TSUKUBA J. MATH.
Vol. 31 No. 1 (2007), 197–204

# THE SERRE DUALITY THEOREM FOR HOLOMORPHIC VECTOR BUNDLES OVER A STRONGLY PSEUDO-CONVEX MANIFOLD

By

Mitsuhiro Itoh and Takanari Saotome

**Abstract.** The Serre duality for a holomorphic vector bundle over a compact, complex manifold still holds over a compact, strongly pseudo-convex manifold $M$. This duality theorem is a vector bundle version of the Serre duality obtained by N. Tanaka in [3] for ordinary $(p,q)$-forms on $M$.

## §1. Introduction

Let $E$ be a holomorphic vector bundle over a compact, complex manifold $M^n$ and $\Omega^p(E)$ be the sheaf of germs of holomorphic $p$-forms with values in $E$. Then we have

$$H^q(M;\Omega^p(E)) \cong H^{n-q}(M;\Omega^{n-p}(E^*)) \tag{1}$$

for any non-negative integers $(p,q)$. Here $E^*$ denotes the dual vector bundle of $E$. We call this isomorphism the Serre duality, which plays an important role in complex geometry. See, for examples, [1], [2]. When we restrict the bundle $E$ as the trivial complex line bundle, the above then reduces to the ordinary duality

$$H^q(M;\Omega^p) \cong H^{n-q}(M;\Omega^{n-p}) \tag{2}$$

On a compact, complex manifold there is an isomorphism between such cohomology groups and the spaces $\boldsymbol{H}^{p,q}(M;E)$ of harmonic forms taking values in $E$ and then the above dualities are verified in terms of $E$-valued $(p,q)$-harmonic forms together with the Hodge star operator(refer to [2]).

N. Tanaka developed the harmonic theory over a compact, strongly pseudo-convex manifold $M$ and derived a similar theorem for the space $\boldsymbol{H}^{p,q}(M)$ of harmonic $(p,q)$-forms on $M$(refer to Theorem 7.3 in [3]);

Received November 17, 2005.
Revised January 26, 2006.

$$\boldsymbol{H}^{p,q}(M) \cong \boldsymbol{H}^{n-p,n-q-1}(M). \tag{3}$$

for any $(p,q)$. Here $\dim M = 2n-1$.

In this article we consider a holomorphic vector bundle $E$ over a compact, strongly pseudo-convex manifold $M$. The sub-ellipticity of the Laplacian holds also for the space of smooth $E$-valued $(p,q)$-forms on $M$ so that the spaces $\boldsymbol{H}^{p,q}(M,E)$ of $E$-valued harmonic $(p,q)$-forms for any $(p,q)$ are finite dimensional whenever $q \neq 0, n-1$.

The aim of this article is to show the duality theorem for a holomorphic vector bundle over a strongly pseudo-convex manifold. We have indeed via the Hodge star operator $\#$.

THEOREM 1. *Let $M$ be a compact strongly pseudo-convex manifold and let $E$ be a holomorphic vector bundle over $M$. Then*

$$\boldsymbol{H}^{p,q}(M;E) \cong \boldsymbol{H}^{n-p,n-q-1}(M;E^*)$$

*for any $(p,q)$, where $E^*$ is the dual bundle of $E$.*

A strongly pseudo-convex manifold $M$ is a smooth manifold of dimension $2n-1$ which carries a strongly pseudo-convex structure $(S,\theta,P,I,g)$, that is, a complex subbundle $S$ of $T^{\mathbf{C}}M$ satisfying $S \cap \bar{S} = 0$ and $[\Gamma(S),\Gamma(S)] \subset \Gamma(S)$ together with a contact form $\theta$ so that $M$ admits the real expression $(P,I)$ of $S$ such that the Levi-form $g$ given by $g(X,Y) = -d\theta(IX,Y)$, $X, Y \in P$ is positive definite.

We notice that our $M$ admits a canonical Riemannian metric $h = g + \theta \otimes \theta$ and the volume form $dv = (n-1)!\theta \wedge (d\theta)^{n-1}$ gives the orientation.

A complex vector bundle $E$ over a strongly pseudo-convex manifold $M$ is said to be *holomorphic*, if there exists a smooth linear differential operator $\bar{\partial}_E = \bar{\partial} : \Gamma(E) \to \Gamma(E \otimes \bar{S}^*)$ satisfying

$$\text{i) } \bar{\partial}(fu) = f\bar{\partial}u + u \otimes d''f, \quad d''f = df|_{\bar{S}}$$

namely, if we set $\bar{\partial}_{\bar{X}}u = \bar{\partial}u(\bar{X})$, then

$$\text{i}') \ \bar{\partial}_{\bar{X}}(fu) = f\bar{\partial}_{\bar{X}}u + (\bar{X}f)u \quad \text{for } u \in \Gamma(E),\ f \in C^{\infty}_{\mathbf{C}}(M),\ X \in \Gamma(S),$$

$$\text{ii) } \bar{\partial}_{\bar{X}}(\bar{\partial}_{\bar{Y}}u) - \bar{\partial}_{\bar{Y}}(\bar{\partial}_{\bar{X}}u) - \bar{\partial}_{[X,Y]}u = 0 \quad \text{for } u \in \Gamma(E),\ X, Y \in \Gamma(S).$$

We call the operator $\bar{\partial}$ on $E$ a holomorphic structure.

Every strongly pseudo-convex manifold $M$ admits canonically a holomorphic vector bundle called the holomorphic tangent bundle $\hat{T}M$ of $M$, the quotient bundle $\hat{T}M = T^{\mathbf{C}}M/\bar{S}$ with the operator $\bar{\partial} = \bar{\partial}_{\hat{T}}$ given by $\bar{\partial}_{\bar{X}}u = \varpi([\bar{X}, Z])$, for $u \in \Gamma(\hat{T}M)$ with $Z \in \Gamma(T^{\mathbf{C}}M)$ such that $\varpi(Z) = u$ and $X \in \Gamma(S)$. Here $\varpi : T^{\mathbf{C}}M \to \hat{T}M$ is the canonical projection.

Notice that like holomorphic vector bundles over a complex manifold the tensor product $E \otimes F$ of holomorphic bundles $E$, $F$, the dual bundle $E^*$ and the exterior product bundle $\Lambda^k E$ of a holomorphic bundle $E$ are also holomorphic.

Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle over a strongly pseudo-convex manifold $M$. We assume that $E$ admits a smooth Hermitian fiber metric $\langle \cdot, \cdot \rangle_E$.

The tensor product $E \otimes \Lambda^p(\hat{T}M)^*$, $0 \le p \le n-1$ carries the holomorphic structure

$$\bar{\partial} = \bar{\partial}_E \otimes id_{\Lambda^p} + id_E \otimes \bar{\partial}_{\Lambda^p}$$

In complex geometry, it is a standard fact that a complex vector bundle $E$ is holomorphic if and only if $E$ admits a locally defined holomorphic frame field around any point. However on a strongly pseudo-convex manifold, it is not obvious whether a holomorphic vector bundle admits a local holomorphic frame fields. With respect to this we have the following theorem ([4]).

Theorem 2 ([4]). *A holomorphic vector bundle $(E, \bar{\partial})$ over a strongly pseudo-convex manifold $M$ with* $\dim M \ge 7$. *Then, for any point $p$ of $M$ there exists an open neighborhood $U$ of $p$ and a smooth local frame $u_1, \dots, u_r \in \Gamma(U, E)$ such that each $u_i$ satisfies $\bar{\partial}u_i = 0$. Here $r =$* rank *$E$.*

**Acknowledgment**

The authors would like to thank Professor Kimio Miyajima and also to the referee for indicating valuable comments.

## § 2. The Proof of Theorem 1

Although the proof of Theorem 1 for a strongly pseudo-convex manifold is quite similar to the proof for a complex manifold, we will give the detailed proof for the sake of readers.

Let $E$ be a holomorphic vector bundle over a compact, strongly pseudo-convex manifold $M$. We denote by $C^{p,q}(E) = \Gamma(M; E \otimes \Lambda^p\hat{T}M^* \otimes \Lambda^q\bar{S}^*)$ the space of smooth $E$-valued $(p,q)$-forms on $M$. Then the holomorphic structure $\bar{\partial} = \bar{\partial}_E$ of $E$ induces an operator for each $p$, $q$ in the ordinary way

$$\bar{\partial}^q : C^{p,q}(E) \to C^{p,q+1}(E)$$

for which we use, in abbreviation, the same symbol $\bar\partial = \bar\partial_E$. For this definition see [3], p. 16.

Let $*$ be the Hodge star operator. Then the operator $*$ is given by the formula

$$h(*\phi, \psi)\, dv = (n-1)!\phi \wedge \psi, \quad \phi \in \Lambda^k T^*M,\ \psi \in \Lambda^{2n-1-k} T^*M.$$

It holds that $*$ is isometric and involutive, that is, $h(*\phi, *\varphi) = h(\phi, \varphi)$ and $*^2 = id$. Moreover, over a strongly pseudo-convex manifold $M$ its complexification exchanges holomorphic forms and anti-holomorphic forms. Thus, $* : \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^* \to \Lambda^{n-p} \hat{T}M^* \otimes \Lambda^{n-q-1} \bar{S}^*$. If we write $\hat{T}M = \mathbf{C}\xi \oplus S$, then the operator $*$ fulfills $* : \mathbf{C}\theta \otimes \Lambda^{p'} S^* \otimes \Lambda^q \bar{S}^* \to \mathbf{C}\theta \otimes \Lambda^{n-q-1} S^* \otimes \Lambda^{n-p'-1} \bar{S}^*$.

For the proof of Theorem 1 we need to introduce an essential machinery, namely, the Hodge star operator $\#$. The complex conjugate Hodge star operator $^- \circ * : \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^* \to \Lambda^{n-p} \hat{T}M^* \otimes \Lambda^{n-q-1} \bar{S}^*$ can be naturally extended over the bundle $E$ as

$$\# : E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^* \to E^* \otimes \Lambda^{n-p} \hat{T}M^* \otimes \Lambda^{n-q-1} \bar{S}^*$$

To be precise, let $\{s_i \mid i = 1, \ldots, r\}$ be a local frame of $E$ defined over $U \subset M$. Here $r = \operatorname{rank} E$. Set the smooth functions $a_{ij} = \langle s_i, s_j \rangle_E \in C^\infty(U; \mathbf{C})$.

By using a local coframe $\{s^j \mid j = 1, \ldots, r\}$, the dual to $\{s_i\}$, we define $\#$ for $\psi = \sum_i \psi^i s_i \in E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^*$,

$$\#\psi = \sum_{j=1}^{r} (\#\psi)_j s^j,$$

where $(\#\psi)_j = \sum_i a_{ji} * \overline{\psi^i}$. Here remark that $\psi^i$ are elements of $\Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^*$, $i = 1, \ldots, r$ and the definition is independent of a choice of local frame. So, $\#\psi \in C^{n-p, n-q-1}(E^*)$ for $\psi \in C^{p,q}(E)$.

Similarly, define $\#^* : E^* \otimes \Lambda^{n-p} \hat{T}M^* \otimes \Lambda^{n-q-1} \bar{S}^* \to E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^*$,

$$\#^*\left(\sum_j \alpha_j s^j\right) = \sum_{j,k} \bar{a}^{kj}\, \overline{*\alpha_j} s_k.$$

Then, it holds $\#^*\#\psi = \psi$ for any $\psi \in E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^*$ at every point of $M$. In fact,

$$\#^*(\#\psi) = \#^*\left\{\left(\sum_{i,j} \overline{a_{ji}} * \overline{\psi^i}\right) s^j\right\}$$

$$= \sum_{i,j,k} \overline{a^{kj}}\, \overline{*(a_{ji} * \overline{\psi^i})} s_k$$

$$= \sum_{i,j,k} a^{jk} a_{ij} \overline{*(\overline{*\psi^i})} s_k$$

$$= \sum \delta_i^k \overline{*(\overline{*\psi^i})} s_k = \sum_i \overline{*(\overline{*\psi^i})} s_i$$

$$= \sum_i \psi^i s_i = \psi.$$

Remark that we have also $\#\#^* = 1$ and then $\# : E \otimes \Lambda^p \hat{T}M^* \otimes \Lambda^q \bar{S}^* \to E^* \otimes \Lambda^{n-p} \hat{T}M^* \otimes \Lambda^{n-q-1} \bar{S}^*$ gives a bundle isomorphism.

In order to define the formal adjoint of the operator $\bar{\partial}^q$ we define an $L^2$-inner product $(\cdot,\cdot)$ on $C^{p,q}(E)$. For $\phi = \sum_i \phi^i s_i$, $\psi = \sum_j \psi^j s_j$ we define a pointwise Hermitian inner product as

$$\langle \phi, \psi \rangle = \sum_{i,j=1}^{r} h(\phi^i, \psi^j) a_{ij},$$

where $h(\phi^i, \psi^j)$ is the inner product of $(p,q)$-forms $\phi^i$, $\psi^j$ defined by

$$h(\phi^i, \psi^j) = \frac{1}{k!} \sum_{i_1,\dots,i_k} \phi^i(X_{i_1},\dots,X_{i_k}) \overline{\psi^j(X_{i_1},\dots,X_{i_k})},$$

where $k = p + q$ and $\{X_i\}$ is a unitary basis of $TM^{\mathbf{C}}$, i.e., $\theta(X_1) = 1$ and $g(X_i, \overline{X}_j) = \delta_{ij}$, $2 \le i, j \le n$. Then we have an $L^2$-inner product on $C^{p,q}(E)$ by integrating over $M$; $(\phi, \psi) = \int_M \langle \phi, \psi \rangle \, dv$.

We denote by $\delta = \delta_E$ the formal adjoint of $\bar{\partial} = \bar{\partial}_E$ with respect to the $L^2$-inner product;

$$\delta : C^{p,q}(E) \to C^{p,q-1}(E).$$

To prove the following lemma, we need to define some notations. If $\phi \in C^{p,q}(E)$, $\alpha \in C^{s,t}(E^*)$ are locally represented by $\phi = \varphi \otimes u$, $\alpha = \omega \otimes \gamma$, where $\varphi \in C^{p,q}(M)$, $\omega \in C^{s,t}(M)$, $u \in \Gamma(E)$, $\gamma \in \Gamma(E^*)$, we define the product $\phi \wedge \alpha \in C^{p+s,q+t}(M)$ as follows.

$$\phi \wedge \alpha = \varphi \wedge \langle u, \gamma \rangle \omega.$$

Here, $\langle \cdot,\cdot \rangle$ is the pairing of $E$ and $E^*$. The property of this product is,

Lemma 1. *For* $\phi \in C^{p,q}(E)$, $\alpha \in C^{s,t}(E^*)$,

$$\bar{\partial}_{\Lambda^{p+s}}(\phi \wedge \alpha) = (-1)^s (\bar{\partial}_E \phi) \wedge \alpha + (-1)^q \phi \wedge (\bar{\partial}_{E^*} \alpha).$$

Proof of Lemma 1. Let $\phi = \varphi \otimes u$, $\alpha = \omega \otimes \gamma$, locally. Then by using the formula $d'' = (-1)^p \bar{\partial}_{\Lambda^p}$,

$$\begin{aligned}
\bar{\partial}_{\Lambda^{p+s}}(\phi \wedge \alpha) &= \bar{\partial}_{\Lambda^{p+s}}(\varphi \wedge \langle u, \gamma \rangle \omega) \\
&= (-1)^{p+s} d''(\varphi \wedge \langle u, \gamma \rangle \omega) \\
&= (-1)^{p+s} d'' \varphi \wedge \langle u, \gamma \rangle \omega + (-1)^{s+q} \varphi \wedge d'' \langle u, \gamma \rangle \omega \\
&\quad + (-1)^{s+q} \varphi \wedge \langle u, \gamma \rangle d'' \omega \\
&= (-1)^s \bar{\partial}_{\Lambda^p} \varphi \wedge \langle u, \gamma \rangle \omega + (-1)^{s+q} \varphi \wedge \langle \bar{\partial}_E u, \gamma \rangle \omega \\
&\quad + (-1)^{s+q} \varphi \wedge \langle u, \bar{\partial}_{E^*} \gamma \rangle \omega + (-1)^q \varphi \wedge \langle u, \gamma \rangle \bar{\partial}_{\Lambda^s} \omega \\
&= (-1)^s (\bar{\partial}_E \phi) \wedge \alpha + (-1)^q \phi \wedge (\bar{\partial}_{E^*} \alpha).
\end{aligned}$$

Lemma 2.

$$\delta_E \psi = (-1)^{n-k} \#^* \bar{\partial}_{E^*}(\# \psi), \quad \psi \in C^{p,q}(E)$$

Proof of Lemma 2. First, we remark that

$$\int_M \bar{\partial}_{\Lambda^n}(\phi \wedge \# \psi) = 0$$

for $\phi \in C^{p,q-1}(E)$, $\psi \in C^{p,q}(E)$. In fact, the form $\phi \wedge \# \psi$ is a globally defined scalar valued $(n, n-2)$-from on $M$, so that $\bar{\partial}_{\Lambda^n}(\phi \wedge \# \psi) = (-1)^n d''(\phi \wedge \# \psi) = (-1)^n d(\phi \wedge \# \psi)$.

Thus integrating both sides of the following

$$\bar{\partial}_{\Lambda^n}(\phi \wedge \# \psi) = (-1)^{n-p} (\bar{\partial}_E \phi) \wedge \# \psi + (-1)^{q-1} \phi \wedge (\bar{\partial}_{E^*} \# \psi),$$

which is given by Lemma 1, we have

$$0 = (-1)^{n-p} (\bar{\partial}_E \phi, \psi) + (-1)^{q-1} (\phi, \#^* (\bar{\partial}_{E^*} \# \psi)),$$

that is,

$$(\bar{\partial}_E \phi, \psi) = (-1)^{n-p-q} (\phi, \#^* (\bar{\partial}_{E^*} \# \psi)).$$

for any $\phi$. This completes the proof of Lemma 2.

Let $\langle \cdot, \cdot \rangle_{E^*}$ be the Hermitian fiber metric on the dual bundle $E^*$ induced from the fiber metric $\langle \cdot, \cdot \rangle_E$ and set $a^{ij} = \langle s^i, s^j \rangle_{E^*}$ with respect to the dual frame

$\{s^i \mid i = 1, \ldots, r\}$ of $E^*$. The space $C^{n-p,n-q-1}(E^*)$ also admits the $L^2$-inner product

$$(\alpha, \beta)_{E^*} = \frac{1}{(n-1)!} \int_M \sum_{i,j} \alpha_i \wedge \overline{*\beta_j} a^{ij}$$

for $\alpha = \sum_i \alpha_i s^i$, $\beta = \sum_j \beta_j s^j$.

Then the Hodge star operator # enjoys being an isometry with respect to the $L^2$-inner products, that is,

$$(\#\phi, \#\psi)_{E^*} = (\psi, \phi)$$

This is shown in a straightforward manner as

$$\begin{aligned}(n-1)!(\#\phi, \#\psi)_{E^*} &= \int_M \sum (\#\phi)_i \wedge \overline{*(\#\psi)_j} a^{ij} \\ &= \int_M \sum (\#\phi)_i \wedge \overline{(*\overline{a^{ij}}(\#\psi)_j)}\end{aligned}$$

which is

$$\int_M \sum (\#\phi)_i \wedge \#^*(\#\psi)^i = \int_M \sum (\#\phi)_i \wedge \psi^i = \int_M \psi^i \wedge (\#\psi)_i,$$

which is written as

$$(n-1)!(\psi, \phi).$$

Moreover, for $\phi \in C^{p,q}(E)$, $\psi \in C^{p,q-1}(E)$

$$\begin{aligned}(\bar{\partial}_{E^*}\#\phi)_i \wedge \overline{*(\#\psi)_j} a^{ij} &= \sum (\bar{\partial}_{E^*}\#\phi)_i \wedge \psi^j \\ &= (\bar{\partial}_{E^*}\#\phi) \wedge \psi \\ &= \psi \wedge (\bar{\partial}_{E^*}\#\phi) \\ &= (-1)^{q-1} \bar{\partial}_{\Lambda^p}(\psi \wedge \#\phi) + (-1)^{n-k}(\bar{\partial}_E \psi) \wedge \#\phi.\end{aligned}$$

Therefore, it turn out that

$$\begin{aligned}(\bar{\partial}_{E^*}\#\phi, \#\psi)_{E^*} &= (-1)^{n-k}(\bar{\partial}_E \psi, \phi)_E \\ &= (-1)^{n-k}(\#\delta\phi, \#\psi)_{E^*} \\ &= (-1)^{n-k}(\#\phi, \#\bar{\partial}_E \psi)_{E^*}.\end{aligned}$$

This implies $\bar{\partial}_{E^*} = (-1)^{n-k} \# \delta_E \#^*$. Hence, the formal adjoint of $\bar{\partial}_{E^*}$ becomes $(-1)^{n-k} \# \bar{\partial}_E \#^*$.

We are now in a position to show Theorem 1.

Take $\psi \in \boldsymbol{H}^{p,q}(M; E)$. Then it holds from definition $\bar{\partial}_E \psi = 0$ and $\delta_E \psi = 0$.

From Lemma 2 we have, since $\# \#^* = id$

$$\bar{\partial}_{E^*}(\# \psi) = (-1)^{n-k} \# \delta_E \psi = 0$$

On the other hand from the above consideration the formal adjoint $\delta_{E^*}$ of $\bar{\partial}_{E^*}$ is $(-1)^{n-k} \# \bar{\partial}_E \#^*$ so that we have

$$\delta_{E^*} \# \psi = (-1)^{n-k} \# \bar{\partial}_E \#^* (\# \psi) = (-1)^{n-k} \# \bar{\partial}_E \psi = 0$$

Therefore we have $\# \psi \in \boldsymbol{H}^{n-p, n-q-1}(E^*)$

The inverse implication is similarly shown.

So we see

$$\psi \in \boldsymbol{H}^{p,q}(M; E) \Leftrightarrow \# \psi \in \boldsymbol{H}^{n-p, n-q-1}(M; E^*).$$

In particular, $\# : \boldsymbol{H}^{p,q}(M; E) \to \boldsymbol{H}^{n-p, n-q-1}(M; E^*)$ is a complex conjugate linear isomorphism.

## References


[1] F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer-Verlag, 1966.

[2] K. Kodaira, Complex Manifolds and Deformation of Complex Structures, Springer-Verlag, 1986.

[3] N. Tanaka, A Differential Geometric Study on Strongly Pseudo-Convex Manifolds (Lectures in Mathematics, Department Mathematics, Kyoto University 9), Kinokuniya, Tokyo, Japan, 1975.

[4] S. M. Webster, The integrability problem for CR vector bundles, Proc. Symposia in Pure Math., **52** (1991), 355–368.



Mitsuhiro Itoh
Institute of Mathematics
University of Tsukuba
305-8751, Tsukuba, Japan
e-mail: itohm@sakura.cc.tsukuba.ac.jp

Takanari Saotome
Graduate School of Pure and Applied Sciences
University of Tsukuba
305-8571, Tsukuba, Japan
e-mail: tsaotome@math.tsukuba.ac.jp