimination. It is evident that we have

$$(1) \qquad 1 \leq a_m(k,n) \leq n$$

and

$$(2) \qquad a_m(1,n) \equiv m \pmod{n},$$

and that

$$a_m(k+1,n+1) \equiv a_m(1,n+1) + a_m(k,n) \pmod{n+1},$$

from which follows at once

$$(3) \qquad a_m(k+1,n+1) \equiv m + a_m(k,n) \pmod{n+1}$$

in view of (2); (3) is the fundamental relation due to P. G. Tait for the Josephus numbers $a_m(k,n)$ (cf. [1; §§1–2]). In effect, the Josephus numbers $a_m(k,n)$ $(1 \leq k \leq n)$ are completely determined by the conditions (1), (2) and (3).

In what follows we devote ourselves to the study of the special case of $k = n$ and write for simplicity's sake $d_m(n) = a_m(n,n)$ as in [1]. We have then $d_m(1) = 1$ for any $m \geq 1$, and the fundamental relation (3) becomes

$$(4) \qquad d_m(n+1) \equiv m + d_m(n) \pmod{n+1}.$$

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Now, in connexion with his study of a Japanese version of the Josephus problem, Seki Takakazu (1642?–1708) called any positive integer n for which one has $d_m(n+1) = 1$, if it exists, a limitative number with respect to the reduction coefficient m; compare [1; §8]. We have formulated there a hypothesis on the infinitude of limitative numbers n for every fixed $m \geq 2$, regarding it as an implicit intention of Seki's. The validity of this hypothesis is easy to prove for $m = 2$ and 3, but for $m \geq 4$ it appears to be difficult to settle it. At present we are able only to show that there are infinitely many integers n satisfying the condition

$$1 \leq d_m(n+1) \leq m - 1$$

for every fixed reduction coefficient $m \geq 2$ (cf. [2; §3]). In this respect it will be of some interest to note that the set of positive integers m for which exist only a bounded number of integers n satisfying $d_m(n+1) = 1$ has natural density 0; in other words, there are unboundedly many limitative numbers n for almost all, so to say, values of the reduction coefficient m (≥ 4) (see §3 below).

In the present note we wish to provide a proof for this metric result as an approach to the original hypothesis mentioned above.

NOTE. Let S be a set of positive integers m. The upper asymptotic density $\bar{\delta}(S)$ of the set S is defined by

$$\bar{\delta}(S) = \limsup_{X \to \infty} \frac{1}{X} \sum_{\substack{m \in S \\ m \leq X}} 1$$

and the lower asymptotic density $\underline{\delta}(S)$ of S is with lim inf in place of lim sup; we always have $\bar{\delta}(S) \geq \underline{\delta}(S)$ and, in case the upper and lower asymptotic densities coincide with each other, say $\bar{\delta}(S) = \delta = \underline{\delta}(S)$, the common value $\delta = \delta(S)$ is the natural density of the set S. If in particular $\bar{\delta}(S) = 0$ then we have naturally $\delta(S) = 0$.

2. Let n, p and q be given positive integers > 1. We denote by $H(n)$ the set of positive integers m for which one has $d_m(n) = 1$ and by $H(p,q)$ the set of positive integers m such that $d_m(p) = d_m(q) = 1$. If $p = q$ then $H(p,q) = H(p,p) = H(p)$.

We set $M_1 = 1$ and for $n > 1$

$$M_n := \text{L.C.M.}(1, 2, \ldots, n).$$

LEMMA 1. *For any* ℓ $(1 \le \ell \le n)$ *the number* $Z(n;\ell)$ *of integers* m $(1 \le m \le M_n)$ *satisfying the condition* $d_m(n) = \ell$ *is given by*

$$Z(n;\ell) = \frac{M_n}{n},$$

so that, in particular, the natural density $\delta(H(n))$ *exists and equals* $1/n$.

This is the special case $k = n$ of Proposition 3 in [2].

LEMMA 2. *Suppose that* p *and* q *be prime numbers,* $p < q$. *Then, for any* ℓ_p $(1 \le \ell_p \le p)$ *and any* ℓ_q $(1 \le \ell_q \le q)$ *the number* $Z(p,q;\ell_p,\ell_q)$ *of integers* m $(1 \le m \le M_q)$ *fulfilling the conditions* $d_m(p) = \ell_p$ *and* $d_m(q) = \ell_q$ *is given by*

$$Z(p,q;\ell_p,\ell_q) = \frac{M_q}{pq},$$

so that, in particular, the natural density $\delta(H(p,q))$ *exists and is equal to* $1/(pq)$.

PROOF. Consider the system of q congruences in m (cf. (4)):

$$m \equiv h_i - h_{i-1} \pmod{i} \quad (i = 1,2,\dots,q), \tag{5}$$

where $h_0 = 0$ and the h_i $(1 \le i \le q)$ are parameters taking some integer values such that

$$1 \le h_i \le i \quad (1 \le i \le q);$$

thus, $h_1 = 1$ and the first congruence in the system (5) is absurd, so that we shall actually deal with (5) only for $2 \le i \le q$.

We fix $h_1 = 1$, $h_p = \ell_p$ and $h_q = \ell_q$. For an arbitrary integer j $(2 \le j \le q)$ we contemplate the subsystem of (5):

$$m \equiv h_i - h_{i-1} \pmod{i} \quad (i = 2,\dots,j). \tag{6}$$

The system of congruences (6) may admit a solution

$$m \equiv m_j \pmod{M_j}$$

under certain conditions, in general, to be imposed on the integers h_i. Anyway there may be several, mutually incongruent solutions $m_j \pmod{M_j}$ of (6), where $m_j = m_j(h_1,h_2,\dots,h_j)$ depends on the ordered j-tuple of integers $(h_1,h_2,\dots,h_j)$, and it is readily seen that if moreover $(h_1', h_2', \dots, h_j')$ is such a j-tuple different from $(h_1, h_2, \dots, h_j)$, then we have

$$m_j(h_1', h_2', \dots, h_j') \not\equiv m_j(h_1, h_2, \dots, h_j) \pmod{M_j}.$$

For $j = 2$ we have plainly with $1 \le h_2 \le 2$

$$m_2 = m_2(h_1, h_2) \equiv h_2 - h_1 = h_2 - 1 \pmod{M_2}.$$

For $j \ge 3$ the solvability condition for the system

(7) $$\begin{cases} m \equiv m_{j-1} & \pmod{M_{j-1}} \\ m \equiv h_j - h_{j-1} & \pmod{j}, \end{cases}$$

which is equivalent to (6), is provided by

(8) $$m_{j-1} \equiv h_j - h_{j-1} \pmod{d_j},$$

where

$$d_i = \text{G.C.D.}(M_{i-1}, i) \quad (i \ge 2).$$

Having determined m_{j-1} modulo M_{j-1} with $(h_1, \dots, h_{j-1})$, we fix h_j to the modulus d_j by $(h_1, \dots, h_{j-1})$ according to the congruence (8), so that the number of possible choices for the value of h_j turns out to be equal primarily to j/d_j.

Setting $Z_1 = M_1 = 1$, we denote by Z_j for $2 \le j \le q$ the number of different (i.e. incongruent) solutions $m_j \pmod{M_j}$ of the system (6), or of the system (7). Clearly $Z_q = Z(p, q; \ell_p, \ell_q)$.

If $2 \le j < p$ then we have

$$Z_j = Z_{j-1} \frac{j}{d_j} = M_j.$$

For $j = p$, a prime, we have $d_p = 1$ and may arbitrarily fix the integer $h_p = \ell_p$ with $1 \le \ell_p \le p$, so that

$$Z_p = Z_{p-1} \cdot 1 = M_{p-1} = \frac{M_p}{p};$$

for $p + 1 \le j \le q$ we find, as above, that

$$Z_j = Z_{j-1} \frac{j}{d_j} = \frac{M_j}{p},$$

and finally for $j = q$, a prime different from p, we have again $d_q = 1$ and, therefore, with $h_q = \ell_q$, $1 \le \ell_q \le q$,

$$Z_q = Z_{q-1} \cdot 1 = \frac{M_{q-1}}{p} = \frac{M_q}{pq},$$

which was to be proved.

Needless to add, our Lemma 2 can naturally be extended to the case in which three or more distinct primes are involved. Given an arbitrary finite set P of prime numbers p and a set (ℓ_p) of prescribed integers ℓ_p with $1 \leq \ell_p \leq p$ $(p \in P)$, the number $Z(P; (\ell_p))$ of integers m $(1 \leq m \leq M_s)$ such that we have

$$d_m(p) = \ell_p \quad \text{for all} \quad p \in P$$

is found to be equal to M_s/D, where s is any integer not less than the maximal prime of the set P and D is the product of all primes $p \in P$.

3. We are now in a position to enunciate and establish our principal result about the hypothesis of Seki, as mentioned in § 1 above. We shall prove the following

Theorem. *For all values of the reduction coefficient* m (>1), *except possibly for a set of integers* m *of natural density* 0, *there exist unboundedly many positive integers* n *satisfying the condition* $d_m(n) = 1$.

Proof. Let A_0 (resp. $A_0(v)$, v being a natural number) the set of positive integers m such that there are only a bounded number (resp. at most v in number) of integers n satisfying $d_m(n) = 1$. We have to show that $\delta(A_0) = 0$; this can be achieved, if we prove that $\delta(A_0(v)) = 0$ however large the bound v $(<+\infty)$ may be, since we have $A_0(v) \subseteq A_0(v')$ if $v < v'$ so that

$$A_0 = \bigcup_{1 \leq v < +\infty} A_0(v) \quad \text{and} \quad \delta(A_0) = \sup_{1 \leq v < +\infty} \delta(A_0(v)) = 0.$$

We define for a fixed positive integer n

$$c_m(n) = \begin{cases} 1 & \text{if } d_m(n) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

this is the characteristic function of the set $H(n)$ of integers m for which holds $d_m(n) = 1$. Denoting by p and q generic primes, we have, in virtue of Lemmas 1 and 2,

$$\delta(H(p)) = \frac{1}{M_s} \sum_{m=1}^{M_s} c_m(p) = \frac{1}{p} \quad \text{if} \quad p \leq s \tag{9}$$

and

$$\delta(H(p,q)) = \frac{1}{M_s}\sum_{m=1}^{M_s} c_m(p)c_m(q) = \begin{cases} \dfrac{1}{pq} & \text{if } p \neq q,\ p,q \leq s, \\ \dfrac{1}{p} & \text{if } p = q \leq s. \end{cases} \tag{10}$$

We now calculate, with a positive real number Q, the dispersion

$$V(Q) := \lim_{X\to\infty} \frac{1}{X}\sum_{m\leq X}\left(\sum_{p\leq Q}\left(c_m(p) - \frac{1}{p}\right)\right)^2, \tag{11}$$

where $\sum_{p\leq Q}$ indicates the summation over the prime numbers $p \leq Q$.

Let s be any integer not less than the largest prime $\leq Q$. Then it follows from (9) and (10) that

$$V(Q) = \frac{1}{M_s}\sum_{m=1}^{M_s}\left(\sum_{p\leq Q}\left(c_m(p) - \frac{1}{p}\right)\right)^2 = \sum_{p\leq Q}\frac{1}{p}\left(1 - \frac{1}{p}\right), \tag{12}$$

which ensures the existence of the limit on the right-hand side of (11).

For any natural number v let us denote by $A(v)$ the set of positive integers m for which we have $d_m(p) = 1$ for at most v primes p in number.

Writing for the sake of brevity

$$S(Q) := \sum_{p\leq Q}\frac{1}{p},$$

we have for every $m \in A(v)$

$$\left|\sum_{p\leq Q}\left(c_m(p) - \frac{1}{p}\right)\right| \geq S(Q) - v.$$

Consequently, however large the bound v $(<+\infty)$ may be, we may choose Q so large as to satisfy $S(Q) > 2v$, which is certainly possible, since $S(Q)$ tends to infinity with Q, as is seen from the well-known inequality

$$S(Q) > \log\log Q - \frac{1}{2} \quad (Q > 2),$$

and we find, by (11),

$$V(Q) \geq \limsup_{X\to\infty} \frac{1}{X} \sum_{\substack{m\leq X \\ m\in A(v)}} \left(\sum_{p\leq Q} \left(c_m(p) - \frac{1}{p} \right) \right)^2$$

$$\geq \left(\frac{1}{2} S(Q) \right)^2 \limsup_{X\to\infty} \frac{1}{X} \sum_{\substack{m\leq X \\ m\in A(v)}} 1 = \frac{1}{4} (S(Q))^2 \bar{\delta}(A(v)).$$

We have $V(Q) < S(Q)$ in view of (12), so that

$$\bar{\delta}(A(v)) \leq \frac{V(Q)}{\frac{1}{4}(S(Q))^2} < \frac{4}{S(Q)},$$

and we may conclude that $\bar{\delta}(A(v)) = 0$, on letting $Q \to +\infty$. We thus have $\delta(A(v)) = 0$ for all $v < +\infty$ and so $\delta(A_0) = 0$, as was noticed above.

This completes our proof of the theorem.

Note that we have actually demonstrated that for almost all values of $m > 1$ there are indefinitely many primes p satisfying $d_m(p) = 1$; here, that the qualifier 'almost' cannot be omitted is clear, as we recall the fact that for $m = 2$ the integers n for which holds $d_2(n) = 1$ are exclusively the powers of 2 (cf. [1; § 8]).

Remark. We note also that if the (upper or lower) asymptotic density were a completely additive probability measure over the subsets of the set of positive integers m, then, in our proof of the theorem, we could have directly appealed to the Borel-Cantelli lemma in probability theory; the density is not a completely additive measure, however.

References

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