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INDEPENDENT FAMILIES OF DESTRUCTIBLE GAPS

By

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Abstract. We investigate the finite support product of forcing notions related to destructible gaps, and prove the existence of a large set of independent destructible gaps under $\diamondsuit$.

1. Introduction and Notation

1.1. Introduction

An ω_1-tree can be considered as a forcing notion adding an uncountable chain. A Suslin tree is a ccc and ω_1-Baire forcing notion (a Suslin algebra). In [8], Kurepa showed that the two-product of one Suslin tree does not have the countable chain condition. This can be proved by the product lemma for forcings and the fact on ccc-forcings because a Suslin tree as a forcing notion adds an uncountable chain and then (if it is normal, i.e. any node has at least two incomparable extensions) it also has an uncountable antichain. But under $\diamondsuit$, for any Suslin tree, we can find another Suslin tree such that the product of these Suslin trees is also ccc. In fact, under $\diamondsuit$, we have several variations of families of Suslin trees ([1]).

In this paper, we deal with destructible gaps. A destructible gap is an (ω_1, ω_1)-gap which can be destroyed by a forcing extension preserving cardinals. A destructible gap has a characterization similar to a Suslin tree ([2]). A Suslin tree is an ω_1-tree having no uncountable chains and antichains. On the other hand, for an (ω_1, ω_1)-pregap $(\mathscr{A}, \mathscr{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ with the set $a_\alpha \cap b_\alpha$ empty for every $\alpha \in \omega_1$, we say here that α and β in ω_1 are compatible if

$$(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \varnothing.$$

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Then by the characterization due to Kunen and Todorčević, we notice that an (ω_1, ω_1)-pregap is a destructible gap iff it has no uncountable pairwise compatible and incompatible subsets of ω_1. (We must notice that from results of Farah and Hirschorn [4, 5], the existence of a destructible gap is independent with the existence of a Suslin tree.)

One of differences from an ω_1-tree is that any (ω_1, ω_1)-pregap have never had an uncountable chain and antichain at the same time. We have forcing notions related to an (ω_1, ω_1)-pregap.

DEFINITION 1.1 [E.g. [3, 7, 10, 11]). *Let* $(\mathscr{A}, \mathscr{B}) = \langle a_\alpha, b_\alpha; \alpha \in \omega_1 \rangle$ *be an* (ω_1, ω_1)*-pregap with* $a_\alpha \cap b_\alpha = \emptyset$ *for every* $\alpha \in \omega_1$.

1. $\mathscr{F}(\mathscr{A}, \mathscr{B}) := \{\sigma \in [\omega_1]^{<\aleph_0}; \forall \alpha \neq \beta \in \sigma, (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset\}$, *ordered by reverse inclusion.*
2. $\mathscr{S}(\mathscr{A}, \mathscr{B}) := \{\sigma \in [\omega_1]^{<\aleph_0}; \bigcup_{\alpha \in \sigma} a_\alpha \cap \bigcup_{\alpha \in \sigma} b_\alpha = \emptyset\}$, *ordered by reverse inclusion.*

We note that $\mathscr{F}(\mathscr{A}, \mathscr{B})$ forces $(\mathscr{A}, \mathscr{B})$ to be indestructible and $\mathscr{S}(\mathscr{A}, \mathscr{B})$ forces $(\mathscr{A}, \mathscr{B})$ to be separated. Using these forcing notions, we can express characterizations of being a gap and destructibility.

THEOREM 1.2 [E.g. [3, 7, 10, 11]). *Let* $(\mathscr{A}, \mathscr{B})$ *be an* (ω_1, ω_1)*-pregap. Then;*

1. $(\mathscr{A}, \mathscr{B})$ *forms a gap iff* $\mathscr{F}(\mathscr{A}, \mathscr{B})$ *has the countable chain condition.*
2. $(\mathscr{A}, \mathscr{B})$ *is destructible (may not be a gap) iff* $\mathscr{S}(\mathscr{A}, \mathscr{B})$ *has the countable chain condition.*

Therefore we say that $(\mathscr{A}, \mathscr{B})$ is a destructible gap if both $\mathscr{F}(\mathscr{A}, \mathscr{B})$ and $\mathscr{S}(\mathscr{A}, \mathscr{B})$ have the ccc. As in the case of a Suslin tree, by the product lemma for forcings, we note that $\mathscr{F}(\mathscr{A}, \mathscr{B}) \times \mathscr{S}(\mathscr{A}, \mathscr{B})$ does not have the ccc, and the referee of the paper [6] has proved that whenever $(\mathscr{A}_i, \mathscr{B}_i)$ is an (ω_1, ω_1)-gap for $i \in I$, $\prod_{i \in I} \mathscr{F}(\mathscr{A}_i, \mathscr{B}_i)$ has the countable chain condition. However it is independent from ZFC that the above statement is true for $\mathscr{S}$, i.e. the following statements are both consistent with ZFC: Whenever $(\mathscr{A}_i, \mathscr{B}_i)$ is a destructible gap for $i \in I$, $\prod_{i \in I} \mathscr{S}(\mathscr{A}_i, \mathscr{B}_i)$ has the countable chain condition; There exists destructible gaps $(\mathscr{A}, \mathscr{B})$ and $(\mathscr{C}, \mathscr{D})$ such that the product $\mathscr{S}(\mathscr{A}, \mathscr{B}) \times \mathscr{S}(\mathscr{C}, \mathscr{D})$ does not have the countable chain condition. (The first consistency is proved to just force by a finite support iteration with the book-keeping argument, and the second consistency is proved using an observation due to Stevo Todorčević as follows: If an indestructible gap is restricted to both a Cohen real and its complement (viewed as subsets of ω), then this pair of new gaps freeze one another. This is pointed by the referee of the paper [?] to me.) We will see that e.g., we may have two destructible gaps $(\mathscr{A},\mathscr{B})$ and $(\mathscr{C},\mathscr{D})$ so that all variations $\mathscr{X}_0(\mathscr{A},\mathscr{B}) \times \mathscr{X}_1(\mathscr{A},\mathscr{B})$ have the ccc. Throughout this paper, we consider families of destructible gaps as follows.

Definition 1.3. *A family* $\{(\mathscr{A}_i,\mathscr{B}_i); i \in I\}$ *of destructible gaps is independent if for every combination* $\langle \mathscr{X}_i; i \in I\rangle$ *where each* $\mathscr{X}_i$ *is either* $\mathscr{S}$ *or* $\mathscr{F}$, *a finite support product* $\prod_{i\in I} \mathscr{X}_i(\mathscr{A}_i,\mathscr{B}_i)$ *has the countable chain condition. Moreover,* $\{(\mathscr{A}_i,\mathscr{B}_i); i \in I\}$ *is a maximal independent family of destructible gaps if it is maximal with the property of independence.*

We note that destructible gaps added by finite approximations are independent, and if κ many Cohen reals are added, then in the extension there is an independent family of κ many destructible gaps (by the similar argument due to Todorčević). So by a book-keeping argument of the ccc-forcings, for any (finite or infinite) cardinal κ, it is consistent with ZFC that there exists a maximal independent family of destructible gaps of size κ.

If $\diamondsuit$ holds, the size of maximal independent families of Suslin trees are quite large. In [13] and [14], Zakrzewski has proved that there exists a family of Suslin trees of size $2^{\aleph_1}$ whose finite support product is also ccc and for any family of Suslin trees of size $\aleph_1$, if the product of members of this family with finite support is also ccc, then it is not maximal with respect to this property. These theorems are also true for destructible gaps. That is,

Theorem 2.1. *Under* $\diamondsuit$, *there exists a family of* $2^{\aleph_1}$ *destructible gaps which is independent.*

Theorem 2.2. *Under* $\diamondsuit$, *every maximal independent family of destructible gaps has size at least* $\aleph_2$.

1.2. Notation

A pregap in $\mathscr{P}(\omega)/\mathrm{fin}$ is a pair $(\mathscr{A},\mathscr{B})$ of subsets of $\mathscr{P}(\omega)$ such that for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$, the set $a \cap b$ is finite. For subsets a and b of ω, we say that a is almost contained in b (and denote $a \subseteq^* b$) if $a \setminus l$ is a subset of b for some $l \in \omega$. If $(\mathscr{A},\mathscr{B})$ is a gap and both ordered sets $\langle \mathscr{A}, \subseteq^*\rangle$ and $\langle \mathscr{B}, \subseteq^*\rangle$ are well ordered and these order type are κ and λ respectively, then we say that a pregap has the type (κ, λ) or a (κ, λ)-pregap. Moreover if $\kappa = \lambda$, we say that the pregap is symmetric. For a pregap $(\mathscr{A}, \mathscr{B})$, we say that $(\mathscr{A}, \mathscr{B})$ is separated if for some $c \in \mathscr{P}(\omega)$, $a \subseteq^* c$ and the set $c \cap b$ is finite for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$. If a pregap is not separated, we say that it is a gap. Moreover if a gap has the type (κ, λ), it is called a (κ, λ)-gap.

For an ordinal α, if we say that $\langle a_\xi, b_\xi; \xi \in \alpha \rangle$ is a pregap, we always assume that

- $\xi < \eta$ in α, $a_\xi \subseteq^* a_\eta$ and $b_\xi \subseteq^* b_\eta$, and
- for every $\xi \in \alpha$, the set $a_\xi \cap b_\xi$ is empty.

In proofs of theorems, all of pregaps will constructed to satisfy the following property.

Definition 1.4 ([12]). *We say that a pregap* $(\mathscr{A}, \mathscr{B}) = \langle a_\gamma, b_\gamma; \gamma \in \alpha \rangle$ *admits finite changes if for any* $\beta \in \alpha$ *with* $\beta = \eta + k$ *for some* $\eta \in \mathsf{Lim} \cap \alpha$ (*where* Lim *is a class of limit ordinals*) *and* $k \in \omega$, $H, J \in [\omega]^{<\aleph_0}$ *with* $H \cap J = \varnothing$ *and* $i > \max(H \cup J)$ *there exists* $n \in \omega$ *so that*

$$a_{\eta+n} \cap i = H, \quad a_{\eta+n} \backslash i = a_\beta \backslash i, \quad b_{\eta+n} \cap i = J, \quad \textit{and} \quad b_{\eta+n} \backslash i = b_\beta \backslash i.$$

We note that an (α, α)-pregap $\langle a_\xi, b_\xi; \xi \in \alpha \rangle$ can be considered a function f from $\alpha \times \omega$ into 3 such that for each $\gamma \in \alpha$,

$$a_\gamma = \{k \in \omega; f(\gamma, k) = 0\} \quad \text{and} \quad b_\gamma = \{k \in \omega; f(\gamma, k) = 1\}.$$

In other words, a function f codes an (α, α)-pregap.

2. Consequences from $\diamondsuit$

Under $\diamondsuit$, there is an independent family of destructible gaps of size $2^{\aleph_1}$. The following proof is a modification of the proof in [13].

Theorem 2.1. *Under* $\diamondsuit$, *there exists a family of* $2^{\aleph_1}$ *destructible gaps which is independent.*

Proof. Let $\langle \mathsf{D}_\alpha; \alpha \in \omega_1 \rangle$ be a $\diamondsuit$-sequence for $\omega_1 \times \omega_1$, i.e. $\mathsf{D}_\alpha \in \mathscr{P}(\alpha) \times \mathscr{P}(\alpha)$ for all $\alpha \in \omega_1$ and for all $\langle \mathsf{E}^0, \mathsf{E}^1 \rangle \in \mathscr{P}(\omega_1) \times \mathscr{P}(\omega_1)$, the set $\{\alpha \in \omega_1; \langle \mathsf{E}^0 \cap \alpha, \mathsf{E}^1 \cap \alpha \rangle = \mathsf{D}_\alpha\}$ is stationary, and say $\mathsf{D}_\alpha = \langle \mathsf{D}^0_\alpha, \mathsf{D}^1_\alpha \rangle$ for each $\alpha \in \omega_1$. In this proof (and the proof of the next theorem), we consider the following coding between pairs $\langle \mathsf{D}, n\rangle$ of subsets of countable ordinals and natural numbers and pairs of sequences of the form $\langle\langle x_i; i\in n\rangle, \langle \mathscr{X}_i; i\in n\rangle\rangle$ such that each x_i is a binary sequence with a fixed length and each $\mathscr{X}_i$ is either $\mathscr{F}$ or $\mathscr{S}$. Let $\alpha\in\omega_1+1$, $\mathsf{D}\subseteq\alpha$ and $n\in\omega$. (D is considered as a function in 2^α by thinking of its characteristic function.) We say that $\langle \mathsf{D}, n\rangle$ codes $\langle\langle x_i; i\in n\rangle, \langle \mathscr{X}_i; i\in n\rangle\rangle$ if each x_i is a binary sequence of length α, each $\mathscr{X}_i$ is either $\mathscr{F}$ or $\mathscr{S}$, and for each $i\in n$, $k\in\omega$ and $\eta\in \mathsf{Lim}\cap\alpha$,

$$\mathscr{X}_i=\mathscr{F}\Leftrightarrow \mathsf{D}(i)=0,$$

$$x_i(k)=\mathsf{D}(n\cdot(k+1)+i),\quad\text{and}\quad x_i(\eta+k)=\mathsf{D}(\eta+n\cdot k+i).$$

A pair $\langle \mathsf{D}, n\rangle$ can be recovered from a pair $\langle\langle x_i; i\in n\rangle, \langle \mathscr{X}_i; i\in n\rangle\rangle$ by this manner. We note that if α and β are limit ordinals in ω_1+1 with $\alpha<\beta$, $\mathsf{D}\subseteq\alpha$, $\mathsf{D}'\subseteq\beta$, $n\in\omega$, and $\langle \mathsf{D}, n\rangle$ and $\langle \mathsf{D}', n\rangle$ code $\langle\langle x_i; i\in n\rangle, \langle \mathscr{X}_i; i\in n\rangle\rangle$ and $\langle\langle x_i'; i\in n\rangle, \langle \mathscr{X}_i'; i\in n\rangle\rangle$ respectively, then $\mathsf{D}=\mathsf{D}'\cap\alpha$ holds iff $x_i'\restriction\alpha=x_i$ and $\mathscr{X}_i'=\mathscr{X}_i$ hold for every $i\in n$. Let $\langle J_n; n\in\omega\rangle$ be a sequence of functions such that each J_n is an injection from $([\omega_1]^{<\aleph_0})^n$ into ω_1 and for distinct natural numbers m and n,

$$J_m[([\omega_1]^{<\aleph_0})^m]\cap J_n[([\omega_1]^{<\aleph_0})^n]=\varnothing,$$

where $J_n[([\omega_1]^{<\aleph_0})^n]$ is a image of the set $([\omega_1]^{<\aleph_0})^n$ by J_n.

We will construct, by recursion on $\alpha\in\omega_1$, a function f from $2^{<\omega_1}\times\omega$ into 3 with the following properties:

1. For any $x\in 2^\alpha$, $f\restriction(\{s\in 2^{<\alpha}; s\subseteq x\}\times\omega)$ codes an (α,α)-pregap with the admission of finite changes, i.e. when we let

$$a_s:=\{k\in\omega; f(s,k)=0\}\quad\text{and}\quad b_s:=\{k\in\omega; f(s,k)=1\}$$

   for each $s\in 2^{<\omega_1}$, $\langle a_{x\restriction\gamma}, b_{x\restriction\gamma}; \gamma\in\alpha\rangle$ forms an (α,α)-pregap and admits finite changes,
2. If α is a limit ordinal and there exists $n\in\omega$ such that
   - $\mathsf{D}^0_\alpha\subseteq J_n[([\omega_1]^{<\aleph_0})^n]$ and $J_n^{-1}[\mathsf{D}^0_\alpha]\subseteq([\alpha]^{<\aleph_0})^n$,
   - the pair $\langle \mathsf{D}^1_\alpha, n\rangle$ codes the pair $\langle\langle x_i; i\in n\rangle, \langle \mathscr{X}_i; i\in n\rangle\rangle$ of the sequence of binary sequences of length α and the sequence of members in $\{\mathscr{F},\mathscr{S}\}$ and the family $J_n^{-1}[\mathsf{D}^0_\alpha]$ is a maximal antichain in $\prod_{i\in n}\mathscr{X}_i(\langle a_{x_i\restriction\gamma}, b_{x_i\restriction\gamma}\rangle_{\gamma\in\alpha})$,

   then for this unique $n\in\omega$, there is an infinite subset S_α of natural numbers such that for every $\tau\in\prod_{i\in n}\mathscr{X}_i(\langle a_{x_i\restriction\gamma}, b_{x_i\restriction\gamma}\rangle_{\gamma\in\alpha})$, there is $m\in\omega$ so that for every $l\in S_\alpha\setminus m$, every finite sequence $\langle H^i, K^i; i\in n\rangle$ with the property that

H^i and K^i are disjoint subsets of l for all $i \in n$ and there exists $\sigma \in J_n^{-1}[\mathsf{D}_\alpha^0]$ such that for all $i \in n$,

- if $\mathscr{X}_i$ is $\mathscr{S}$, then

$$\left(\bigcup_{\xi \in \sigma(i)} a_{x_i \restriction \xi} \cap K^i\right) \cup \left(H^i \cap \bigcup_{\xi \in \sigma(i)} b_{x_i \restriction \xi}\right) = \emptyset,$$

$$\bigcup_{\xi \in \sigma(i)} a_{x_i \restriction \xi} \setminus l \subseteq a_{x_i} \setminus l \quad \text{and} \quad \bigcup_{\xi \in \sigma(i)} b_{x_i \restriction \xi} \setminus l \subseteq b_{x_i} \setminus l,$$

  and
- if $\mathscr{X}_i$ is $\mathscr{F}$, then $\sigma(i) \cup \tau(i)$ is a condition in $\mathscr{F}(\langle a_{x_i \restriction \gamma}, b_{x_i \restriction \gamma} \rangle_{\gamma \in \alpha})$ and for any $\xi \in \sigma(i)$,

$$(a_{x_i \restriction \xi} \cap (b_{x_i} \setminus l)) \cup ((a_{x_i} \setminus l) \cap b_{x_i \restriction \xi}) \neq \emptyset.$$

By the property 1, the construction at successor stages are trivial. Assume that α is a limit ordinal and satisfies the assumption of the property 2 for an $n \in \omega$ and say $\langle \mathsf{D}_\alpha^1, n\rangle$ codes a sequence $\langle x_i; i \in n\rangle$ of functions in 2^α and a sequence $\langle \mathscr{X}_i; i \in n\rangle$. Let $\langle \mu_h; h \in \omega\rangle$ enumerate conditions in $\prod_{i \in n} \mathscr{X}_i(\langle a_{x_i \restriction \gamma}, b_{x_i \restriction \gamma}\rangle_{\gamma \in \alpha})$ and let $I_\alpha := \{i \in n; \mathscr{X}_i = \mathscr{S}\}$. For all $x \in 2^\alpha \setminus \{x_i; i \in n\}$, we simply take $f \restriction \{x\} \times \omega$, such that $a_x \cap b_x = \emptyset$, both $\omega \setminus (a_x \cup b_x)$, $a_x \setminus a_{x \restriction \gamma}$ and $b_x \setminus b_{x \restriction \gamma}$ are infinite, and $a_{x \restriction \gamma} \subseteq^* a_x$ and $b_{x \restriction \gamma} \subseteq^* b_x$ for every $\gamma \in \alpha$. We construct $f \restriction (\{x_i; i \in n\} \times \omega)$ which satisfies the property 2 as follows.

By recursion on $k \in \omega$, we will construct $\langle \zeta_k^i; i \in n\rangle \in \alpha^n$ and $l_k \in \omega$ such that

- $l_k < l_{k+1}$ for every $k \in \omega$,
- for each $i \in n$, the sequence $\langle \zeta_k^i; k \in \omega\rangle$ is cofinal in α, and
- $a_{x_i \restriction \zeta_{k-1}^i} \cap l_{k-1} = a_{x_i \restriction \zeta_k^i} \cap l_{k-1}$, $a_{x_i \restriction \zeta_{k-1}^i} \setminus l_{k-1} \subseteq a_{x_i \restriction \zeta_k^i}$, $b_{x_i \restriction \zeta_{k-1}^i} \cap l_{k-1} = b_{x_i \restriction \zeta_k^i} \cap l_{k-1}$ and $b_{x_i \restriction \zeta_{k-1}^i} \setminus l_{k-1} \subseteq b_{x_i \restriction \zeta_k^i}$, for every $i \in n$ and $k \in \omega$.

Assume that we have already constructed ζ_h^i and l_h for all $h \in k$ and $i \in n$. Let $\{\langle\langle H_j^i, K_j^i; i \in n\rangle, \tau_j\rangle; j \in N_k\}$ enumerate all pairs of sequences $\langle H^i, K^i; i \in n\rangle$ so that H^i and K^i are disjoint subsets of l_{k-1} and members of the set $\{\mu_h; h \in k\}$. (So $N_k = 2^{l_{k-1} \times n} \times k$.) By the induction hypothesis of the property 1 and our assumption, we can choose $\langle \eta_j^i; i \in n\rangle \in \alpha^n$ and $\sigma_j \in J_n^{-1}[\mathsf{D}_\alpha^0]$ by recursion on $j \in N_k$ such that

- for each $i \in n$, put $\sigma_{-1}(i) := \emptyset$ and $\eta_{-1}^i := \zeta_{k-1}^i$,
- we take large enough $\eta_j^i \in \alpha$ for each $i \in n$ such that
  - if $i \in I_\alpha$, then

$$a_{x_i\restriction\eta_j^i}\cap l_{k-1}=H_j^i,\quad b_{x_i\restriction\eta_j^i}\cap l_{k-1}=K_j^i,$$

$$a_{x_i\restriction\eta_j^i}\setminus l_{k-1}\supseteq\left(\bigcup_{\xi\in\sigma_{j-1}(i)}a_{x_i\restriction\xi}\cup a_{x_i\restriction\eta_{j-1}^i}\right)\Big\backslash\, l_{k-1},$$

and

$$b_{x_i\restriction\eta_j^i}\setminus l_{k-1}\supseteq\left(\bigcup_{\xi\in\sigma_{j-1}(i)}b_{x_i\restriction\xi}\cup b_{x_i\restriction\eta_{j-1}^i}\right)\Big\backslash\, l_{k-1},$$

and

– if $i\in n\setminus I_\alpha$, then we take not only η_j^i but also a natural number $e_j^i\geq l_{k-1}$ and a condition $v_j(i)$ in $\mathscr{F}(\langle a_{x_i\restriction\gamma},b_{x_i\restriction\gamma}\rangle_{\gamma\in\alpha})$, which extends $\tau_j(i)$ such that
  * $\eta_j^i>\max(\tau_j(i))$,

$$a_{x_i\restriction\eta_j^i}\setminus l_{k-1}\supseteq a_{x_i\restriction\eta_{j-1}^i}\setminus l_{k-1},$$

  and

$$b_{x_i\restriction\eta_j^i}\setminus l_{k-1}\supseteq b_{x_i\restriction\eta_{j-1}^i}\setminus l_{k-1}$$

  (in this case, we don't need take care what are $a_{x_i\restriction\eta_j^i}\cap l_{k-1}$ and $b_{x_i\restriction\eta_j^i}\cap l_{k-1}$),
  * for every $\xi\in v_j(i)$,

$$a_{x_i\restriction\xi}\setminus e_j^i\subseteq a_{x_i\restriction\eta_j^i}\setminus e_j^i\quad\text{and}\quad b_{x_i\restriction\xi}\setminus e_j^i\subseteq b_{x_i\restriction\eta_j^i}\setminus e_j^i,$$

  * for any $\langle H,K\rangle\in\mathscr{P}(e_j^i)^2$ with $H\cap K=\emptyset$, there exists $\xi\in v_j(i)$ such that

$$(a_{x_i\restriction\xi}\cap K)\cup(H\cap b_{x_i\restriction\xi})=\emptyset,$$

  and
  * $\{\langle a_{x_i\restriction\xi}\setminus e_j^i,b_{x_i\restriction\xi}\setminus e_j^i\rangle;\xi\in v_j(i)\}=\{\langle a_{x_i\restriction\xi}\setminus e_j^i,b_{x_i\restriction\xi}\setminus e_j^i\rangle;\xi\in\tau_j(i)\}$,

  and

– σ_j is compatible with

$$\langle\{\eta_j^i\},v_j(i');i\in I_\alpha\ \&\ i'\in n\setminus I_\alpha\rangle,$$

  in $\prod_{i\in n}\mathscr{X}_i(\langle a_{x_i\restriction\gamma},b_{x_i\restriction\gamma}\rangle_{\gamma\in\alpha})$.

(In the case that $i\in n\setminus I_\alpha$, we take η_j^i, e_j^i and $v_j(i)$ as follows: At first, we take $e_j^i>l_{k-1}$ such that for every $\xi\neq\xi'$ in $\tau_j(i)$,

$$((a_{x_i\restriction\xi}\cap b_{x_i\restriction\xi'})\cup(a_{x_i\restriction\xi'}\cap b_{x_i\restriction\xi}))\subseteq e_j^i.$$

Next, we take $\eta_j^i>\max(\tau_j(i))$ such that for every $\xi\in\tau_j(i)$,

$$a_{x_i\restriction\xi}\backslash e_j^i \subsetneq a_{x_i\restriction\eta_j^i}\backslash e_j^i \quad \text{and} \quad b_{x_i\restriction\xi}\backslash e_j^i \subsetneq b_{x_i\restriction\eta_j^i}\backslash e_j^i.$$

Then we take a family $\mathscr{H}$ of pairs of subsets of e_j^i such that

- $\mathscr{H}$ contains all $\langle a_{x_i\restriction\xi}\cap e_j^i, b_{x_i\restriction\xi}\cap e_j^i\rangle$ for $\xi\in\tau_j(i)$,
- for every $\langle H,K\rangle\neq\langle H',K'\rangle$ in $\mathscr{H}$, $H\cap K=\emptyset$ and

$$(H\cap K')\cup(H'\cap K)\neq\emptyset,$$

  and
- for every pair $\langle H,K\rangle$ of subsets of e_j^i with $H\cap K$ empty, there exists $\langle H',K'\rangle\in\mathscr{H}$ such that

$$(H\cap K')\cup(H'\cap K)=\emptyset.$$

Fixing any $\xi_0\in\tau_j(i)$, for each $\langle H,K\rangle\in\mathscr{H}\backslash\{\langle a_{x_i\restriction\xi}\cap e_j^i, b_{x_i\restriction\xi}\cap e_j^i\rangle;\xi\in\tau_j(i)\}$, let $\rho_{\langle H,K\rangle}\in\alpha$ such that

$$a_{x_i\restriction\rho_{\langle H,K\rangle}}\cap e_j^i=H,\quad b_{x_i\restriction\rho_{\langle H,K\rangle}}\cap e_j^i=K,$$

$$a_{x_i\restriction\rho_{\langle H,K\rangle}}\backslash e_j^i=a_{x_i\restriction\xi_0}\backslash e_j^i \quad \text{and} \quad b_{x_i\restriction\rho_{\langle H,K\rangle}}\backslash e_j^i=b_{x_i\restriction\xi_0}\backslash e_j^i.$$

We have to note that every $\rho_{\langle H,K\rangle}$ is diffrent from η_j^i. At last, let

$$v_j(i):=\tau_j(i)\cup\{\rho_{\langle H,K\rangle};\langle H,K\rangle\in\mathscr{H}\backslash\{\langle a_{x_i\restriction\xi}\cap e_j^i, b_{x_i\restriction\xi}\cap e_j^i\rangle;\xi\in\tau_j(i)\}\}.$$

We should note that η_j^i doesn't belong to $\sigma_j(i)$ in this case, because $v_j(i)$ and $\{\eta_j^i\}$ are incompatible in $\mathscr{F}(\langle a_{x_i\restriction\gamma}, b_{x_i\restriction\gamma}\rangle_{\gamma\in\alpha})$.)

We must notice in the construction $\langle\eta_j^i;i\in n\rangle$ that for each $j\in N_k$ and

- for each $i\in I_\alpha$,

$$\left(\bigcup_{\xi\in\sigma_j(i)}a_{x_i\restriction\xi}\cap K_j^i\right)\cup\left(H_j^i\cap\bigcup_{\xi\in\sigma_j(i)}b_{x_i\restriction\xi}\right)=\emptyset,$$

$$\left(\left(\bigcup_{\xi\in\sigma_j(i)}a_{x_i\restriction\xi}\cup a_{x_i\restriction\eta_j^i}\right)\Big\backslash l_{k-1}\right)\cap\left(\left(\bigcup_{\xi\in\sigma_j(i)}b_{x_i\restriction\xi}\cup b_{x_i\restriction\eta_j^i}\right)\Big\backslash l_{k-1}\right)=\emptyset,$$

  and
- for each $i\in n\backslash I_\alpha$, $\sigma_j(i)\cup\tau_j(i)\in\mathscr{F}(\langle a_{x_i\restriction\gamma}, b_{x_i\restriction\gamma}\rangle_{\gamma\in\alpha})$ and for every $\xi\in\sigma_j(i)$,

$$(a_{x_i\restriction\xi}\cap(b_{x_i\restriction\eta_j^i}\backslash e_j^i))\cup((a_{x_i\restriction\eta_j^i}\backslash e_j^i)\cap b_{x_i\restriction\xi})\neq\emptyset,$$

  therefore

$$(a_{x_i\restriction\xi}\cap(b_{x_i\restriction\eta_j^i}\backslash l_{k-1}))\cup((a_{x_i\restriction\eta_j^i}\backslash l_{k-1})\cap b_{x_i\restriction\xi})\neq\emptyset.$$

By the property 1 again, we take a large enough ordinal $\zeta_k^i\in\alpha$ such that

- for each $i \in I_\alpha$,

$$a_{x_i \restriction \zeta_k^i} \cap l_{k-1} = a_{x_i \restriction \zeta_{k-1}^i} \cap l_{k-1}, \quad \left(\bigcup_{\xi \in \sigma_{N_k-1}(i)} a_{x_i \restriction \xi} \cup a_{x_i \restriction \eta_{N_k-1}^i} \right) \Big\backslash l_{k-1} \subseteq a_{x_i \restriction \zeta_k^i},$$

$$b_{x_i \restriction \zeta_k^i} \cap l_{k-1} = b_{x_i \restriction \zeta_{k-1}^i} \cap l_{k-1}, \quad \text{and} \quad \left(\bigcup_{\xi \in \sigma_{N_k-1}(i)} b_{x_i \restriction \xi} \cup b_{x_i \restriction \eta_{N_k-1}^i} \right) \Big\backslash l_{k-1} \subseteq b_{x_i \restriction \zeta_k^i},$$

  and
- for each $i \in n \backslash I_\alpha$,

$$a_{x_i \restriction \zeta_k^i} \cap l_{k-1} = a_{x_i \restriction \zeta_{k-1}^i} \cap l_{k-1}, \quad a_{x_i \restriction \eta_{N_k-1}^i} \backslash l_{k-1} \subseteq a_{x_i \restriction \zeta_k^i},$$
$$b_{x_i \restriction \zeta_k^i} \cap l_{k-1} = b_{x_i \restriction \zeta_{k-1}^i} \cap l_{k-1}, \quad \text{and} \quad b_{x_i \restriction \eta_{N_k-1}^i} \backslash l_{k-1} \subseteq b_{x_i \restriction \zeta_k^i}.$$

Then we choose large enough $l_k > l_{k-1}$ such that

- for all $i \in n$,

$$|(\omega \backslash (a_{x_i \restriction \zeta_k^i} \cup b_{x_i \restriction \zeta_k^i})) \cap l_k| \geq k$$

  and
- for all $i \in n \backslash I_\alpha$, $j \in N_k$ and $\xi \in \sigma_j(i)$,

$$((a_{x_i \restriction \xi} \cap b_{x_i \restriction \zeta_k^i})) \cup (a_{x_i \restriction \zeta_k^i} \cap b_{x_i \restriction \xi})) \cap (l_k \backslash l_{k-1}) \neq \emptyset,$$

which completes the construction of ζ_k^i and l_k.

We define

$$a_{x_i} := \bigcup_{k \in \omega} a_{x_i \restriction \zeta_k^i} \quad \text{and} \quad b_{x_i} := \bigcup_{k \in \omega} b_{x_i \restriction \zeta_k^i}$$

and $S_\alpha := \{l_k; k \in \omega\}$, which complete the construction of f.

The rest of the proof is that the family

$$\{\langle a_{x \restriction \alpha}, b_{x \restriction \alpha}; \alpha \in \omega_1 \rangle; x \in 2^{\omega_1}\}$$

is independent. It suffices to prove that for every $n \in \omega$, $\langle x_i; i \in n \rangle \in (2^{\omega_1})^n$ and $\langle \mathscr{X}_i; i \in n \rangle$, the product forcing $\prod_{i \in n} \mathscr{X}_i(\langle a_{x_i \restriction \alpha}, b_{x_i \restriction \alpha} \rangle_{\alpha \in \omega_1})$ has the countable chain condition because of the following well known statement: If a finite support product $\prod_{\gamma \in \Gamma} \boldsymbol{P}_\gamma$ of forcing notions has an uncountable antichain, then there is $\Gamma' \in [\Gamma]^{<\aleph_0}$ such that the product $\prod_{\gamma \in \Gamma'} \boldsymbol{P}_\gamma$ also has an uncountable antichain. (This can be shown by considering a Δ-system refinement of the set of supports of conditions in the antichain.)

Assume that a subset Y of $\prod_{i\in n}\mathscr{X}_i(\langle a_{x_i\restriction\alpha}, b_{x_i\restriction\alpha}\rangle_{\alpha\in\omega_1})$ is a maximal antichain. Let $\mathsf{E}^0 := J_n[Y]$ and a subset E^1 of ω_1 such that $\langle \mathsf{E}^1, n\rangle$ codes a pair of sequences $\langle x_i; i\in n\rangle$ and $\langle \mathscr{X}_i; i\in n\rangle$. By $\diamondsuit$, we can find a limit ordinal α such that

- $J_n[([\alpha]^{<\aleph_0})^n] = J_n[([\omega_1]^{<\aleph_0})^n]\cap\alpha$,
- $\mathsf{E}^0\cap\alpha = \mathsf{D}^0_\alpha$ and $\mathsf{E}^1\cap\alpha = \mathsf{D}^1_\alpha$, and
- $Y\cap([\alpha]^{<\aleph_0})^n$ is a maximal antichain in $\prod_{i\in n}\mathscr{X}_i(\langle a_{x_i\restriction\gamma}, b_{x_i\restriction\gamma}\rangle_{\gamma\in\alpha})$.

We will conclude that $Y\cap([\alpha]^{<\aleph_0})^n$ is a maximal antichain in $\prod_{i\in n}\mathscr{X}_i(\langle a_{x_i\restriction\alpha}, b_{x_i\restriction\alpha}\rangle_{\alpha\in\omega_1})$ as below.

Let $I := \{i\in n; \mathscr{X}_i = \mathscr{S}\}$. Take any condition τ in

$$\prod_{i\in n}\mathscr{X}_i(\langle a_{x_i\restriction\alpha}, b_{x_i\restriction\alpha}\rangle_{\alpha\in\omega_1})\setminus([\alpha]^{<\aleph_0})^n$$

and let $\langle \eta_i; i\in I\rangle\in\omega_1^I$ be such that for each $i\in I$,

$$a_{x_i\restriction\eta_i} = \bigcup_{\xi\in\tau(i)} a_{x_i\restriction\xi} \quad\text{and}\quad b_{x_i\restriction\eta_i} = \bigcup_{\xi\in\tau(i)} b_{x_i\restriction\xi}.$$

In considering the property 2 for a condition $\langle \tau(i)\cap\alpha; i\in n\rangle$, we take a large enough $l\in S_\alpha$ such that for all $i\in n$

- if $i\in I$ and η_i is smaller than α, then
$$a_{x_i\restriction\eta_i}\setminus l\subseteq a_{x_i\restriction\alpha}\setminus l \quad\text{and}\quad b_{x_i\restriction\eta_i}\setminus l\subseteq b_{x_i\restriction\alpha}\setminus l,$$
- if $i\in I$ and η_i is not smaller than α,
$$a_{x_i\restriction\alpha}\setminus l\subseteq a_{x_i\restriction\eta_i}\setminus l \quad\text{and}\quad b_{x_i\restriction\alpha}\setminus l\subseteq b_{x_i\restriction\eta_i}\setminus l,$$
and
- if $i\in n\setminus I$, then for all $\xi\in\tau(i)\setminus\alpha$,
$$a_{x_i\restriction\alpha}\setminus l\subseteq a_{x_i\restriction\xi}\setminus l \quad\text{and}\quad b_{x_i\restriction\alpha}\setminus l\subseteq b_{x_i\restriction\xi}\setminus l.$$

For each $i\in n$, let

$$H^i := a_{x_i\restriction\eta_i}\cap l \quad\text{and}\quad K^i := b_{x_i\restriction\eta_i}\cap l.$$

Then applying the property 2 of the construction above to these α, n, l, $\langle H^i, K^i; i\in n\rangle$ and $\langle \tau(i)\cap\alpha; i\in n\rangle$, we get $\sigma\in J_n^{-1}[\mathsf{D}^0_\alpha]\subseteq Y\cap([\alpha]^{<\aleph_0})^n$ such that

- if $i\in I$, then
$$\left(\bigcup_{\xi\in\sigma(i)} a_{x_i\restriction\xi}\cap K^i\right)\cup\left(H^i\cap\bigcup_{\xi\in\sigma(i)} b_{x_i\restriction\xi}\right) = \varnothing,$$

$$\bigcup_{\xi \in \sigma(i)} a_{x_i \restriction \xi} \setminus l \subseteq a_{x_i \restriction \alpha} \setminus l \quad \text{and} \quad \bigcup_{\xi \in \sigma(i)} b_{x_i \restriction \xi} \setminus l \subseteq b_{x_i \restriction \alpha} \setminus l,$$

and

- if $i \in n \setminus I$, then $\sigma(i) \cup (\tau(i) \cap \alpha)$ is a condition in $\mathscr{F}(\langle a_{x_i \restriction \gamma}, b_{x_i \restriction \gamma} \rangle_{\gamma \in \alpha})$ and for any $\xi \in \sigma(i)$

$$(a_{x_i \restriction \xi} \cap (b_{x_i \restriction \alpha} \setminus l)) \cup ((a_{x_i \restriction \alpha} \setminus l) \cap b_{x_i \restriction \xi}) \neq \emptyset.$$

Then

- if $i \in I$, then

$$\left(\bigcup_{\xi \in \sigma(i)} a_{x_i \restriction \xi} \cap b_{x_i \restriction \eta_i} \right) \cup \left(a_{x_i \restriction \eta_i} \cap \bigcup_{\xi \in \sigma(i)} b_{x_i \restriction \xi} \right) = \emptyset,$$

- if $i \in n \setminus I$, then for every $\xi \in \sigma(i)$ and $\xi' \in \tau(i) \setminus \alpha$,

$$(a_{x_i \restriction \xi} \cap b_{x_i \restriction \xi'}) \cup (a_{x_i \restriction \xi'} \cap b_{x_i \restriction \xi}) \neq \emptyset,$$

therefore τ and σ are compatible in $\prod_{i \in n} \mathscr{X}_i(\langle a_{x_i \restriction \alpha}, b_{x_i \restriction \alpha} \rangle_{\alpha \in \omega_1})$. □

To show the following theorem, we prove that for any independent family Γ of $\aleph_1$ many destructible gaps, using $\diamondsuit$, we find a gap which is independent from Γ. The following proof is also similar to the proof in [14].

THEOREM 2.2. *Under $\diamondsuit$, every maximal independent family of destructible gaps has size at least $\aleph_2$.*

PROOF. This proof is similar to a proof of Theorem 2.1. Let $\langle \mathsf{D}_\alpha ; \alpha \in \omega_1 \rangle$ be a $\diamondsuit$-sequence on ω_1 and $\langle J_n ; n \in \omega \rangle$ be as in this proof of Theorem 2.1. (But in the proof, we assume that each J_n is an injection from $([\omega_1]^{<\aleph_0} \times \omega_1 \times 2)^n$ into ω_1.) And assume that $\Gamma := \{(\mathscr{A}_\xi, \mathscr{B}_\xi) ; \xi \in \omega_1 \setminus \{0\}\}$ is an independent family of destructible gaps. We denote that $(\mathscr{A}_\xi, \mathscr{B}_\xi) = \langle a^\xi_\zeta, b^\xi_\zeta ; \zeta \in \omega_1 \rangle$. By recursion, we construct a function f from $\omega_1 \times \omega$ into 3 with the following property:

1. For any $\alpha \in \omega_1$, $f \restriction (\alpha \times \omega)$ codes a pregap as in the proof of Theorem 2.1.
2. For any $\alpha \in \omega_1$, the pregap decoded by $f \restriction (\alpha \times \omega)$ admits finite changes.
3. If α is a limit ordinal and there exists $n \in \omega$ such that
   - $\mathsf{D}_\alpha \subseteq J_n[([\omega_1]^{<\aleph_0} \times \omega_1 \times 2)^n]$ and $J_n^{-1}[\mathsf{D}_\alpha] \subseteq ([\alpha]^{<\aleph_0} \times \alpha \times 2)^n$,
   - for all $\xi, \eta \in \mathsf{D}_\alpha$ and $i \in n$, letting

$$J_n^{-1}(\xi) =: \langle \sigma_\xi^\alpha(i), \gamma_\xi^\alpha(i), h_\xi^\alpha(i); i \in n \rangle,$$

$h_\xi^\alpha(i) = h_\eta^\alpha(i)$, $\gamma_\xi^\alpha(i) = \gamma_\eta^\alpha(i)$ and $\gamma_\xi^\alpha(0) = \gamma_\eta^\alpha(0) = 0$, and

- letting $I_\alpha := \{\gamma_\xi^\alpha(i); 1 \le i \in n\}$ for some (any) $\xi \in \mathsf{D}_\alpha$ and $\mathscr{X}_i = \mathscr{F}$ if $h_\xi^\alpha(i) = 0$ and $\mathscr{X}_i = \mathscr{S}$ if $h_\xi^\alpha(i) = 1$, the family $\{\langle \sigma_\xi^\alpha(i); i \in n \rangle; \xi \in \mathsf{D}_\alpha\}$ is a maximal antichain in $\prod_{i \in n} \mathscr{X}_i(\mathscr{A}_{\gamma_\xi^\alpha(i)} \restriction \alpha, \mathscr{B}_{\gamma_\xi^\alpha(i)} \restriction \alpha)$, where $(\mathscr{A}_0 \restriction \alpha, \mathscr{B}_0 \restriction \alpha) = \langle a_\zeta^0, b_\zeta^0; \zeta \in \alpha \rangle$ is the pregap decoded by $f \restriction (\alpha \times \omega)$,

then for this unique $n \in \omega$, there is an infinite subset S_α of natural numbers so that for every $l \in S_\alpha$ and finite sequence $\langle H_i, K_i; i \in n \rangle$ with the property that H_i and K_i are disjoint subsets of l for all $i \in n$, there exists $\sigma \in J_n^{-1}[\mathsf{D}_\alpha^0]$ such that for all $i \in n$,

- if $\mathscr{X}_i$ is $\mathscr{S}$, then for some (any) $\xi \in \mathsf{D}_\alpha$,

$$\left(\bigcup_{\zeta \in \sigma(i)} a_\zeta^{\gamma_\xi^\alpha(i)} \cap K_i \right) \cap \left(\bigcup_{\zeta \in \sigma(i)} b_\zeta^{\gamma_\xi^\alpha(i)} \cap H_i \right) = \varnothing,$$

$$\bigcup_{\zeta \in \sigma(i)} a_\zeta^{\gamma_\xi^\alpha(i)} \backslash l \subseteq a_\alpha^0 \backslash l \quad \text{and} \quad \bigcup_{\zeta \in \sigma(i)} b_\zeta^{\gamma_\xi^\alpha(i)} \backslash l \subseteq b_\alpha^0 \backslash l,$$

and

- if $\mathscr{X}_i$ is $\mathscr{F}$, then for some (any) $\xi \in \mathsf{D}_\alpha$ and for any $\zeta \in \sigma(i)$,

$$(a_\zeta^{\gamma_\xi^\alpha(i)} \cap (b_\alpha^0 \backslash l)) \cup ((a_\alpha^0 \backslash l) \cap b_\zeta^{\gamma_\xi^\alpha(i)}) \neq \varnothing.$$

By a similar argument in the proof of Theorem 2.1, we can see that f decoded a destructible gap which is independent from Γ. □

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