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NOTE ON WEIGHTED STRICHARTZ ESTIMATES FOR KLEIN-GORDON EQUATIONS WITH POTENTIAL

By

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Abstract. In this paper we prove a mixed weighted Strichartz inequality for the solution of

$$(\partial_t^2 - \Delta_x + V(x) + 1)u(t,x) = F(t,x),$$

where $x \in \boldsymbol{R}^3$ and V is a Hölder continuous non-negative potential satisfying the inequality

$$V(x) \leq C(1 + |x|)^{-3-\delta}$$

with some constants C, $\delta > 0$.

1. Introduction

We consider the Cauchy problem for the Klein-Gordon equation with a non-negative potential:

$$\begin{cases} (\partial_t^2 - \Delta_x + V(x) + 1)u(t,x) = F(t,x), & x \in \boldsymbol{R}^3,\ t > 0, \\ u(0,x) = f(x),\ u_t(0,x) = g(x). \end{cases} \tag{1.1}$$

The aim of this work is to establish weighted Strichartz estimates under suitable assumptions on the potential $V(x)$. In the unperturbed case $V(x) \equiv 0$, such estimates have been studied by Lindblad and Sogge [11]. Among other things, they observe that the dispersive property of the solution is well-exploited by using the foliation of the light cone with hyperboloid instead of the foliation of the whole space by means of hyperplanes $\{t = \text{const.}\}$. More precisely, for given $G : \boldsymbol{R}_+ \times \boldsymbol{R}^3 \to \boldsymbol{R}$ and $1 \leq q < +\infty$, they consider the following mixed norm:

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$$\|G\|_{L^qL^2} := \left(\int_1^{+\infty}\left(\int_{\boldsymbol{R}^3}|G(r\langle z\rangle, rz)|^2\frac{\mathrm{d}z}{\langle z\rangle}\right)^{q/2} r^3\,\mathrm{d}r\right)^{1/q}, \tag{1.2}$$

where $\langle z\rangle = \sqrt{1+|z|^2}$. Then the solution of the problem (1.1) with $V \equiv 0$ satisfies

$$\|u\|_{L^qL^2} + \sup_{t\ge 0}\|u(t,\cdot)\|_{H^{1/2}} \le C(\|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}} + \|F\|_{L^{q'}L^2}), \tag{1.3}$$

provided $q > 8/3$ and $\operatorname{supp} F \subset \{(t,x) \mid t^2 - |x|^2 \ge 1\}$, where $1/q + 1/q' = 1$. Their proof is based on the Fourier representation of the solution and the invariance of the free Klein-Gordon equation under the hyperbolic rotation.

Now we turn our attention to the perturbed Klein-Gordon equation. We immediately lose the favorable properties mentioned above. As for the representation formula of the solution, we make use of the Generalized Fourier Transform related to $H = -\Delta + V(x)$, which is a self-adjoint non-negative operator on L^2. In order to introduce the transform, we first consider the Lippmann-Schwinger equation:

$$\omega(x,\xi) = -\int_{\boldsymbol{R}^3} V(y)(\omega(y,\xi)+1)\frac{\mathrm{e}^{\mathrm{i}(|\xi|\,|x-y|+\xi\cdot(x-y))}}{4\pi|x-y|}\,\mathrm{d}y, \quad x,\xi\in\boldsymbol{R}^3, \tag{1.4}$$

which is the integral equation of the stationary problem corresponding to (1.1). If $V(x)$ is a real-valued Hölder continuous function decaying faster than $|x|^{-2}$, then for any $\xi \ne 0$ there is a unique solution $\omega(x,\xi)$ of (1.4) such that $\omega(x,\xi)\in\mathscr{C}(\boldsymbol{R}_x^3)$ and $\omega(x,\xi)$ tends uniformly to 0 as $|x|\to+\infty$ (see Theorem 3 of [10], also [2]). Then we are ready to define the generalized Fourier transform related to H and its inverse as follows:

$$\mathscr{F}f(\xi) = (2\pi)^{-3/2}\int_{\boldsymbol{R}_x^3}\mathrm{e}^{-\mathrm{i}x\cdot\xi}(1+\omega(x,\xi))f(x)\,\mathrm{d}x,$$

$$\mathscr{F}^*f(x) = (2\pi)^{-3/2}\int_{\boldsymbol{R}_\xi^3}\mathrm{e}^{\mathrm{i}x\cdot\xi}(1+\bar{\omega}(x,\xi))f(\xi)\,\mathrm{d}\xi.$$

We refer to e.g. Theorem 5 of [10] or [1] about the standard properties for the generalized Fourier transform. Especially, for any Borel function α one has

$$\alpha(H)f(x) = \mathscr{F}^*[\alpha(|\cdot|^2)\mathscr{F}f(\cdot)](x). \tag{1.5}$$

In addition, for given $s\in\boldsymbol{R}$, we introduce the Sobolev norm of order s associated with H:

$$\|f\|_{\mathscr{H}^s(\boldsymbol{R}^3)} = \|(1+H)^{s/2}f\|_{L^2(\boldsymbol{R}^3)}.$$

Since the function $e^{-ix\cdot\xi}(1+\omega(x,\xi))$ is a generalized eigenfunction to H, that is

$$(-\Delta_x + V(x))(e^{-ix\cdot\xi}(1+\omega(x,\xi))) = |\xi|^2(e^{-ix\cdot\xi}(1+\omega(x,\xi))),$$

we see that the solution to (1.1) takes the form

$$u(t,x) = \mathscr{U}_V'(t)[f](x) + \mathscr{U}_V(t)[g](x) + \int_0^t \mathscr{U}_V(t-s)[F(s,\cdot)](x)\,ds,$$

where

$$\mathscr{U}_V(t)[g](x) = \frac{\sin(t\sqrt{1+H})}{\sqrt{1+H}}g(x).$$

In this way we can overcome the difficulty caused by $V(x)$ in the Fourier representation of the solution. On the contrary, the lack of the invariance of the Klein-Gordon equation with a potential $V(x)$ with respect to the hyperbolic rotation is crucial. We extend the definition of the mixed norm (1.2) as follows:

$$\|G\|_{L^qL^2_s} := \left(\int_1^{+\infty}\left(\int_{\boldsymbol{R}^3}|G(r\langle z\rangle, rz)|^2\langle z\rangle^s\frac{dz}{\langle z\rangle}\right)^{q/2} r^3\,dr\right)^{1/q}, \tag{1.6}$$

where $1 \le q < +\infty$ and $s \in \boldsymbol{R}$.

Now we are in a position to state the main result of this paper.

Theorem 1. *Let $V(x)$ be a Hölder continuous non-negative function such that*

$$V(x) \le C_0(1+|x|)^{-3-\delta'} \quad \text{for } x \in \boldsymbol{R}^3 \tag{1.7}$$

with some C_0, $\delta' > 0$. Suppose that $\operatorname{supp} F \subset \{(t,x) \mid t^2 - |x|^2 \ge 1\}$, $s > 0$, $\delta > 1$ *and $4 < q < +\infty$. Let $u(t,x)$ be the solution of (1.1). Then there exists $C = C(s,q,\delta) > 0$ such that*

$$\begin{aligned}\|(1+H)^{-\delta/2}u\|_{L^qL^2_{-s}} &+ \sup_{t\ge 0}\|u(t,\cdot)\|_{\mathscr{H}^{(1-\delta)/2}} \\ &\le C(\|f\|_{\mathscr{H}^{(1-\delta)/2}} + \|g\|_{\mathscr{H}^{(-1-\delta)/2}} + \|F\|_{L^{q'}L^2_s}),\end{aligned} \tag{1.8}$$

whenever the norms on the right side of this inequality are finite. Here $1/q + 1/q' = 1$.

Let us compare this result with the unperturbed case considered in [11]. In our estimate the loss in the weight s and in the derivatives δ is due to the lack of Lorentzian invariance for the operator $\partial_t^2 - \Delta + 1 + V(x)$.

We conclude this introduction comparing our theorem with other works concerning L^p-L^q estimates for the Klein-Gordon equation with potential. In [14] the one dimensional case is analyzed. On the contrary, in [12] the space dimension is $n \geq 4$. Finally, Yajima in [15] considers the 3-dimensional case with $F = 0$ and he gives an estimate for $\|u(t,\cdot)\|_{L^p(\boldsymbol{R}^3)}$ with the stronger assumption $|V(x)| \leq C(1+|x|)^{-5-\delta}$.

The plan of the paper is the following. In Section 2 we give some preliminary results on oscillatory integrals. Section 3 and Section 4 are devoted to the proof of Theorem 1. In particular, in Section 3 we reduce the inequality (1.8) to an estimate on the unit hyperboloid. This estimate is established in Section 4 by the aid of a stationary phase argument. In the appendix we prove L^∞ and L^2 estimates for the generalized eigenfunction $\omega(x,\xi)$ by modifying the argument used in [4] for the wave equation with potential. The role of such estimates in our proof is crucial. We prefer to separate them since, to our knowledge, they have some interest also independently of this application.

1.1. Notation.

- By $f \lesssim g$ we mean $f \leq Cg$ where C is a positive constant independent of any variable of the functions f, g. Similarly, $f \simeq g$ stands for $f = Cg$.
- The inner product of $\xi, x \in \boldsymbol{R}^3$ is denoted by $\xi \cdot x$.
- As usual, for any $x \in \boldsymbol{R}^3$, the symbol $\langle x \rangle$ stands for $\sqrt{1+|x|^2}$.
- Pair of conjugate exponents are written as q, q' where $q > 1$ and $1/q' + 1/q = 1$.
- Assume $w : \boldsymbol{R}^3 \to \boldsymbol{R}$ be a positive function. The norm of the weighted space $L^2(w)$ is given by $\|f\|_{L^2(w)}^2 := \int_{\boldsymbol{R}^3} |f(x)|^2 w(x)\,\mathrm{d}x$. In the case $w(x) = \langle x \rangle^\gamma$, we put $\|f\|_{L^2(w)} =: \|f\|_{2,\gamma}$. Finally, for any $1 \leq p \leq +\infty$, L^p stands for $L^p(\boldsymbol{R}^3)$ endowed with the norm $\|f\|_{L^p} := \|f\|_p$.
- The unit hyperboloid $\boldsymbol{H}_+^3 = \{(t,x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^3 \mid t^2 - |x|^2 = 1\}$ will be endowed with the Riemannian metric induced by the Minkowski metric on $\boldsymbol{R}^4$ with signature $(-1,1,1,1)$. The Lorentz group is denoted by $SH(4)$. Finally, the projection from $\boldsymbol{H}_+^3$ to $\boldsymbol{R}^3$ is defined by $\Pi(X_0,X_1,X_2,X_3) = (X_1,X_2,X_3)$ and its inverse by Π^*.

2. Preliminary Results

A variant of Young's inequality is the following.

LEMMA 2.1. *Let* $w : \boldsymbol{R}^n \to \boldsymbol{R}_+$, $\mu, \nu \in \boldsymbol{R}_+$. *Consider*

$$T_w[f](x) = \int_{\boldsymbol{R}^n} k(x, y) f(y) w(y)\, \mathrm{d}y,$$

where $k(x, y)$ *is a measurable function on* $\boldsymbol{R}^n \times \boldsymbol{R}^n$. *Assume that, for any* $x, y \in \boldsymbol{R}^n$

$$\int_{\boldsymbol{R}_y^n} |k(x, y)| w(y)\, \mathrm{d}y \le C_1 \langle x \rangle^{\mu}, \quad \int_{\boldsymbol{R}_x^n} |k(x, y)| w(x)\, \mathrm{d}x \le C_2 \langle y \rangle^{\nu}.$$

Then, for any $f \in L^2(\langle \cdot \rangle^{\nu} w)$, *one has*

$$\|\langle \cdot \rangle^{-\mu/2} T_w f\|_{L^2(w)} \le C_1^{1/2} C_2^{1/2} \|\langle \cdot \rangle^{\nu/2} f\|_{L^2(w)}.$$

Proof. Applying Hölder inequality, in $L_y^1(w)$, we find

$$\begin{aligned}
&\int_{\boldsymbol{R}_x^n} \langle x \rangle^{-\mu} |T_w[f](x)|^2 w(x)\, \mathrm{d}x \\
&\quad \le \int_{\boldsymbol{R}_x^n} \langle x \rangle^{-\mu} \int_{\boldsymbol{R}_y^n} |k(x, y)| w(y)\, \mathrm{d}y \int_{\boldsymbol{R}_y^n} |k(x, y)|\, |f(y)|^2 w(y)\, \mathrm{d}y w(x)\, \mathrm{d}x \\
&\quad \le C_1 \int_{\boldsymbol{R}_x^n} \int_{\boldsymbol{R}_y^n} |k(x, y)|\, |f(y)|^2 w(y) w(x)\, \mathrm{d}y \mathrm{d}x \\
&\quad \le C_1 C_2 \int_{\boldsymbol{R}_y^n} \langle y \rangle^{\nu} |f(y)|^2 w(y)\, \mathrm{d}y.
\end{aligned}$$

This corresponds to our thesis. □

For completeness, we present the proof of a simple inequality needed in what follows.

Lemma 2.2. *Let* $x, \xi \in \boldsymbol{R}^n$. *For any* $j = 1, \dots, n$ *one has*

$$\langle x \rangle \langle \xi \rangle - |x| \xi_j \ge \frac{\langle \xi \rangle}{2 \langle x \rangle}. \tag{2.1}$$

Proof. Taking $\eta = \xi_j / \langle \xi \rangle$, we see that (2.1) is equivalent to $1 + 2|x|^2 - 2\langle x \rangle |x| \eta \ge 0$ with $x, \eta \in \boldsymbol{R}$, $|\eta| \le 1$. We can assume $0 \le \eta \le 1$. We have $2|x| \langle x \rangle \eta \le |x|^2 |\eta|^2 + \langle x \rangle^2 \le 1 + 2|x|^2$, that is our conclusion. □

Next, we recall the following lemma.

Lemma 2.3. *Let* $g(x) = x^m$ *with* $m \in \boldsymbol{R}$. *Let* $f \in \mathscr{C}^{\infty}(\boldsymbol{R})$ *satisfy*

$$|D_x^l f(x)| \lesssim \langle x\rangle^{-l} f(x), \quad x \in \boldsymbol{R},\ l \geq 0.$$

Then we have

$$|D_x^k g(f(x))| \lesssim \langle x\rangle^{-k} g(f(x)), \quad x \in \boldsymbol{R},\ k \geq 0.$$

Next, we establish an estimate (2.3) below for an oscillatory integral

$$A(\Lambda, x, \eta) = \int_{\boldsymbol{R}^2} \mathrm{e}^{\pm \mathrm{i}\Lambda\langle\eta\rangle(\langle z\rangle - 1)} \frac{\langle\eta\rangle}{(\langle x\rangle\langle\eta\rangle\langle z\rangle - |x|\eta)^{\delta}\langle z\rangle}\, \mathrm{d}z, \tag{2.2}$$

where $\delta > 0$ and $\Lambda \in \boldsymbol{R}$.

Proposition 2.1. *Let $k \geq 0$. Then $A(\Lambda, x, \eta)$ defined by (2.2) satisfies*

$$|\partial_\eta^k A(\Lambda, x, \eta)| \lesssim |\Lambda|^{-1}\langle\eta\rangle^{-k} \tag{2.3}$$

for $x \in \boldsymbol{R}^3$, $\eta \in \boldsymbol{R}$ and $\Lambda \neq 0$.

Proof. We can assume $\Lambda > 0$. By using the polar coordinates $z = r\omega$, we can write the integral as

$$A(\Lambda, x, \eta) = 2\pi \int_0^{+\infty} \mathrm{e}^{\pm \mathrm{i}\Lambda\langle\eta\rangle(\langle r\rangle - 1)} \frac{\langle\eta\rangle}{(\langle x\rangle\langle\eta\rangle\langle r\rangle - |x|\eta)^{\delta}} \frac{r\, \mathrm{d}r}{\langle r\rangle}.$$

Changing the variables as $s = \langle x\rangle\langle\eta\rangle(\langle r\rangle - 1)$, we have

$$A(\Lambda, x, \eta) = 2\pi\langle x\rangle^{-1} I(\Lambda, x, \eta), \tag{2.4}$$

where

$$I(\Lambda, x, \eta) = \int_0^{+\infty} \mathrm{e}^{\pm \mathrm{i}\Lambda\langle x\rangle^{-1}s}(s + b(x, \eta))^{-\delta}\, ds$$

with $b(x, \eta) = \langle x\rangle\langle\eta\rangle - |x|\eta$. A simple integration by parts gives

$$I(\Lambda, x, \eta) = \pm \mathrm{i}\langle x\rangle\Lambda^{-1}\left[(b(x, \eta))^{-\delta} - \delta\int_0^{+\infty} \mathrm{e}^{\pm \mathrm{i}\Lambda\langle x\rangle^{-1}s}(s + b(x, \eta))^{-1-\delta}\, ds\right]. \tag{2.5}$$

Suppose we have found

$$|\partial_\eta^l b(x, \eta)| \lesssim \langle\eta\rangle^{-l} b(x, \eta), \quad l \geq 0. \tag{2.6}$$

Then for $k \geq 0$, $s \geq 0$, Lemma 2.3 implies

$$|\partial_\eta^k (b(x, \eta))^{-\delta}| \lesssim \langle\eta\rangle^{-k}(b(x, \eta))^{-\delta}, \quad |\partial_\eta^k (s + b(x, \eta))^{-1-\delta}| \lesssim \langle\eta\rangle^{-k}(s + b(x, \eta))^{-1-\delta}.$$

Since $(\langle x\rangle\langle\eta\rangle - |x|\eta)^2 - (\langle x\rangle\eta - |x|\langle\eta\rangle)^2 = 1$, we see that

$$b(x,\eta) \geq 1, \quad b(x,\eta) \geq |\langle x\rangle\eta - |x|\langle\eta\rangle|. \tag{2.7}$$

Therefore (2.5) yields $|\partial_\eta^k I(\Lambda, x, \eta)| \lesssim \langle x\rangle\Lambda^{-1}\langle\eta\rangle^{-k}$. Hence (2.3) is proved.

It remains to check (2.6). We put $c(x,\eta) = \langle x\rangle\eta - |x|\langle\eta\rangle$. Then we see that

$$\partial_\eta b(x,\eta) = \langle\eta\rangle^{-1}c(x,\eta), \quad \partial_\eta c(x,\eta) = \langle\eta\rangle^{-1}b(x,\eta).$$

By (2.7) we have $|c(x,\eta)| \lesssim b(x,\eta)$. Therefore we get inductively

$$|\partial_\eta^l b(x,\eta)| \lesssim \langle\eta\rangle^{-l}b(x,\eta), \quad |\partial_\eta^l c(x,\eta)| \lesssim \langle\eta\rangle^{-l}b(x,\eta), \quad l \geq 0.$$

Thus we have proved (2.6). This completes the proof. $\square$

We conclude this section by collecting some useful lemmas which enable us to bound integrals of type

$$\int_{\boldsymbol{R}^3} \langle z\rangle^{-r}|x-z|^{-s_1}|y-z|^{-s_2}\,\mathrm{d}z.$$

We start proving, for completeness, an estimate that can be found in [2].

Lemma 2.4. *Let $s, r \in \boldsymbol{R}$ such that $0 < s < n$, $r \geq 0$, $s + r > n$. There exists a constant $C = C_{r,s} > 0$ such that*

$$\int_{\boldsymbol{R}^n} \langle z\rangle^{-r}|x-z|^{-s}\,\mathrm{d}z \leq C, \quad x \in \boldsymbol{R}^n.$$

Proof. In the case $x = 0$, passing in polar coordinates we have

$$\int_{\boldsymbol{R}^n} \langle z\rangle^{-r}|z|^{-s}\,\mathrm{d}z \lesssim \int_0^1 \rho^{n-s-1}\,\mathrm{d}\rho + \int_1^{+\infty} \rho^{n-s-r-1}\,\mathrm{d}\rho.$$

Due to the assumptions on s, r, last integrals converge.

Suppose $|x| \leq 1$. This implies $\langle x-z\rangle \lesssim \langle z\rangle$. Thus we obtain

$$\int_{\boldsymbol{R}^n} \langle z\rangle^{-r}|x-z|^{-s}\,\mathrm{d}z \lesssim \int_{\boldsymbol{R}^n} \langle x-z\rangle^{-r}|x-z|^{-s}\,\mathrm{d}z = \int_{\boldsymbol{R}^n} \langle z\rangle^{-r}|z|^{-s}\,\mathrm{d}z \leq C.$$

Next, we take $|x| \geq 1$. We divide the integral region $\boldsymbol{R}^n$ into $D_1 := \{z : |z| \leq |x|/2\}$ and $D_2 := \{z : |z| > |x|/2\}$. Since in the first region $|x-z| \geq |z|$, we have

$$\int_{D_1} \langle z\rangle^{-r}|x-z|^{-s}\,\mathrm{d}z \leq \int_{D_1} \langle z\rangle^{-r}|z|^{-s}\,\mathrm{d}z \leq C.$$

Finally, it holds that

$$\int_{D_2} \langle z\rangle^{-r}|x-z|^{-s}\,\mathrm{d}z$$

$$\leq + \int_{\substack{|z|\geq|x|/2 \\ 1/2\leq|x-z|\leq|z|}} \langle x-z\rangle^{-r-s}\,\mathrm{d}z + \int_{\substack{|z|\geq|x|/2 \\ |x-z|\geq|z|}} \langle z\rangle^{-r}|z|^{-s}\,\mathrm{d}z \leq C.$$

Hence, we get the conclusion. □

A variant of this lemma is the following.

LEMMA 2.5. *Let $s, r \in \boldsymbol{R}$ such that $0 < s < n$, $r > n$. There exists a constant $C = C_{r,s} > 0$ such that*

$$\int_{\boldsymbol{R}^n} \langle z\rangle^{-r}|x-z|^{-s}\,\mathrm{d}z \leq C\langle x\rangle^{-s}, \quad x \in \boldsymbol{R}^n.$$

PROOF. The previous lemma gives the statement in the case $|x| \leq 1$. Assuming $|x| \geq 1$, we split the integral region $\boldsymbol{R}^n$ into $D_1 := \{z : |z| \leq |x|/2\}$ and $D_2 := \{z : |z| > |x|/2\}$. For any $z \in D_1$, we have $|x-z| \geq |x|/2$, hence

$$\int_{D_1} \langle z\rangle^{-r}|x-z|^{-s}\,\mathrm{d}z \lesssim |x|^{-s}\int_{\boldsymbol{R}^n} \langle z\rangle^{-r}\,\mathrm{d}z \lesssim \langle x\rangle^{-s}.$$

On the contrary, in D_2 we get $|x| \lesssim \langle z\rangle$; the previous lemma implies

$$\int_{D_2} \langle z\rangle^{-r}|x-z|^{-s}\,\mathrm{d}z \lesssim \langle x\rangle^{-s}\int_{\boldsymbol{R}^n} \langle z\rangle^{-r+s}|x-z|^{-s}\,\mathrm{d}z \leq C\langle x\rangle^{-s}.$$

Combining these estimates we conclude the proof. □

Next result is a multi-variable version of Lemma 2.4.

LEMMA 2.6. *Let $s, r_1, r_2 > 0$, $0 < s < n$ such that $s + r_1 + r_2 > 2n$. There exists a constant $C = C_{r_1,r_2,s} > 0$ such that*

$$\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n} \langle x\rangle^{-r_1}\langle y\rangle^{-r_2}|x-y|^{-s}\,\mathrm{d}x\mathrm{d}y \leq C.$$

PROOF. Let $\varepsilon > 0$ such that $s + r_1 + r_2 = 2n + 2\varepsilon$. We can assume $r_1 < n$ and $r_2 < n$. The opposite case will be a consequence of this. In particular we have

$r_1 \le n + \varepsilon$ and $r_2 \le n + \varepsilon$. Then we put $\alpha_i = n + \varepsilon - r_i \ge 0$ for $i = 1, 2$. By using Lemma 2.5, we have

$$\begin{aligned}
&\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n} \langle x\rangle^{-r_1}\langle y\rangle^{-r_2}|x-y|^{-s}\,\mathrm{d}x\mathrm{d}y \\
&\qquad \le \iint_{|x|\ge|y|} |x-y|^{-s}\langle x\rangle^{-r_1+\alpha_2}\langle y\rangle^{-r_2-\alpha_2}\,\mathrm{d}x\mathrm{d}y \\
&\qquad\quad + \iint_{|y|\ge|x|} |x-y|^{-s}\langle x\rangle^{-r_1-\alpha_1}\langle y\rangle^{-r_2+\alpha_1}\,\mathrm{d}x\mathrm{d}y \\
&\qquad \lesssim \int_{\boldsymbol{R}^n} \langle x\rangle^{-r_1+\alpha_2-s}\,\mathrm{d}x + \int_{\boldsymbol{R}^n} \langle y\rangle^{-r_2+\alpha_1-s}\,\mathrm{d}x \le C.
\end{aligned}$$

By the choice of α_i, the last integrals are bounded and the proof is completed. $\square$

In [2], one can also find the following statement.

Lemma 2.7. *Let $s_1, s_2 \in \boldsymbol{R}$ such that $0 < s_1, s_2 < n$, $s_1 + s_2 > n$. There exists a constant $C > 0$ such that for all $x, y \in \boldsymbol{R}^n$ with $x \ne y$ one has*

$$\int_{\boldsymbol{R}^n} |x-z|^{-s_1}|y-z|^{-s_2}\,\mathrm{d}z \le C|x-y|^{n-s_1-s_2}.$$

In the next lemma we see that the case $s_1 + s_2 < n$ can be treated if a term $\langle z\rangle^{-r}$ is involved.

Lemma 2.8. *Let $s_1, s_2, r > 0$ such that $s_1 + s_2 < n$, $s_1 + s_2 + r > n$. There exists a constant $C = C_{s_1, s_2, r} > 0$ such that*

$$\int_{\boldsymbol{R}^n} \langle z\rangle^{-r}|x-z|^{-s_1}|y-z|^{-s_2}\,\mathrm{d}z \le C, \quad x, y \in \boldsymbol{R}^n.$$

Proof. The thesis follows by using Lemma 2.4, splitting $\boldsymbol{R}^n$ into $\{z : |z-x| \ge |y-z|\}$ and $\{z : |z-x| \le |y-z|\}$. $\square$

Lemma 2.9. *Let $s_1, s_2, r \in \boldsymbol{R}$ such that $s_1, s_2 > 0$, $s_1 + s_2 < n$, $r > n$. There exists a constant $C = C_{s_1, s_2, r} > 0$ such that*

$$\int_{\boldsymbol{R}^n} \langle z\rangle^{-r}|x-z|^{-s_1}|y-z|^{-s_2}\,\mathrm{d}z \le C\langle x\rangle^{-s_1}\langle y\rangle^{-s_2}, \quad x, y \in \boldsymbol{R}^n.$$

Proof. Let us fix $x, y \in \boldsymbol{R}^n$. For simmetry we can assume $|x| \le |y|$. Applying previous lemma, we find the thesis when $|y| \le 1$.

Let $|y| \ge 1$; in particular $\langle y \rangle \lesssim |y|$. First we use Lemma 2.5 and obtain

$$\int_{|z| \le |y|/2} \langle z \rangle^{-r} |x-z|^{-s_1} |y-z|^{-s_2}\, \mathrm{d}z \lesssim |y|^{-s_2} \int_{\boldsymbol{R}^n} \langle z \rangle^{-r} |x-z|^{-s_1}\, \mathrm{d}z \lesssim \langle x \rangle^{-s_1} \langle y \rangle^{-s_2}.$$

In the region $|z| \ge |y|/2$, it holds $|z| \ge |x|/2$, then

$$\begin{aligned} &\int_{|z| \ge |y|/2} \langle z \rangle^{-r} |x-z|^{-s_1} |y-z|^{-s_2}\, \mathrm{d}z \\ &\qquad \lesssim \langle x \rangle^{-s_1} \langle y \rangle^{-s_2} \int \langle z \rangle^{-r+s_2+s_1} |x-z|^{-s_1} |y-z|^{-s_2}\, \mathrm{d}z. \end{aligned}$$

Lemma 2.8 implies that last integral is bounded. This concludes the proof. $\square$

3. Proof of Theorem 1 (I): Duality Argument

In this section we reduce the proof of Theorem 1 to an inequality on the unit hyperboloid. This requires a duality argument. More precisely, our proof is based on the following abstract lemma (see for example [8]).

Lemma 3.1. *Let $\mathscr{H}$ be an Hilbert space. Let X be a Banach space with dual X^*. Let $A : X \to \mathscr{H}$ be a linear map and let $A^* : \mathscr{H} \to X^*$ be its adjoint, defined by*

$$\langle A^* v, f \rangle_{XX^*} = \langle v, Af \rangle_{\mathscr{H}}, \quad \forall f \in X, \forall v \in \mathscr{H}.$$

Then the following three conditions are equivalent.

i) *There exists $C \ge 0$ such that for all $f \in X$ one has $\|Af\|_{\mathscr{H}} \le C\|f\|_X$.*
ii) *There exists $C \ge 0$ such that for all $v \in \mathscr{H}$ one has $\|A^* v\|_{X^*} \le C\|v\|_{\mathscr{H}}$.*
iii) *There exists $C \ge 0$ such that for all $f \in X$ one has $\|A^*Af\|_{X^*} \le C^2\|f\|_X$.*

The constant C is the same in all three sentences.

By virtue of the Duhamel principle, we can write the solution of (1.1) as $u = u_0 + \underline{u}$, where u_0 solves

$$\begin{cases} (\partial_t^2 - \Delta_x + V(x) + 1)u_0(t,x) = 0, \quad x \in \boldsymbol{R}^3,\ t > 0, \\ u_0(0,x) = f(x),\ (u_0)_t(0,x) = g(x), \end{cases}$$

hence $\underline{u}$ is a solution of the non-homogeneous problem with zero initial data.

We start estimating $\|(1+H)^{-\delta/2}u_0\|_{L^qL^2_{-s}} + \sup_{t\geq 0}\|u_0(t,\cdot)\|_{\mathscr{H}^{(1-\delta)/2}}$. For any $b \in \boldsymbol{R}$, we have the conservation of the energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int |\langle\xi\rangle^b \mathscr{F}\partial_t u_0(t,\xi)|^2 \,\mathrm{d}\xi + \int |\langle\xi\rangle^{b+1}\mathscr{F}u_0(t,\xi)|^2 \,\mathrm{d}\xi\right) = 0.$$

Hence, for any $\delta > 0$, we get

$$\|u_0(t,\cdot)\|_{\mathscr{H}^{(1-\delta)/2}} \lesssim \|f\|_{\mathscr{H}^{(1-\delta)/2}} + \|g\|_{\mathscr{H}^{(-1-\delta)/2}}. \tag{3.1}$$

On the other hand, we have

$$u_0(t,x) = \frac{\sin(t\sqrt{1+H})}{\sqrt{1+H}} g(x) + \cos(t\sqrt{1+H})f(x).$$

In order to prove

$$\|(1+H)^{-\delta/2}u_0\|_{L^qL^2_{-s}} \lesssim \|f\|_{\mathscr{H}^{(1-\delta)/2}} + \|g\|_{\mathscr{H}^{(-1-\delta)/2}}, \tag{3.2}$$

it suffices to find a constant $C = C(q,s,\delta) > 0$ such that

$$\|\mathrm{e}^{\pm\mathrm{i}r\langle z\rangle\sqrt{1+H}}\Phi(rz)\|_{L^qL^2_{-s}} \leq C(q,s,\delta)\|\Phi\|_{\mathscr{H}^{(1+\delta)/2}} \tag{3.3}$$

for suitable q, s, δ. Let $0 \leq k < 3$. If $q > 8/(3-k)$, then (3.3) will follow from

$$\int_{\boldsymbol{R}^3} |\mathrm{e}^{\pm\mathrm{i}r\langle z\rangle\sqrt{1+H}}\Phi(rz)|^2 \frac{\mathrm{d}z}{\langle z\rangle^{1+s}} \leq C(s,\delta)r^{-3+k}\|\Phi\|^2_{\mathscr{H}^{(1+\delta)/2}}, \tag{3.4}$$

with $C(s,\delta) \geq 0$, after integration in r on the interval $(1,+\infty)$.

Let us explain the reason why (3.4) can be deduced from the inequality (3.6) below. For $r,s,\delta > 0$, by means of the generalized Fourier transform, which is a unitary operator from L^2 to itself, we can define an operator $U_{r,\pm}$ from $L^2(\langle\cdot\rangle^{1+s})$ to L^2 through the following formula:

$$\mathscr{F}U_{r,\pm}[h](\xi) = r^{3/2}\langle\xi\rangle^{-(1+\delta)/2}\int_{\boldsymbol{R}^3_z} \mathrm{e}^{-\mathrm{i}rz\cdot\xi}(1+\omega(rz,\xi))\mathrm{e}^{\mp\mathrm{i}r\langle z\rangle\langle\xi\rangle}h(z)\,\mathrm{d}z.$$

On the other hand, noting $(L^2(\langle\cdot\rangle^{1+s}))^* = L^2(\langle\cdot\rangle^{-1-s})$, we can introduce the adjoint operator $(U_{r,\pm})^*$ of $U_{r,\pm}$ by

$$(U_{r,\pm})^*[g](z) = r^{3/2}\int_{\boldsymbol{R}^3_\xi} \mathrm{e}^{\mathrm{i}rz\cdot\xi}(1+\overline{\omega}(rz,\xi))\mathrm{e}^{\pm\mathrm{i}r\langle z\rangle\langle\xi\rangle}\langle\xi\rangle^{-(1+\delta)/2}\mathscr{F}[g](\xi)\,\mathrm{d}\xi.$$

In addition, we have

$$(U_{r,\pm})^* U_{r,\pm}[h](z)$$

$$= r^3 \int_{\boldsymbol{R}^3_\xi} \int_{\boldsymbol{R}^3_y} e^{ir[(z-y)\cdot\xi \pm (\langle z\rangle - \langle y\rangle)\langle\xi\rangle]} (1+\bar{\omega}(rz,\xi))(1+\omega(ry,\xi))\langle\xi\rangle^{-1-\delta} h(y)\,dy d\xi.$$

Note that the inequality (3.4) is a consequence of

$$\|(U_{r,\pm})^*[g]\|^2_{2,-1-s} \le C(s,\delta) r^k \|g\|_2^2, \tag{3.5}$$

by taking $g = (1+H)^{(1+\delta)/4}\Phi$. In view of Lemma 3.1, we see that this estimate is equivalent to

$$\|(U_{r,\pm})^* U_{r,\pm}[h]\|^2_{2,-1-s} \le C(s,\delta) r^k \|h\|^2_{2,1+s}. \tag{3.6}$$

We leave the proof of (3.6) to Section 4.

It remains to estimate $\underline{u}$ which is the solution of $(\partial_t^2 - \Delta_x + V(x) + 1)\underline{u}(t,x) = F(t,x)$ with zero initial data. In what follows we denote by $\chi_t(s)$ the characteristic function of the interval $[0,t]$, and $\bar{\chi}(t,x)$ the characteristic function of $\{t^2 - |x|^2 \ge 1\}$. Recalling that $\operatorname{supp} F \subset \{t^2 - |x|^2 \ge 1\}$, we can write $\underline{u}$ explicitly:

$$\underline{u}(t,x) = \frac{1}{2i}\sum_{\pm} \pm \int_0^t e^{\pm i(t-\tau)\sqrt{1+H}}(1+H)^{-1/2}[F(\tau,\cdot)](x)\,d\tau$$

$$= \frac{1}{2i}\sum_{\pm} \pm \int_{\boldsymbol{R}} e^{\pm i(t-\tau)\sqrt{1+H}}(1+H)^{-1/2}[\bar{\chi}(\tau,\cdot)\chi_t(\tau)F(\tau,\cdot)](x)\,d\tau. \tag{3.7}$$

As we have seen before, if (3.6) holds, then we get (3.3), and hence

$$\|\bar{\chi} e^{\pm ir\langle z\rangle\sqrt{1+H}}(1+H)^{-(1+\delta)/4}[\Phi](rz)\|_{L^q L^2_{-s}} \le C(q,s,\delta)\|\Phi\|_2 \tag{3.8}$$

for any $\Phi \in L^2$. Let us introduce an operator from $L^{q'}L^2_s$ to L^2:

$$A_\pm[f](x) = \int_{\boldsymbol{R}_t} e^{\mp i\tau\sqrt{1+H}}(1+H)^{-(1+\delta)/4}[(\bar{\chi}f)(\tau,\cdot)](x)\,d\tau$$

for any $\delta > 0$ and $f = f(t,x) \in L^{q'}L^2_s$. Noting $(L^q L^2_{-s})^* = L^{q'}L^2_s$, we can define the adjoint operator $A^*_\pm$ of $A_\pm$, which maps L^2 into $L^q L^2_{-s}$ as follows:

$$A^*_\pm[v](t,x) = \bar{\chi}(t,x) e^{\pm it\sqrt{1+H}}(1+H)^{-(1+\delta)/4}[v](x) \tag{3.9}$$

for $v \in L^2$ and $f \in L^{q'}L^2_s$. Furthermore, $A^*_\pm A_\pm$ maps $L^{q'}L^2_s$ into $L^q L^2_{-s}$, and we have

$$A^*_\pm A_\pm[f](t,x) = \bar{\chi}(t,x)\int_{\boldsymbol{R}_t} e^{\pm i(t-\tau)\sqrt{1+H}}(1+H)^{-(1+\delta)/2}[(\bar{\chi}f)(\tau,\cdot)](x)\,d\tau.$$

Now combining (3.8)–(3.9), we get $\|A_\pm^*[v]\|_{L^qL^2_{-s}} \lesssim \|v\|_{L^2}$. Hence this estimate and Lemma 3.1 imply that

$$\|A_\pm[f]\|_{L^2} \lesssim \|f\|_{L^{q'}L^2_s}, \tag{3.10}$$

$$\|A_\pm^* A_\pm[f]\|_{L^qL^2_{-s}} \lesssim \|f\|_{L^{q'}L^2_s}. \tag{3.11}$$

Turning our attention back to (3.7), we can write $\underline{u}$ by using $A_\pm$ and $A_\pm^*$ in two ways:

$$(1+H)^{(1-\delta)/4}\underline{u}(t,x) = \frac{1}{2\mathrm{i}}\sum_\pm \pm \mathrm{e}^{\pm \mathrm{i}t\sqrt{1+H}} A_\pm[\chi_t F](x), \tag{3.12}$$

$$\bar{\chi}(t,x)(1+H)^{-\delta/2}\underline{u}(t,x) = \frac{1}{2\mathrm{i}}\sum_\pm \pm A_\pm^* A_\pm[\chi_t F](t,x). \tag{3.13}$$

Therefore, thanks to (3.10) and (3.12), we obtain, for $t > 0$,

$$\begin{aligned}\|\underline{u}(t,\cdot)\|_{\mathscr{H}^{(1-\delta)/2}} &= \|(1+H)^{(1-\delta)/4}\underline{u}(t,\cdot)\|_2 \\ &\lesssim \sum_\pm \|\mathrm{e}^{\pm \mathrm{i}t\sqrt{1+H}} A_\pm[\chi_t F]\|_2 \lesssim \|A_\pm[\chi_t F]\|_2 \\ &\lesssim \|\chi_t F\|_{L^{q'}L^2_s} \lesssim \|F\|_{L^{q'}L^2_s}. \end{aligned} \tag{3.14}$$

On the other hand, thanks to (3.11) and (3.13), we obtain, for $t > 0$,

$$\begin{aligned}\|(1+H)^{-\delta/2}\underline{u}\|_{L^qL^2_{-s}} &= \|\bar{\chi}(1+H)^{-\delta/2}\underline{u}\|_{L^qL^2_{-s}} \\ &\lesssim \sum_\pm \|A_\pm^* A_\pm[\chi_t F]\|_{L^qL^2_{-s}} \lesssim \|\chi_t F\|_{L^{q'}L^2_s} \lesssim \|F\|_{L^{q'}L^2_s}. \end{aligned} \tag{3.15}$$

Summarizing (3.14)–(3.15) and (3.1)–(3.2), we complete the proof of Theorem 1, once we have established the estimate (3.6). The proof of (3.6) will be the object of the next section.

4. Proof of Theorem 1 (II): A Weighted Estimate on the Unit Hyperboloid

In this section we shall prove (3.6). For $\lambda > 1$, $\varepsilon > 0$ and $\delta > 1$ we set

$$\begin{aligned} W_\pm^\lambda[h](x) = \lambda^3 \int_{\mathbf{R}^3_\xi}\int_{\mathbf{R}^3_y} & \mathrm{e}^{\mathrm{i}\lambda[(x-y)\cdot\xi \pm (\langle x\rangle - \langle y\rangle)\langle\xi\rangle]} \\ & \times (1+\bar{\omega}(\lambda x,\xi))(1+\omega(\lambda y,\xi))h(y)\frac{\mathrm{d}y}{\langle y\rangle^{1+\varepsilon}}\frac{\mathrm{d}\xi}{\langle\xi\rangle^{1+\delta}}. \end{aligned}$$

Then (3.6) follows from

$$\|W_{\pm}^{\lambda}[h]\|_{2,-1-2\varepsilon}^{2} \le C\lambda^{k}\|h\|_{2,-1}^{2} \tag{4.1}$$

with C independent of λ, by taking $2\varepsilon \le s$. We shall prove that (4.1) is valid for any $\lambda > 1$, $\varepsilon > 0$ and $\delta > 1$. To this aim we define $W_{i,\pm}^{\lambda}$ such that $W_{\pm}^{\lambda}(x) = \sum_{i=1}^{4} W_{i,\pm}^{\lambda}$. More precisely $W_{2,\pm}^{\lambda}$ contains the term $\bar{\omega}(\lambda x, \xi)$, $W_{3,\pm}^{\lambda}$ contains $\omega(\lambda y, \xi)$, and $W_{4,\pm}^{\lambda}$ contains the product $\bar{\omega}(\lambda x, \xi)\omega(\lambda y, \xi)$.

4.1. The estimate for first term of W^{λ}. We start discussing the estimate for

$$W_{1,\pm}^{\lambda}[h](x) = \lambda^{3}\int_{\boldsymbol{R}_{\xi}^{3}}\int_{\boldsymbol{R}_{y}^{3}} \mathrm{e}^{\mathrm{i}\lambda[(x-y)\cdot\xi \pm (\langle x\rangle - \langle y\rangle)\langle\xi\rangle]} h(y)\frac{\mathrm{d}y}{\langle y\rangle^{1+\varepsilon}}\frac{\mathrm{d}\xi}{\langle\xi\rangle^{1+\delta}},$$

having kernel

$$K_{1,\pm}^{\lambda}(x,y) = \lambda^{3}\int_{\boldsymbol{R}_{\xi}^{3}} \mathrm{e}^{\mathrm{i}\lambda[(x-y)\cdot\xi \pm (\langle x\rangle - \langle y\rangle)\langle\xi\rangle]}\frac{\mathrm{d}\xi}{\langle\xi\rangle^{1+\delta}}.$$

Following [11], we consider the distance on the hyperboloid

$$d(x,y) := \log(\langle x\rangle\langle y\rangle - x\cdot y + \sqrt{(\langle x\rangle\langle y\rangle - x\cdot y)^{2} - 1}).$$

We take $C_0 > 0$ and split $\boldsymbol{R}_{y}^{3}$ in the regions $d(x,y) \le C_0$ and the remainder. We put

$$W_{0}^{\lambda}[h](x) = \int_{d(x,y)\le C_0} K_{1,\pm}^{\lambda}(x,y)h(y)\frac{\mathrm{d}y}{\langle y\rangle^{1+\varepsilon}}. \tag{4.2}$$

With some modification with respect to the local argument in [11], one finds that for suitable large $C_0 > 0$ it holds that

$$\|W_{0}^{\lambda}h\|_{2,-1} \le C\|h\|_{2,-1} \tag{4.3}$$

with C independent of λ. More precisely, we put $X_0 = (1,0,0,0)$ and choose $\{X_j\} \subset \boldsymbol{H}_{+}^{3}$ such that $B_j := \{X \in \boldsymbol{H}_{+}^{3} \mid d(X,X_j) \le C_0\}$ satisfy $\boldsymbol{H}_{+}^{3} = \bigcup B_j$ and have uniformly finite overlap. Besides we put $B_j^{*} = \{X \in \boldsymbol{H}_{+}^{3} \mid d(X,X_j) \le 2C_0\}$. Given a function $f(X)$ on $\boldsymbol{H}_{+}^{3}$, we set

$$\tilde{W}_{0}^{\lambda}[f](X) = W_{0}^{\lambda}[f\circ\Pi^{*}](\Pi X).$$

Since $\operatorname{supp}\tilde{W}_{0}^{\lambda}f \subset B_j^{*}$ when $\operatorname{supp} f \subset B_j$, the inequality (4.3) reduces to

$$\|\tilde{W}_{0}^{\lambda}f\|_{L^{2}(B_j^{*})} \le C\|f\|_{L^{2}(B_j)}, \quad \operatorname{supp} f \subset B_j \tag{4.4}$$

with C independent of j and λ. The next step is to make a rotation on the unit hyperboloid. Suppose we have found that for any $T \in SH(4)$, it holds

$$\|\tilde{W}_0^\lambda[f \circ T^{-1}] \circ T\|_{L^2(B_0^*)} \le C\|f\|_{L^2(B_0)}, \quad \operatorname{supp} f \subset B_0 \tag{4.5}$$

with a constant C independent of λ and T. Then we see that (4.5) yields (4.4). Let us denote by χ_x the characteristic function of $\{y \in \boldsymbol{R}^3 \mid d(x, y) \le C_0\}$. Thanks to the invariance of the phase function under the hyperbolic rotation, explicitly we have

$$\tilde{W}_0^\lambda[f \circ T^{-1}](TX) = \lambda^3 \int_{\boldsymbol{R}_y^3} \int_{\boldsymbol{R}_\xi^3} \frac{e^{\mp i\lambda[(x-y)\cdot\xi - (\langle x\rangle - \langle y\rangle)\langle\xi\rangle]}\, d\xi}{\langle \tilde{T}\xi\rangle^\delta \langle\xi\rangle} \frac{\chi_x(y) f(\Pi^* y)\, dy}{\langle \tilde{T}y\rangle^\varepsilon \langle y\rangle}.$$

Here and in the sequel of this section, we set $x = \Pi X$ and $\tilde{T} = \Pi T \Pi^*$. In the unperturbed case $V \equiv 0$, one can take $\delta = \varepsilon = 0$, hence T disappears. On the contrary, we lose the Lorentz invariance, so we will be careful about the dependence on T.

The euclidean version of (4.5) is obtained by taking $\beta \in \mathscr{C}_0^\infty(\boldsymbol{R}^3)$ such that $\beta(x) \equiv 1$ if $(\langle x\rangle, x) \in B_0^*$ and introducing the operator

$$W_T^\lambda[h](x) = \lambda^3 \beta(x) \int_{\boldsymbol{R}_y^3} \int_{\boldsymbol{R}_\xi^3} \frac{e^{\mp i\lambda[(x-y)\cdot\xi - (\langle x\rangle - \langle y\rangle)\langle\xi\rangle]}\, d\xi}{\langle \tilde{T}\xi\rangle^\delta \langle\xi\rangle} \beta(y) h(y)\, dy.$$

In particular we have

$$W_T^\lambda[\langle \tilde{T}\cdot\rangle^{-\varepsilon} \langle\cdot\rangle^{-1} \chi_x f \circ \Pi^*](x) = \beta(x) \tilde{W}_0^\lambda[\beta f \circ T^{-1}](TX).$$

Hence, the inequality (4.5) will be obtained by

$$\|W_T^\lambda h\|_2 \le C\|h\|_2 \tag{4.6}$$

with C independent of λ and T. Proceeding as in [11], taking C_0 in (4.2) appropriately large, we deduce that the operator

$$U_T^\lambda h(x) := \lambda^{3/2} \beta(x) \int_{\boldsymbol{R}_\xi^3} e^{\mp i\lambda(x\cdot\xi - \langle x\rangle\langle\xi\rangle)} \hat{h}(\xi) \frac{d\xi}{\langle \tilde{T}\xi\rangle^{\delta/2} \langle\xi\rangle^{1/2}}$$

is L^2 bounded uniformly in λ and T. Let $(U_T^\lambda)^*$ be the adjoint of U_T^λ. One has $W_T^\lambda = U_T^\lambda (U_T^\lambda)^*$. Invoking again Lemma 3.1, we see that (4.6) is satisfied. In turn, this gives (4.5), (4.4) and finally (4.3).

Main point is the estimate for

$$W_\infty^\lambda[h](x) = \int_{d(x,y) \ge C_0} K_{1,\pm}^\lambda(x, y) h(y) \frac{dy}{\langle y\rangle^{1+\varepsilon}}$$

in $L^2(\langle x\rangle^{-1-2\varepsilon})$. This will be done by means of L^∞ estimate of

$$I_\pm(\lambda,x)=\int_{d(x,y)\ge C_0}|K^\lambda_{1,\pm}(x,y)|\frac{dy}{\langle y\rangle^{1+\varepsilon}}.$$

Suppose we have found $C_1>0$, independent of λ, such that

$$I_\pm(\lambda,x)\le C_1\langle x\rangle^\varepsilon, \tag{4.7}$$

then the modified Young inequality, given in Lemma 2.1, implies

$$\|W^\lambda_\infty h\|_{2,-1-2\varepsilon}=\|\langle\cdot\rangle^{-\varepsilon/2}W^\lambda_\infty h\|_{2,-1-\varepsilon}\le C\|\langle\cdot\rangle^{\varepsilon/2}h\|_{2,-1-\varepsilon}.$$

Combining this with (4.3), we obtain

$$\|W^\lambda_{1,\pm}h\|_{2,-1-2\varepsilon}\le C\|h\|_{2,-1} \tag{4.8}$$

with C independent of λ.

We turn to the proof of (4.7). For fixed $x\in\boldsymbol{R}^3$, there exists $T_x\in SH(4)$ such that $T_xX_0=(\langle x\rangle,x)$ with $X_0=(1,0,0,0)$. One can construct such T_x as follows: let A_x be a unitary matrix such that $A_x(|x|,0,0)=x$; we put

$$T_1=\begin{pmatrix}1&0\\0&A_x\end{pmatrix},\quad T_2=\begin{pmatrix}\langle x\rangle&-|x|&0\\|x|&-\langle x\rangle&\\0&&I_2\end{pmatrix},\quad T_x=T_1\circ T_2,$$

with I_2 the identical 2×2 matrix. In particular, for any $z\in\boldsymbol{R}^3$, we have $\langle\widetilde{T_x}(z)\rangle=\langle x\rangle\langle z\rangle-|x|z_1$. Then

$$I_\pm(\lambda,x)=\lambda^3\int_{\langle y\rangle+|y|\ge e^{C_0}}\left|\int_{\boldsymbol{R}^3_\xi}e^{\pm i\lambda((1-\langle y\rangle)\langle\xi\rangle+y\cdot\xi)}\frac{d\xi}{(\langle x\rangle\langle\xi\rangle-|x|\xi_1)^\delta\langle\xi\rangle}\right|$$
$$\times\frac{dy}{(\langle x\rangle\langle y\rangle-|x|y_1)^\varepsilon\langle y\rangle}.$$

By using Lemma 2.2 with $\xi=y$, we get

$$I_\pm(\lambda,x)\lesssim\lambda^3\langle x\rangle^\varepsilon\int_{|y|\ge C'}\left|\int_{\boldsymbol{R}^3_\xi}e^{\pm i\lambda((1-\langle y\rangle)\langle\xi\rangle+y\cdot\xi)}\frac{d\xi}{\langle\xi\rangle(\langle x\rangle\langle\xi\rangle-|x|\xi_1)^\delta}\right|\frac{dy}{\langle y\rangle^{1+\varepsilon}}$$

for suitable large $C'>1$. It remains to find

$$\left|\int_{\boldsymbol{R}^3_\xi}e^{\pm i\lambda((1-\langle y\rangle)\langle\xi\rangle+y\cdot\xi)}\frac{d\xi}{\langle\xi\rangle(\langle x\rangle\langle\xi\rangle-|x|\xi_1)^\delta}\right|\le C\lambda^{-3}|y|^{-2+\kappa} \tag{4.9}$$

for any $0 < \kappa < \varepsilon$. The expressions (4.7) and (4.9) are invariant under orthogonal transformations with respect to y. Taking $\mathcal{O}y = (|y|, 0, 0)$ we see that it suffices to show

$$\left| \int_{\boldsymbol{R}^3_\xi} e^{\pm i\lambda((1-\langle y\rangle)\langle\xi\rangle+|y|\xi_1)} \frac{d\xi}{\langle\xi\rangle(\langle x\rangle\langle\xi\rangle - |x|\xi_1)^\delta} \right| \le C\lambda^{-3}|y|^{-2+\kappa}.$$

After change of variables $\xi = (\eta, \langle\eta\rangle z_1, \langle\eta\rangle z_2)$, we write previous integral in the form

$$J(\lambda, x, y) = \int_{-\infty}^{+\infty} \int_{\boldsymbol{R}^2} e^{\pm i\lambda((1-\langle y\rangle)\langle\eta\rangle\langle z\rangle+|y|\eta)} \frac{\langle\eta\rangle}{\langle z\rangle(\langle x\rangle\langle\eta\rangle\langle z\rangle - |x|\eta)^\delta}\, dz d\eta.$$

Having in mind (2.2), we have

$$J(\lambda, x, y) = -\int_{-\infty}^{+\infty} e^{\mp i\lambda\varphi(y,\eta)} A(-\lambda(\langle y\rangle - 1), x, -\eta)\, d\eta.$$

The phase is given by

$$\varphi(y, \eta) = (\langle y\rangle - 1)\langle\eta\rangle + |y|\eta$$

and it satisfies $\partial_\eta^2\varphi(y, \eta) = (\langle y\rangle - 1)\langle\eta\rangle^{-3}$ and

$$\partial_\eta\varphi(y, \eta) \ge \frac{1}{2} + \frac{\langle y\rangle}{2\langle\eta\rangle^2} \tag{4.10}$$

for $|y| \ge C'$ with large $C' > 1$. Indeed, by rewriting $\partial_\eta\varphi$ as

$$\partial_\eta\varphi(y, \eta) = 1 + |y| - \langle y\rangle + (\langle y\rangle - 1)\left(\frac{\eta}{\langle\eta\rangle} + 1\right)$$

and noting $(\eta/\langle\eta\rangle) + 1 \ge \langle\eta\rangle^{-2}/2$, we find (4.10).

Now, integrating by parts, N times, we find

$$\begin{aligned} |J(\lambda, x, y)| &\lesssim \lambda^{-N} \int_{-\infty}^{+\infty} |(L^*)^N A(-\lambda(\langle y\rangle - 1), x, -\eta)|\, d\eta \\ &\lesssim \lambda^{-N} \int_{-\infty}^{+\infty} \sum_{k=0}^{N} |\partial_\eta^k((\partial_\eta\varphi(y, \eta))^{-N})|\, |\partial_\eta^{N-k} A(-\lambda(\langle y\rangle - 1), x, -\eta)|\, d\eta, \end{aligned}$$

where

$$L^* = i\frac{\partial}{\partial\eta}\frac{1}{\partial_\eta\varphi(y, \eta)}.$$

From Proposition 3.1, it follows that

$$|\partial_\eta^{N-k} A(-\lambda(\langle y\rangle - 1), x, -\eta)| \lesssim \lambda^{-1}(\langle y\rangle - 1)^{-1}\langle \eta\rangle^{-N+k}, \quad 0 \le k \le N.$$

Moreover, for any $l \ge 2$ we have $|\partial_\eta^l \varphi(y,\eta)| \lesssim \langle \eta\rangle^{-l}\partial_\eta\varphi(y,\eta)$. Hence, Lemma 2.3 yields

$$|\partial_\eta^k((\partial_\eta\varphi(y,\eta))^{-N})| \lesssim \langle\eta\rangle^{-k}(\partial_\eta\varphi(y,\eta))^{-N}, \quad k \ge 0.$$

Therefore, we arrive at

$$\begin{aligned} |J(\lambda,x,y)| &\lesssim \lambda^{-N-1}(\langle y\rangle - 1)^{-1}\int_{-\infty}^{+\infty} \langle\eta\rangle^{-N}(\partial_\eta\varphi(y,\eta))^{-N}\,\mathrm{d}\eta \\ &\lesssim \lambda^{-N-1}(\langle y\rangle - 1)^{-1}\int_{-\infty}^{+\infty} \langle\eta\rangle^{-N}(\partial_\eta\varphi(y,\eta))^{-(1-\kappa)}\,\mathrm{d}\eta, \end{aligned}$$

since $N > 1-\kappa$ for any $\kappa > 0$ and $\partial_\eta\varphi(y,\eta) \ge 1/2$ by (4.10). Thus we conclude

$$|J(\lambda,x,y)| \lesssim \lambda^{-N-1}|y|^{-2+\kappa}\int_{-\infty}^{+\infty}\langle\eta\rangle^{-N+2-2\kappa}\,\mathrm{d}\eta \lesssim \lambda^{-N-1}|y|^{-2+\kappa},$$

provided that $N \ge 3$. This means that (4.9) holds. We underline that in this estimate $\delta \ge 0$. This means that for the free term we do not lose derivatives and we can take the exponent $q > 8/3$.

4.2. The estimates for the second and third terms of W^λ. Before dealing with

$$W_{2,\pm}^\lambda[h](x) = \lambda^3\int_{\boldsymbol{R}_\xi^3}\int_{\boldsymbol{R}_y^3} \mathrm{e}^{\mathrm{i}\lambda[(x-y)\cdot\xi \pm (\langle x\rangle - \langle y\rangle)\langle\xi\rangle]}\overline{\omega}(\lambda x,\xi)h(y)\frac{\mathrm{d}y}{\langle y\rangle^{1+\varepsilon}}\frac{\mathrm{d}\xi}{\langle\xi\rangle^{1+\delta}},$$

we claim that if

$$\|W_{2,\pm}^\lambda h\|_{2,-1-\varepsilon} \le C\lambda^k\|h\|_{2,-1-\varepsilon}, \tag{4.11}$$

then

$$\|W_{3,\pm}^\lambda h\|_{2,-1-\varepsilon} \le C\lambda^k\|h\|_{2,-1-\varepsilon}, \tag{4.12}$$

where

$$W_{3,\pm}^\lambda[h](x) = \lambda^3\int_{\boldsymbol{R}_\xi^3}\int_{\boldsymbol{R}_y^3} \mathrm{e}^{\mathrm{i}\lambda[(x-y)\cdot\xi \pm (\langle x\rangle - \langle y\rangle)\langle\xi\rangle]}\omega(\lambda y,\xi)h(y)\frac{\mathrm{d}y}{\langle y\rangle^{1+\varepsilon}}\frac{\mathrm{d}\xi}{\langle\xi\rangle^{1+\delta}}.$$

We start observing that

$$(W_{2,\pm}^\lambda g \mid h)_{L^2(\langle\cdot\rangle^{-1-\varepsilon})} = (g \mid W_{3,\mp}^\lambda h)_{L^2(\langle\cdot\rangle^{-1-\varepsilon})}.$$

This means that we can apply Lemma 3.1 with $\mathscr{H} = L^2(\langle\cdot\rangle^{-1-\varepsilon}) = X^*$ and $A = W^{\lambda}_{3,\mp}$. With respect to $L^2(\langle\cdot\rangle^{-1-\varepsilon})$ product, the duality gives $X = L^2(\langle\cdot\rangle^{-1-\varepsilon})$ and $A^* = W^{\lambda}_{2,\pm}$. From (4.11) we have $\|A^*h\|_{X^*} \lesssim \lambda^k \|h\|_{\mathscr{H}}$. Hence we conclude that $\|Ah\|_{\mathscr{H}} \lesssim \lambda^k \|h\|_X$, i.e., (4.12) holds.

The proof of (4.11) is the core of this paper, since the estimates for generalized eigenfunctions come into play. We can write

$$W^{\lambda}_{2,\pm}[h](x) = \int_{\boldsymbol{R}^3_y} K^{\lambda}_{2,\pm}(x, y) h(y) \frac{\mathrm{d}y}{\langle y\rangle^{1+\varepsilon}},$$

where

$$K^{\lambda}_{2,\pm}(x, y) = \lambda^3 \int_{\boldsymbol{R}^3_\xi} \mathrm{e}^{\mathrm{i}\lambda[(x-y)\cdot\xi \pm (\langle x\rangle - \langle y\rangle)\langle\xi\rangle]} \overline{\omega}(\lambda x, \xi) \frac{\mathrm{d}\xi}{\langle\xi\rangle^{1+\delta}}. \tag{4.13}$$

We reduce our matter to establish that

$$|K^{\lambda}_{2,\pm}(x, y)| \leq \lambda^k |x|^{-1} |y|^{-1} \quad \text{for } x \neq 0,\ y \neq 0. \tag{4.14}$$

In fact, assuming this inequality and combining Hölder inequality with Lemma 2.4, we gain

$$|W^{\lambda}_{2,\pm} h(x)| \lesssim \lambda^k |x|^{-1} \|h\|_{2,-1-\varepsilon}.$$

Using once more Lemma 2.4, we obtain (4.11).

In order to prove (4.14) we make use of Theorem A.1. Let us recall that the free resolvent operator is given by

$$R_0(|\xi|^2 + \mathrm{i}0)[f](x) = \lim_{\varepsilon\to 0^+} R_0(|\xi|^2 + \mathrm{i}\varepsilon)[f](x) = \int_{\boldsymbol{R}^3} \frac{\mathrm{e}^{\mathrm{i}|\xi|\,|x-z|}}{4\pi|x-z|} f(z)\,\mathrm{d}z. \tag{4.15}$$

Then (1.4) can be rewritten as

$$v(x, \xi) = -R_0(|\xi|^2 + \mathrm{i}0)[Vv(\cdot, \xi)](x) + R_0(|\xi|^2 + \mathrm{i}0)[V_\xi](x), \tag{4.16}$$

where $v(x, \xi) = \mathrm{e}^{-\mathrm{i}x\cdot\xi}\omega(x, \xi)$ and $V_\xi(x) = -\mathrm{e}^{-\mathrm{i}x\cdot\xi} V(x)$. Now, passing to the polar coordinates: $\xi = \rho\sigma$, $\rho \geq 0$, $\sigma \in S^2$, we see from (4.13) that

$$|K^{\lambda}_{2,\pm}(x, y)| \leq \mathscr{A}(x, y),$$

where

$$\mathscr{A}(x, y) = \lambda^3 \int_0^{+\infty} \left| \int_{S^2} \mathrm{e}^{\mathrm{i}\lambda\rho y\cdot\sigma} v(\lambda x, \rho\sigma)\,\mathrm{d}\sigma \right| \frac{\rho^2\,\mathrm{d}\rho}{\langle\rho\rangle^{1+\delta}}. \tag{4.17}$$

Substituting (4.16) into (4.17) we get

$$\mathscr{A}(x,y) \le I_1(x,y) + I_2(x,y), \tag{4.18}$$

where

$$I_1(x,y) = \lambda^3 \int_0^{+\infty} \left| \int_{S^2} e^{i\lambda\rho y\cdot\sigma} R_0(\rho^2 + i0)[V_{\rho\sigma}](\lambda x)\, d\sigma \right| \frac{\rho^2\, d\rho}{\langle\rho\rangle^{1+\delta}},$$

$$I_2(x,y) = \lambda^3 \int_0^{+\infty} \left| \int_{S^2} e^{i\lambda\rho y\cdot\sigma} R_0(\rho^2 + i0)[Vv(\cdot,\rho\sigma)](\lambda x)\, d\sigma \right| \frac{\rho^2\, d\rho}{\langle\rho\rangle^{1+\delta}}.$$

First we evaluate $I_1(x,y)$. Recalling that $\int_{S^2} e^{ix\cdot\sigma}\, d\sigma \simeq |x|^{-1}\sin|x|$, the inner integral is explicitly written as

$$-\int_{S^2} e^{i\lambda\rho y\cdot\sigma} \int_{\boldsymbol{R}_z^3} \frac{e^{i\rho|\lambda x - z|}}{4\pi|\lambda x - z|} V(z) e^{-i\rho\sigma\cdot z}\, dz d\sigma \simeq \int_{\boldsymbol{R}_z^3} \frac{V(z)e^{i\rho|\lambda x - z|}\sin(\rho|\lambda y - z|)}{|\lambda x - z|\,|\lambda y - z|\rho}\, dz.$$

Therefore, by our assumption (1.7) we have

$$I_1(x,y) \lesssim \lambda^3 \int_0^{+\infty} \int_{\boldsymbol{R}_z^3} \frac{dz}{|\lambda x - z|\,|\lambda y - z|\langle z\rangle^{3+\delta'}} \frac{\rho\, d\rho}{\langle\rho\rangle^{1+\delta}}.$$

In this estimate the loss of derivatives and the exponent k appear: for the convergence in ρ we require $\delta > 1$ and we pay a factor λ. In fact, by using Lemma 2.9, we get

$$|x|\,|y| I_1(x,y) \lesssim \lambda^3 |x|\,|y| \langle\lambda x\rangle^{-1}\langle\lambda y\rangle^{-1} \lesssim \lambda. \tag{4.19}$$

In order to estimate $I_2(x,y)$, we employ the following propositions.

PROPOSITION 4.1. *For any $x \in \boldsymbol{R}^3$, we have*

$$|x| \left| \int_{S^2} e^{-ix\cdot\sigma} g(\sigma)\, d\sigma \right| \lesssim \sup_{\sigma\in S^2} |g(\sigma)| + \sum_{\substack{j<k \\ k=1,2,3}} \sup_{\sigma\in S^2} |\Omega_{j,k} g(\sigma)|.$$

Here $\Omega_{j,k} := \sigma_j\partial_k - \sigma_k\partial_j$ are the tangential vector fields to the sphere. Besides $g : S^2 \to \boldsymbol{R}$ is a $\mathscr{C}^1$ function such that the right side is finite.

PROOF. We may assume $x = (0, 0, |x|)$ without loss of generality. In the polar coordinates

$$\sigma_1 = \sin\theta\cos\phi, \quad 0 \le \theta \le \pi$$

$$\sigma_2 = \sin\theta\sin\phi, \quad 0 \le \phi \le 2\pi$$

$$\sigma_3 = \cos\theta,$$

the integral in the left hand side is equal to

$$\int_0^{2\pi}\int_0^{\pi} \mathrm{e}^{-\mathrm{i}|x|\cos\theta} g(\sigma(\theta,\phi)) \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = \int_0^{2\pi}\int_0^{\pi} \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{e}^{-\mathrm{i}|x|\cos\theta}}{\mathrm{i}|x|}\right) g(\sigma)\,\mathrm{d}\theta\mathrm{d}\phi.$$

The desired estimate follows from the relation $\partial_\theta g = -\cos\phi\Omega_{1,3}g - \sin\phi\Omega_{2,3}g$ after one integration by parts. □

Proposition 4.2. *Let* $x, \xi \in \boldsymbol{R}^3$. *Let* $|V(x)| \leq C_0(1+|x|)^{-3-\delta'}$ *with some* C_0, $\delta' > 0$. *Then the following estimate holds*:

$$|R_0(|\xi|^2 + \mathrm{i}0)[Vf](x)| \lesssim \langle x\rangle^{-1}\|f\|_{2,-a} \tag{4.20}$$

for $a < 3 + 2\delta'$ *and any function* f *such that the right side is finite.*

Proof. By (1.7) we have

$$\begin{aligned}|R_0(\rho^2+\mathrm{i}0)[Vf](x)| &\lesssim \int_{\boldsymbol{R}^3_y} \frac{|f(y)|}{|x-y|\langle y\rangle^{3+\delta'}}\,\mathrm{d}y \\ &\lesssim \|f\|_{2,-a}\left(\int_{\boldsymbol{R}^3_y}\frac{1}{|x-y|^2\langle y\rangle^{6-a+2\delta'}}\,\mathrm{d}y\right)^{1/2}\end{aligned}$$

Due to Lemma 2.5, this implies (4.20). □

Recalling (4.15) and the fact that Ω fields act on σ, the application of Proposition 4.1 gives

$$|x|\,|y|I_2(x,y) \lesssim \sum_{|\alpha|\leq 1}\lambda^2|x|\int_0^{+\infty}\sup_{\sigma\in S^2}|R_0(\rho^2+\mathrm{i}0)[V\Omega^\alpha(v(\cdot,\rho\sigma))](\lambda x)|\frac{\rho\,\mathrm{d}\rho}{\langle\rho\rangle^{1+\delta}}. \tag{4.21}$$

In order to establish $|x|\,|y|I_2(x,y) \lesssim \lambda$, it suffices to find

$$|R_0(\rho^2+\mathrm{i}0)[V\Omega^\alpha(v(\cdot,\rho\sigma))](\lambda x)| \lesssim \langle\lambda x\rangle^{-1}\langle\rho\rangle^{-1+|\alpha|} \tag{4.22}$$

for $x \in \boldsymbol{R}^3$, $\rho \geq 0$, $|\alpha| = 0, 1$.

The case $\alpha = 0$ is a consequence of $|v(\lambda x, \rho\sigma)| = |\omega(\lambda x, \rho\sigma)|$ and (4.20). More precisely, for $\rho \leq 1$ we take $a = 3 + \delta'$ and use (A.2). For $\rho \geq 1$, we fix $a = 1 + \delta'$ and employ (A.3).

Similarly, in order to establish (4.22) in the case $|\alpha| = 1$, we apply (A.5) when $\rho \leq 1$ and (A.6) if $\rho \geq 1$.

Since $|x|\,|y|I_2(x,y) \lesssim \lambda$ and (4.18) and (4.19) hold, this concludes the proof of (4.14) with $k = 1$.

4.3. The estimate for last term of W^λ. Here we prove

$$\|W^\lambda_{4,\pm}h\|_{2,-1-\varepsilon} \le C\lambda\|h\|_{2,-1-\varepsilon}, \tag{4.23}$$

where

$$\begin{aligned}W^\lambda_{4,\pm}[h](x) &= \lambda^3\int_{\boldsymbol{R}^3_\xi}\int_{\boldsymbol{R}^3_y} e^{i\lambda[(x-y)\cdot\xi\pm(\langle x\rangle-\langle y\rangle)\langle\xi\rangle]}\bar{\omega}(\lambda x,\xi)\omega(\lambda y,\xi)h(y)\frac{dy}{\langle y\rangle^{1+\varepsilon}}\frac{d\xi}{\langle\xi\rangle^{1+\delta}}\\ &=: \int_{\boldsymbol{R}^3_y} K^\lambda_{4,\pm}(x,y)h(y)\frac{dy}{\langle y\rangle^{1+\varepsilon}}.\end{aligned}$$

From (4.16) we get

$$|K^\lambda_{4,\pm}(x,y)| \le I_3(x,y) + I_4(x,y),$$

where

$$I_3(x,y) = \lambda^3\int_0^{+\infty}\left|\int_{S^2}\bar{v}(\lambda x,\rho\sigma)R_0(\rho^2+i0)[V_{\rho\sigma}](\lambda y)\,d\sigma\right|\frac{\rho^2\,d\rho}{\langle\rho\rangle^{1+\delta}},$$

$$I_4(x,y) = \lambda^3\int_0^{+\infty}\left|\int_{S^2}\bar{v}(\lambda x,\rho\sigma)R_0(\rho^2+i0)[Vv(\cdot,\rho\sigma)](\lambda y)\,d\sigma\right|\frac{\rho^2\,d\rho}{\langle\rho\rangle^{1+\delta}}.$$

Explicitly, from (4.15) and (4.17), we have

$$\begin{aligned}|x|\,|y|I_3(x,y) &\le |x|\,|y|\int_{\boldsymbol{R}^3_z}\frac{V(z)}{4\pi|\lambda y-z|}\mathscr{A}\left(x,\frac{z}{\lambda}\right)dz\\ &\lesssim \lambda^2|y|\int_{\boldsymbol{R}^3_z}\frac{dz}{|\lambda y-z|\,|z|\langle z\rangle^{3+\delta'}} \lesssim C\lambda,\end{aligned}$$

as it follows from $|x|\,|y|\mathscr{A}(x,y)\lesssim\lambda$ and Lemma 2.9. In order to evaluate $I_4(x,y)$, we use (A.4) and get $|v(\lambda x,\rho\sigma)|\lesssim\langle\lambda x\rangle^{-1}$. Therefore using again (4.22) with $\alpha=0$, we find

$$|x|\,|y|I_4(x,y)\lesssim\lambda\int_0^{+\infty}\frac{\rho^2\,d\rho}{\langle\rho\rangle^{2+\delta}}\lesssim\lambda.$$

Hence (4.23) holds. This completes the proof.

END OF PROOF OF (4.1). Gathering (4.8), (4.11), (4.12) with $k=1$, and (4.23), we see that (4.1) holds good for all $\lambda>1$, $\varepsilon>0$ and $\delta>1$.

Appendix

We prove the following L^∞ and L^2 estimates for $\omega(x,\xi)$ used in the previous subsections.

THEOREM A.1. *Let $V(x)$ be a Hölder continuos non-negative function such that*

$$V(x) \le C_0(1+|x|)^{-2-\delta'} \quad \text{for } x \in \boldsymbol{R}^3 \tag{A.1}$$

with some C_0, $\delta' > 0$. Let $\omega(x,\xi)$ be the solution of (1.4). *Then, there exists a constant $C > 0$, independent of ξ, such that for any $0 < \varepsilon < \delta'$, it holds*

$$\|\omega(x,\xi)\|_{L_x^\infty} \le C \quad \text{for } \xi \in \boldsymbol{R}^3, \tag{A.2}$$

$$\|\langle x\rangle^{(-1-\varepsilon)/2}\omega(x,\xi)\|_{L_x^2} \le C|\xi|^{-1} \quad \text{for } |\xi| \ge 1. \tag{A.3}$$

Suppose in addition that

$$V(x) \le C_0(1+|x|)^{-3-\delta'} \quad \text{for } x \in \boldsymbol{R}^3.$$

Denote $\xi = \rho\sigma$ with $\rho \ge 0$ and $\sigma \in S^2$. Then, there exists a constant $C > 0$, independent of ξ, such that for any $0 < \varepsilon < \delta'$, $x, \xi \in \boldsymbol{R}^3$, it holds

$$|\omega(x,\xi)| \le C\langle x\rangle^{-1} \tag{A.4}$$

$$\|\Omega_{j,k}(\mathrm{e}^{-\mathrm{i}\rho x\cdot\sigma}\omega(x,\rho\sigma))\|_{L_x^\infty} \le C\rho \quad \text{for } 1 \le j,k \le 3, \tag{A.5}$$

$$\|\langle\cdot\rangle^{(-1-\varepsilon)/2}\Omega_{j,k}(\mathrm{e}^{-\mathrm{i}\rho x\cdot\sigma}\omega(x,\rho\sigma))\|_{L_x^2} \le C \quad \text{for } \rho \ge 1,\ 1 \le j,k \le 3, \tag{A.6}$$

where $\Omega_{j,k} = \sigma_j\partial_k - \sigma_k\partial_j$.

A slightly different version of the L^∞ bounds (A.2), (A.4), (A.5) can be found in [5]. In that paper, the author obtains $|\omega(x,\xi)| \le C\langle\xi\rangle^{-1}$ requiring a stronger decay for the potential $V(x)$. We prefer to minimize the assumption on the potential since these estimates are enough to our aim. Besides, we underline that the assumption that V is a non-negative function enables us to avoid any hypothesis about the presence of resonances for the operator $H = -\Delta + V(x)$.

To show the theorem, we study an integral equation related to (1.4):

$$\psi(x) = -R_0(|\xi|^2 + \mathrm{i}0)[V\psi](x) + R_0(|\xi|^2 + \mathrm{i}0)[f](x), \tag{A.7}$$

where the free resolvent operator has been defined in (4.15). In Theorem A.2 below we shall find the unique solvability of (4.15) when $f \in L^2(\langle\cdot\rangle^{1+\varepsilon})$. In that case we put $\psi(x) = \psi_{|\xi|}[f](x)$. Moreover we find L^∞ and weighted L^2 estimates for this function. At the end of the appendix, we use these estimates to prove Theorem A.1.

THEOREM A.2. *Let $V(x)$ be a Hölder continuous non-negative function satisfying* (A.1). *Then for all $\xi \in \boldsymbol{R}^3$ and $f \in L^2(\langle\cdot\rangle^{1+\varepsilon})$ with $0 < \varepsilon \le \delta'$, there is a unique solution of* (A.7) *denoted by $\psi(x) = \psi_{|\xi|}[f](x)$, $\psi_{|\xi|}[f](x) \in L^2(\langle\cdot\rangle^{-3-\varepsilon})$. Moreover there exists a constant $C > 0$, independent of ξ and f, such that*

$$\|\psi_{|\xi|}[f]\|_\infty \le C\|f\|_{2,1+\varepsilon} \quad \textit{for } \xi \in \boldsymbol{R}^3, \tag{A.8}$$

$$\|\psi_{|\xi|}[f]\|_{2,-1-\varepsilon} \le C|\xi|^{-1}\|f\|_{2,1+\varepsilon} \quad \textit{for } |\xi| \ge 1. \tag{A.9}$$

A more general version of (A.8) is given in Proposition 6.3 of [4] under stronger decay assumption for the potential V together with its derivatives. In particular, in that paper, the authors obtain L^∞ estimates for $\partial_\xi^\alpha(\mathrm{e}^{-\mathrm{i}|x|\,|\xi|-\mathrm{i}x\cdot\xi}\omega(x,\xi))$.

In order to prove Theorem A.2, we prepare some preliminary results.

LEMMA A.1. *Assume $A(x)$ and $B(x)$ are continuous functions satisfying*

$$0 < A(x) \le C\langle x\rangle^{-(3+\delta')/2}, \quad |B(x)| \le C\langle x\rangle^{-(1+\delta')/2} \quad x \in \boldsymbol{R}^3 \tag{A.10}$$

for some C, $\delta' > 0$. Suppose that for $V := AB$ the following condition holds:

$$\textit{if} \quad U(x) \in L^2(\langle\cdot\rangle^{-1-\delta_0}) \quad \textit{and} \quad (-\Delta + V)U = 0, \quad \textit{then} \quad U \equiv 0 \tag{A.11}$$

for any $0 < \delta_0 < 2\delta'$. Then for $\xi \in \boldsymbol{R}^3$ and $v_0 \in L^2$, there is a unique solution $v \in L^2$ of the following equation:

$$v(x) = -\int_{\boldsymbol{R}^3} v(y)A(x)\frac{\mathrm{e}^{\mathrm{i}|\xi|\,|x-y|}}{4\pi|x-y|}B(y)\,\mathrm{d}y + v_0(x). \tag{A.12}$$

Moreover there exists a constant $C > 0$, independent of ξ and v_0, such that

$$\|v\|_2 \le C\|v_0\|_2. \tag{A.13}$$

PROOF. We pose the problem in an abstract setting. Let $z \in \boldsymbol{C}$. We define

$$W(z)[g](x) = \int_{\boldsymbol{R}_y^3} K(x,y;z)g(y)\,\mathrm{d}y,$$

$$K(x,y;z) = -A(x)\frac{\mathrm{e}^{\mathrm{i}z|x-y|}}{4\pi|x-y|}B(y).$$

The relation (A.12) is rewritten as

$$v(x) = W(|\xi|)[v](x) + v_0(x).$$

By using the assumption (A.10) and Lemma 2.5, we see that the Hilbert-Schmidt norm of $W(z)$

$$\|W(z)\|_{HS} = \left(\iint |K(x, y; z)|^2 \, \mathrm{d}x\mathrm{d}y\right)^{1/2}$$

is finite and uniformly bounded in z, hence $W(z)$ is a compact operator. It suffices to prove that $I - W(k)$ is invertible and $\|(I - W(k))^{-1}\|$ is uniformly bounded for $k \geq 0$. Here $\|A\|$ denotes the $\mathscr{L}(L^2, L^2)$ norm for the operator.

Step 1. Let us prove the existence of $(I - W(k))^{-1}$. It is clear that the map $z \mapsto W(z)$ is analytic for $\Im z > 0$ and continuous for $\Im z \geq 0$; moreover $\|W(z)\| \to 0$ as $\Im z \to +\infty$. Hence $(I - W(z))^{-1}$ is represented by the Neumann series for $\Im z > 0$ large. We see that it is well defined and analytic for such z. Thus, it follows from analytic Fredholm theory (see e.g. [13]) that the family of operators $(I - W(k))^{-1}$ exists and depends continuously on $k \in \boldsymbol{R}$ outside a closed set $\mathscr{E} = \{k \in \boldsymbol{R} : \{0\} \neq \mathrm{Ker}(I - W(k)) \subset L^2\}$ whose measure is zero. For our aim it suffices to show that $\mathscr{E} \subset (-\infty, 0)$.

Suppose that there exist a positive number k and a solution $v \in L^2$ of the homogeneous equation $v(x) = W(k)[v](x)$. Setting $u(x) = v(x)(A(x))^{-1}$, we have

$$u(x) = -\int_{\boldsymbol{R}^3_y} v(y) \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|} B(y) \, \mathrm{d}y = -\int_{\boldsymbol{R}^3_y} u(y) \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|} (AB)(y) \, \mathrm{d}y. \quad \text{(A.14)}$$

Combining Schwartz inequality and Lemma 2.4, we find $u \in L^\infty$. An application of Lemma 4.4 in [10] shows that $u \equiv 0$, hence $v \equiv 0$. Making use of the Fredholm alternative, we conclude that any strictly positive number does not belong to $\mathscr{E}$.

It remains to prove that $0 \notin \mathscr{E}$. Here the resonance assumption (A.11) comes into play. As before, from $v \in L^2$ we get $u \in L^2(\langle \cdot \rangle^{-3-\delta'})$. Moreover from (A.14), with $k = 0$ we deduce $(-\Delta + AB)u = 0$. Suppose we have found a particular positive $\delta_0 < 2\delta'$ such that

$$u \in L^2(\langle \cdot \rangle^{-1-\delta_0}), \quad \text{(A.15)}$$

then (A.11) gives $u \equiv 0$. As before, we find $0 \notin \mathscr{E}$.

Now, we show (A.15) with $\delta_0 = \delta'$. For any $h \in \boldsymbol{Z}$, we have

$$\|u\|^2_{2,-3+(h+1)\delta'} = \int_{\boldsymbol{R}^3_x} \langle x\rangle^{-3+(h+1)\delta'} \left(\int_{\boldsymbol{R}^3_y} \frac{(AB)(y)|u(y)|}{4\pi|x-y|}\,\mathrm{d}y\right)^2 \mathrm{d}x$$

$$\lesssim \|u\|^2_{2,-3+h\delta'} \int_{\boldsymbol{R}^3_x}\int_{\boldsymbol{R}^3_y} \langle x\rangle^{-3+(h+1)\delta'} \langle y\rangle^{-1-(2+h)\delta'} |x-y|^{-2}\,\mathrm{d}y\mathrm{d}x.$$

Lemma 2.6 assures $\|u\|_{2,-3+(h+1)\delta'} \lesssim \|u\|_{2,-3+h\delta'}$ whenever $-1-2\delta' < h\delta' < 3-\delta'$. We know $u \in L^2(\langle\cdot\rangle^{-3-\delta'})$; this means we consider $h \geq -1$. The only requirement becomes $(h+1)\delta' < 3$. In particular for $2-2\delta' \leq h\delta'$, we arrive at (A.15). It is possible to choose such h, once we suppose $\delta' < 2$. This condition is not restrictive, since we are interested in small δ'.

Step 2. By the closedness of $\mathscr{E}$ and the continuity of $k \mapsto \|(I-W(k))^{-1}\|$, we have that the equi-boundedness of $\|(I-W(k))^{-1}\|$, for $k \geq 0$, is a consequence of $\|W(k)\| \to 0$ for $k \to +\infty$. It suffices to prove the punctual limit

$$\lim_{k\to+\infty} \|W(k)[g]\|_2 = 0 \quad g \in L^2. \tag{A.16}$$

Since, for any $r \in \boldsymbol{R}$, $\mathscr{C}^1_0$ is dense in $L^2(\langle\cdot\rangle^r)$ we can use a density argument. We put

$$\overline{W}(k)[h](x) = \int \frac{A(x)\mathrm{e}^{\mathrm{i}k|x-y|}h(y)}{4\pi|x-y|}\,\mathrm{d}y = A(x)\int_0^{+\infty}\int_{|\omega|=1} \mathrm{e}^{\mathrm{i}kr} h(x-r\omega)r\,\mathrm{d}\omega\mathrm{d}r. \tag{A.17}$$

In particular $\overline{W}(k)[Bg](x) = -W(k)[g](x)$. By means of Lemma 2.4, we have

$$\|\overline{W}(k)h\|^2_2 \lesssim \int\left(\int \frac{A(x)h(y)}{|x-y|}\,\mathrm{d}y\right)^2 \mathrm{d}x \lesssim \|h\|^2_{2,1+\delta'} \int \langle x\rangle^{-3-\delta'} \int \frac{\langle y\rangle^{-1-\delta'}}{|x-y|^2}\,\mathrm{d}y\mathrm{d}x$$

$$\lesssim \|h\|^2_{2,1+\delta'}$$

for any $h \in L^2(\langle\cdot\rangle^{1+\delta'})$. Suppose we have found

$$\lim_{k\to+\infty} \overline{W}(k)[h](x) = 0 \tag{A.18}$$

for a.e. $x \in \boldsymbol{R}^3$ and any $h \in \mathscr{C}^1_0$. Lebesgue's dominant convergence theorem imply

$$\lim_{k\to+\infty} \|\overline{W}(k)[h]\|_2 = 0$$

for any $h \in \mathscr{C}^1_0$. In order to gain (A.16), we take $h \in \mathscr{C}^1_0$ which approximate $Bg \in L^2(\langle\cdot\rangle^{1+\delta'})$ and observe that

$$\begin{aligned}\|W(k)[g]\|_2 = \|\overline{W}(k)[Bg]\|_2 &\le \|\overline{W}(k)[Bg-h]\|_2 + \|\overline{W}(k)[h]\|_2 \\ &\le \|Bg-h\|_{2,1+\delta'} + \|\overline{W}(k)[h]\|_2.\end{aligned}$$

It remains to establish (A.18). Since $h(y) \in \mathscr{C}_0^1$, we can integrate by parts with respect to r in (A.17). We get

$$\begin{aligned}|k|\,|\overline{W}(k)[h](x)| &\lesssim \left|\int \frac{e^{ik|\eta|}}{|\eta|^2} h(x-\eta)\,d\eta\right| + \left|\int \frac{e^{ik|\eta|}}{|\eta|^2}\eta\cdot\nabla h(x-\eta)\,d\eta\right| \\ &\lesssim \|\langle\cdot\rangle^{1+\epsilon}h\|_\infty \int \frac{\langle y\rangle^{-1-\epsilon}}{|x-y|^2}\,dy + \|\langle\cdot\rangle^{2+\epsilon}|\nabla h|\,\|_\infty \int \frac{\langle y\rangle^{-2-\epsilon}}{|x-y|}\,dy \le C,\end{aligned}$$

where C is independent of k and x. Thus we obtain (A.18) and conclude the proof. □

The following result can be established with a proof similar to the previous lemma.

Lemma A.2. *Assume $A(x)$ and $B(x)$ are continuous functions on $\boldsymbol{R}^3$ satisfying*

$$0 < A(x) \le C\langle x\rangle^{-(1+\delta')/2}, \quad |B(x)| \le C\langle x\rangle^{-(3+\delta')/2} \tag{A.19}$$

for some C, $\delta' > 0$. Letting $V := AB$, we suppose that (A.11) *for any $0 < \delta_0 < 2\delta'$ holds. Then for $\xi \in \boldsymbol{R}^3$ and $v_0 \in L^2$, there is a unique solution $v \in L^2$ of* (A.12) *verifying* (A.13).

For completeness we give the proof of the next lemma that enable us to avoid the resonance assumption (A.11). This type of results has been proved by Georgiev and Visciglia in [7].

Lemma A.3. *Let $V \in C(\boldsymbol{R}^3, \boldsymbol{R})$ be a non-negative potential such that $V(x) \le C\langle x\rangle^{-2-\delta'}$ for some C, $\delta' > 0$. Then the condition* (A.11) *holds for any $0 < \delta_0 < 2\delta'$.*

Proof. Let $U \in L^2(\langle\cdot\rangle^{-1-\delta_0})$ be a solution of $-\Delta U = -VU$, hence

$$U(x) = \int_{\boldsymbol{R}^3} \frac{U(y)V(y)}{4\pi|x-y|}\,dy \tag{A.20}$$

In particular

$$|U(x)|^2 \lesssim \|U\|^2_{L^2(\langle\cdot\rangle^{-1-\delta_0})} \int_{\mathbf{R}^3} \langle y\rangle^{-3-2\delta'+\delta_0}|x-y|^{-2}\,\mathrm{d}y.$$

Since $\delta_0 < 2\delta'$, we find

$$|U(x)| \lesssim \langle x\rangle^{-1}. \tag{A.21}$$

This inequality gives $\Delta U = VU \in L^1$. Similarly, we have

$$|\partial_{x_j} U(x)| \lesssim \int_{\mathbf{R}^3} |U(y)|V(y)|x-y|^{-2}\,\mathrm{d}y \lesssim \|\langle\cdot\rangle U\|_\infty \int_{\mathbf{R}^3} \langle y\rangle^{-3-\delta'}|x-y|^{-2}\,\mathrm{d}y.$$

This implies

$$|\nabla U(x)| \lesssim \langle x\rangle^{-2}. \tag{A.22}$$

Let be $R > 1$ and φ is a smooth function such that $\varphi(x) = 1$ if $|x| \le 1$ and $\varphi(x) = 0$ if $|x| \ge 2$. We multiply the equation $(-\Delta + V)U = 0$ by the function $\varphi(R^{-1}x)U(x)$. After integration by parts, we get

$$\int_{\mathbf{R}^3} (|\nabla U|^2 + V(x)|U(x)|^2)\varphi(R^{-1}x)\,\mathrm{d}x + \frac{1}{R}\int_{\mathbf{R}^3} \nabla U(x)\cdot(\nabla\varphi)(R^{-1}x)U(x)\,\mathrm{d}x = 0.$$

This yields

$$\begin{aligned}\int_{|x|\le 2R} &(|\nabla U|^2 + V(x)|U(x)|^2)\varphi(R^{-1}x)\,\mathrm{d}x \\ &\le \frac{\|\nabla\varphi\|_\infty}{R}\int_{R<|x|<2R} |\nabla U(x)|\,|U(x)|\,\mathrm{d}x.\end{aligned} \tag{A.23}$$

Combining (A.21) and (A.22), we see that there exists $C > 0$, independent of R, such that

$$\int_{R<|x|<2R} |\nabla U(x)|\,|U(x)|\,\mathrm{d}x \le C.$$

Taking the limit in (A.23), we find

$$\int_{\mathbf{R}^3} (|\nabla U|^2 + V(x)|U(x)|^2)\,\mathrm{d}x = 0. \tag{A.24}$$

This implies that U is piece-wise constant and $V(x)|U(x)|^2 = 0$. Coming back to the fundamental solution (A.20) we arrive at $U = 0$ and complete the proof. $\square$

End of Proof of Theorem A.2. First we take $A(x) = \langle x \rangle^{-(3+\delta')/2}$, $B(x) = A^{-1}(x)V(x)$ and $v_0(x) = A(x)R_0(|\xi|^2 + \mathrm{i}0)[f](x)$. Due to (A.1) and to Lemma A.3, the assumptions (A.10) and (A.11) are satisfied. Moreover by using Lemma 2.4, we see that

$$\|R_0(|\xi|^2 + \mathrm{i}0)[f]\|_\infty \lesssim \|f\|_{2,1+\varepsilon}$$

for any $\varepsilon > 0$. Hence $v_0 \in L^2$ and $\|v_0\|_2 \lesssim \|f\|_{2,1+\varepsilon}$. Thus Lemma A.1 yields the existence of a unique solution $v \in L^2$ of (A.12) such that

$$\|v\|_2 \lesssim \|v_0\|_2 \lesssim \|f\|_{2,1+\varepsilon}.$$

If we set $\psi(x) = A^{-1}(x)v(x)$, then we see from (A.12) that $\psi(x)$ solves (A.7), and satisfies

$$\psi(x) = -\int_{\boldsymbol{R}_y^3} v(y) \frac{\mathrm{e}^{\mathrm{i}|\xi|\,|x-y|}}{4\pi|x-y|} B(y)\,\mathrm{d}y + R_0(|\xi|^2 + \mathrm{i}0)[f](x). \tag{A.25}$$

Since $|B(y)| \leq \langle y \rangle^{-(1+\delta')/2}$, we gain

$$|\psi(x)| \lesssim \|v\|_2 \left(\int_{\boldsymbol{R}_y^3} \langle y \rangle^{-1-\delta'} |x-y|^{-2}\,\mathrm{d}y \right)^{1/2} + |R_0(|\xi|^2 + \mathrm{i}0)[f](x)| \lesssim \|f\|_{2,1+\varepsilon}.$$

This means that (A.8) holds.

Let us fix $A(x) = \langle x \rangle^{-(1+\delta')/2}$, $B(x) = A^{-1}(x)V(x)$, and $v_0(x) = A(x)R_0(|\xi|^2 + \mathrm{i}0)[f](x)$. Due to (A.1) and to Lemma A.3, the assumptions (A.19) and (A.11) are satisfied. In order to prove $v_0 \in L^2$, we observe that

$$\begin{aligned} \|v_0\|_2 &\lesssim \|R_0(|\xi|^2 + \mathrm{i}0)[f](x)\|_{2,-1-\delta'} \\ &\lesssim \|R_0(|\xi|^2 + \mathrm{i}0)[f](x)\|_{2,-1-\varepsilon} \lesssim |\xi|^{-1}\|f\|_{2,1+\varepsilon} \end{aligned}$$

for any $\varepsilon > 0$ with $0 < \varepsilon \leq \delta'$ and $|\xi| \geq 1$. In the last line we used a well known estimate for the free resolvent. In a very general version this can be found in [1] Appendix A, Remark 2.

Coming back to our proof, Lemma A.2 yields the existence of a unique solution $\tilde{v} \in L^2$ of (A.12) such that

$$\|\tilde{v}\|_2 \lesssim \|v_0\|_2 \lesssim |\xi|^{-1}\|f\|_{2,1+\varepsilon}.$$

Hence $\psi(x) = A^{-1}(x)\tilde{v}(x)$ is the unique solution of (A.7). Since (A.25) with v replaced by $\tilde{v}$ holds, and now $|B(y)| \leq \langle y \rangle^{-(3+\delta')/2}$, using Lemma 2.5, we obtain

$$|\psi(x)| \lesssim \langle x \rangle^{-1}\|\tilde{v}\|_2 + |R_0(|\xi|^2 + \mathrm{i}0)[f](x)|.$$

Therefore we have

$$\|\psi\|_{2,-1-\varepsilon} \lesssim \|\tilde{v}\|_2 + \|R_0(|\xi|^2 + \mathrm{i}0)[f](x)\|_{2,-1-\varepsilon} \lesssim |\xi|^{-1}\|f\|_{2,1+\varepsilon}.$$

Thus (A.9) is shown and the proof is completed.

Proof of Theorem A.1. According to (1.4), we note that $\omega(x,\xi) = \mathrm{e}^{\mathrm{i}x\cdot\xi}\psi_{|\xi|}[V_\xi](x)$ with $V_\xi(x) = -V(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}$. In particular (A.1) implies $V_\xi(x) \in L^2(\langle\cdot\rangle^{1+\varepsilon})$.

The relation (A.2) is a direct consequence of (A.8). Similarly (A.3) follows from (A.9).

For proving (A.4) we combine (A.8) and (A.20) with $a = 3 + \delta'$. In fact, from (A.7) we have

$$\begin{aligned}|\omega(x,\xi)| &\le \|\psi_{|\xi|}[V_\xi]\|_\infty \le \|R_0(|\xi|^2 + \mathrm{i}0)[V\psi_{|\xi|}[V_\xi]]\|_\infty + \|R_0(|\xi|^2 + \mathrm{i}0)[V_\xi]\|_\infty \\ &\lesssim \langle x\rangle^{-1}(\|V\psi_{|\xi|}[V_\xi]\|_{2,3+\delta'} + \|V_\xi\|_{2,3+\delta'}) \\ &\lesssim \langle x\rangle^{-1}(\|\psi_{|\xi|}[V_\xi]\|_\infty + 1)\|V\|_{2,-3-\delta'} \lesssim \langle x\rangle^{-1}.\end{aligned}$$

In order to prove (A.5) and (A.6) we note that $\Omega_{j,k}(\mathrm{e}^{-\mathrm{i}\rho\sigma\cdot x}\omega(x,\rho\sigma))$ solves (A.7) with $f \equiv f_{j,k}(\rho,\sigma,y) = -\mathrm{i}\rho(\sigma_j y_k - \sigma_k y_j)V_{\rho\sigma}(y)$. In particular $f_{j,k}(\rho,\sigma,\cdot) \in L^2(\langle\cdot\rangle^{1+\varepsilon})$ if $V(y) \lesssim \langle y\rangle^{-3-\varepsilon}$. Hence, (A.8) and (A.9) give the conclusion. □

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