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IDENTIFICATION OF THE ABSENT SPECTRAL GAPS IN A CLASS OF GENERALIZED KRONIG-PENNEY HAMILTONIANS

By

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Abstract. We study the spectral gaps of the generalized Kronig-Penney Hamiltonians which possesses two point interactions in the basic period cell. We suppose that each interaction is given by a rotation. We determine whether or not the jth spectral gap of the Hamiltonian is absent for a given $j \in \boldsymbol{N}$.

1. Introduction

In this paper we study the spectral gaps of the one-dimensional Schrödinger operators with particular periodic point interactions. We fix

$$\kappa \in (0, 2\pi).$$

Let

$$\Gamma_1 = 2\pi\boldsymbol{Z}, \quad \Gamma_2 = \{\kappa\} + 2\pi\boldsymbol{Z}, \quad \Gamma = \Gamma_1 \cup \Gamma_2, \quad \tau = 2\pi - \kappa.$$

For $\theta_1, \theta_2 \in [-\pi/2, \pi/2)$ and

$$A_1, A_2 \in SO(2)\backslash\{\pm I\}, \tag{1}$$

we define the operator $H = H(A_1, A_2, \theta_1, \theta_2)$ in $L^2(\boldsymbol{R})$ as follows.

$$(Hy)(x) = -\frac{d^2}{dx^2}y(x), \quad x \in \boldsymbol{R}\backslash\Gamma,$$

$$\mathrm{Dom}(H) = \left\{ y \in H^2(\boldsymbol{R}\backslash\Gamma) \,\middle|\, \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix}, x \in \Gamma_j, j = 1, 2 \right\}.$$

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In Proposition 2.1 we prove that H is self-adjoint. Since the interaction is 2π-periodic, the operator admits a direct integral decomposition (see [9, Section XIII.16]). For $\mu \in \boldsymbol{R}$, we define the Hilbert space

$$\mathscr{H}_\mu = \{u \in L^2_{loc}(\boldsymbol{R}) \mid u(x+2\pi) = e^{i\mu} u(x) \text{ a.e. } x \in \boldsymbol{R}\}$$

equipped with the inner product

$$(u, v)_{\mathscr{H}_\mu} = \int_0^{2\pi} u(x)\overline{v(x)}\, dx, \quad u, v \in \mathscr{H}_\mu.$$

We introduce the fiber operator $H_\mu = H_\mu(A_1, A_2, \theta_1, \theta_2)$ in $\mathscr{H}_\mu$ given by

$$(H_\mu y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \boldsymbol{R} \backslash \Gamma,$$

$$\mathrm{Dom}(H_\mu) = \Big\{ y \in \mathscr{H}_\mu \mid y \in H^2((0, 2\pi)\backslash\{\kappa\}),$$
$$\begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \text{ for } x \in \Gamma_j, j = 1, 2 \Big\}.$$

Furthermore, we define the unitary operator

$$\mathscr{U} : \mathrm{L}^2(\boldsymbol{R}) \to \int_0^{2\pi} \oplus \mathscr{H}_\mu \, d\mu$$

as

$$(\mathscr{U}u)(x, \mu) = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} e^{il\mu} u(x - 2l\pi).$$

Then we have

$$\mathscr{U} H \mathscr{U}^{-1} = \int_0^{2\pi} \oplus H_\mu \, d\mu.$$

For $j \in \boldsymbol{N} = \{1, 2, 3, \ldots\}$, let $\lambda_j(\mu)$ be the j-th eigenvalue of H_μ counted with multiplicity. Since $A_j \in SO(2)\backslash\{\pm I\}$, we can write the elements of A_j as

$$A_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix} = \begin{pmatrix} \cos \alpha_j & -\sin \alpha_j \\ \sin \alpha_j & \cos \alpha_j \end{pmatrix}, \quad \alpha_j \in (-\pi, 0) \cup (0, \pi). \tag{2}$$

The basic spectral properties of H are described as follows.

Proposition 1.1.

(a) *The function* $\lambda_j(\cdot)$ *is continuous on* $[0, 2\pi]$.

(b) *It holds that* $\lambda_j(\mu) = \lambda_j(-\mu + 2\theta_1 + 2\theta_2)$.

(c) *If* $\mu - (\theta_1 + \theta_2) \notin \pi\boldsymbol{Z}$, *then every eigenvalue of* H_μ *is simple.*

(d) *The spectrum of* $H(A_1, A_2, \theta_1, \theta_2)$ *is given by*

$$\begin{aligned}\sigma(H(A_1, A_2, \theta_1, \theta_2)) &= \bigcup_{\mu \in [\theta_1+\theta_2, \theta_1+\theta_2+\pi]} \sigma(H_\mu(A_1, A_2, \theta_1, \theta_2)) \\ &= \bigcup_{j=1}^{\infty} \lambda_j([\theta_1 + \theta_2, \theta_1 + \theta_2 + \pi]) \\ &= \bigcup_{j=1}^{\infty} \bigcup_{\mu \in [\theta_1+\theta_2, \theta_1+\theta_2+\pi]} \{\lambda_j(\mu)\}.\end{aligned}$$

(e) *The set* $\sigma(H(A_1, A_2, \theta_1, \theta_2))$ *is independent of* θ_1 *and* θ_2.

(f) *If* $b_1 b_2 > 0$ *and* $\theta_1 = \theta_2 = 0$, *then the function* $\lambda_j(\cdot)$ *is strictly monotone increasing* (*respectively*, *decreasing*) *function on* $[0, \pi]$ *for odd* (*respectively*, *even*) j.

(g) *If* $b_1 b_2 < 0$ *and* $\theta_1 = \theta_2 = 0$, *then the function* $\lambda_j(\cdot)$ *is strictly monotone increasing* (*respectively*, *decreasing*) *function on* $[0, \pi]$ *for even* (*respectively*, *odd*) j.

Since $\sigma(H)$ is independent of θ_1 and θ_2, we hereafter discuss only the case where

$$\theta_1 = \theta_2 = 0,$$

which does not cause any loss of generality. We define

$$G_j = \begin{cases} (\lambda_j(\pi), \lambda_{j+1}(\pi)) & \text{for } j \text{ odd}, \\ (\lambda_j(0), \lambda_{j+1}(0)) & \text{for } j \text{ even} \end{cases}$$

in the case where $b_1 b_2 > 0$, while we put

$$G_j = \begin{cases} (\lambda_j(\pi), \lambda_{j+1}(\pi)) & \text{for } j \text{ even}, \\ (\lambda_j(0), \lambda_{j+1}(0)) & \text{for } j \text{ odd} \end{cases}$$

if $b_1 b_2 < 0$. Moreover we set $B_j = \lambda_j([0, \pi])$. The open interval G_j is called the j-th gap of the spectrum of H, the closed interval B_j the j-th band. The aim of this papar is to determine whether or not the j-th gap is absent for a given $j \in \boldsymbol{N}$. Throughout this paper we use the notations

$$a \equiv b \quad \text{if } a - b \in \pi\boldsymbol{Z}, \qquad a \not\equiv b \quad \text{if } a - b \notin \pi\boldsymbol{Z}$$

for $a, b \in \boldsymbol{R}$. For convenience we adopt the following classification of the parameters α_1 and α_2.

(I) $\alpha_1 - \alpha_2 \not\equiv 0,\ \alpha_1 + \alpha_2 \not\equiv 0$.
(II) $\alpha_1 - \alpha_2 \not\equiv 0,\ \alpha_1 + \alpha_2 \equiv 0$.
(III) $\alpha_1 - \alpha_2 \equiv 0,\ \alpha_1 + \alpha_2 \equiv 0$, i.e.,

$$(\alpha_1, \alpha_2) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right),\quad \left(\frac{\pi}{2}, -\frac{\pi}{2}\right),\quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\quad \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right),$$

(IV) $\alpha_1 - \alpha_2 \equiv 0,\ \alpha_1 + \alpha_2 \not\equiv 0$.

Our main results are the following three theorems.

Theorem 1.2. *Let* $\kappa \neq \pi$. *If the condition* (I) *holds, then*

$$G_j \neq \emptyset \quad \text{for } j \in \boldsymbol{N}.$$

Theorem 1.3. *Let* $\kappa \neq \pi$. *Suppose that either* (II) *or* (III) *is valid.*
(1) *Let* $\kappa/\pi \notin \boldsymbol{Q}$. *Then we have*

$$G_j = \emptyset \quad \text{if and only if} \quad j = 3.$$

(2) *If* $\kappa/2\pi = q/p$, $(p, q) \in \boldsymbol{N}^2$ *and* $\gcd(p, q) = 1$, *then*

$$\{j \in \boldsymbol{N} \mid G_j = \emptyset\} = \{3\} \cup \{pk + 1 \mid k \in \boldsymbol{N}\}.$$

Though it is hard to identify the indices of the absent spectral gaps in the case (IV), we can still determine the positions of them in the next theorem.

Theorem 1.4. *Let* $\kappa \neq \pi$. *Assume that* (IV) *is valid. We put* $\eta_j = \pi^2 j^2 / 4(\pi - \kappa)^2$ *for* $j \in \boldsymbol{N}$. *Then it holds that*

$$\bigcup_{k=1}^{\infty} B_k \cap B_{k+1} = \left\{ \eta_j \,\middle|\, -2\left(\sqrt{\eta_j} + \frac{1}{\sqrt{\eta_j}}\right)^{-1} \cot \kappa\sqrt{\eta_j} = \tan \alpha_1 \ \text{and} \ j \in \boldsymbol{N} \right\}. \tag{3}$$

In order to describe the motivation and background of our study, we give a review on the related works [1, 4, 5, 8, 10, 12]. The concept of the Schrödinger operators with periodic point interactions was first inspired by Kronig and Penny in 1931. They introduced and discussed in [8] the Hamiltonian which is formally expressed as

$$L_1 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta(x - 2\pi l) \quad \text{in} \ L^2(\boldsymbol{R}),$$

where $\delta(x)$ is the Dirac δ-function at the origin and $\beta \in \boldsymbol{R}\backslash\{0\}$. This operator is nowadays called the Kronig-Penney Hamiltonian and is frequently referred as the most fundamental model in solid states physics. The precise definition of this operator is given through boundary conditions on the lattice $2\pi\boldsymbol{Z}$ as follows.

$$(L_1 y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \boldsymbol{R}\backslash 2\pi\boldsymbol{Z},$$

$$\mathrm{Dom}(L_1) = \left\{ y \in H^2(\boldsymbol{R}\backslash 2\pi\boldsymbol{Z}) \middle| \begin{pmatrix} y(t+0) \\ y'(t+0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} y(t-0) \\ y'(t-0) \end{pmatrix}, \ t \in 2\pi\boldsymbol{Z} \right\}.$$

They illustrated the graph of the band function of this operator. With the advance of the theory of point interactions, this Hamiltonian was widely generalized. Gesztesy, Holden, and Kirsch inspired a new class of point interactions. They studied in [4] and [5] the operator in $L^2(\boldsymbol{R})$ of the form

$$(L_2 y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \boldsymbol{R}\backslash 2\pi\boldsymbol{Z},$$

$$\mathrm{Dom}(L_2) = \left\{ y \in H^2(\boldsymbol{R}\backslash 2\pi\boldsymbol{Z}) \middle| \begin{pmatrix} y(t+0) \\ y'(t+0) \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(t-0) \\ y'(t-0) \end{pmatrix}, \ t \in 2\pi\boldsymbol{Z} \right\},$$

where $\beta \in \boldsymbol{R}\backslash\{0\}$. This operator has the formal expression

$$L_2 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta'(x - 2\pi l) \quad \text{in} \ L^2(\boldsymbol{R}).$$

They showed that the j-th gap of $\sigma(L_2)$ is absent if and only if $j = 1$ and $\beta = -2\pi$. They also proved that every gap of $\sigma(L_1)$ is present. The δ'-interaction was generalized by Šeba [10] (see also [1]); he proved that the domain of any self-adjoint extension of $(-d^2/dx^2)|_{C_0^\infty(\boldsymbol{R}\backslash\{0\})}$ in $L^2(\boldsymbol{R})$ of coupled type is expressed as

$$\left\{ y \in H^2(\boldsymbol{R}\backslash\{0\}) \middle| \begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = cA \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix} \right\}$$

with $A \in SL_2(\boldsymbol{R})$, $c \in \boldsymbol{C}$, and $|c| = 1$, and vice versa. Šeba [10] and Chernoff and Hughes [1] discussed particular classes of self-adjoint extensions of $(-d^2/dx^2)|_{C_0^\infty(\boldsymbol{R}\backslash\{0\})}$ in $L^2(\boldsymbol{R})$ which can be approximated by the Schrödinger operators with local short-range potentials in the strong resolvent sense. In

[7] Hughes studied the Schrödinger operator in $L^2(\boldsymbol{R})$ with generalized point interactions of the form

$$(L_3 y)(x) = -\frac{d^2}{dx^2} y(x), \quad x \in \boldsymbol{R} \backslash 2\pi \boldsymbol{Z},$$

$$\mathrm{Dom}(L_3) = \left\{ y \in H^2(\boldsymbol{R} \backslash 2\pi \boldsymbol{Z}) \middle| \begin{pmatrix} y(t+0) \\ y'(t+0) \end{pmatrix} = cA \begin{pmatrix} y(t-0) \\ y'(t-0) \end{pmatrix} \text{ for } t \in 2\pi \boldsymbol{Z} \right\}.$$

She gave the Floquet-Bloch decomposition of this operator. We note that the operator $H(A_1, A_2, \theta_1, \theta_2)$ is unitarily equivalent to L_3 if $(A_1, A_2) \in \{(A, I), (A, -I), (I, A), (-I, A)\}$. The operators discussed in [4, 5, 8] involve only one point interaction in the basic period cell $[0, 2\pi)$. In [12] Yoshitomi investigated the operators

$$P_0 = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} (\beta_1 \delta(x - \kappa - 2\pi l) + \beta_2 \delta(x - 2\pi l)) \quad \text{in } L^2(\boldsymbol{R}),$$

$$P_1 = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} (\beta_1 \delta'(x - \kappa - 2\pi l) + \beta_2 \delta'(x - 2\pi l)) \quad \text{in } L^2(\boldsymbol{R})$$

which admit two point interactions in the basic period cell. He proved for $j \in \boldsymbol{N}$ and $k \in \{0, 1\}$ that $\sigma(P_k)$ has an absent gap if and only if both $\beta_1 + \beta_2 = 0$ and $\kappa/\pi \in \boldsymbol{Q}$ hold. He also showed that if $\beta_1 + \beta_2 = 0$, $\kappa/2\pi = m/n$, $(m, n) \in \boldsymbol{N}^2$, and $\gcd(n, m) = 1$, then the j-th gap of $\sigma(P_k)$ is absent if and only if $j - k \in n\boldsymbol{N}$. The results in [1, 4, 5, 8, 10, 12] draw our interest in the spectral gaps of the general operator

$$(H_0 y)(x) := -\frac{d^2}{dx^2} y(x), \quad x \in \boldsymbol{R} \backslash \Gamma,$$

$$\mathrm{Dom}(H_0) = \left\{ y \in H^2(\boldsymbol{R} \backslash \Gamma) \middle| \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} C_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \text{ for } x \in \Gamma_j, j = 1, 2 \right\}$$

with $C_1, C_2 \in SL_2(\boldsymbol{R})$. However, the spectral gaps of this operator are too hard to analyze, because it involves ten real parameters. It seems for the author that the absence of gaps in this general setting cannot be determined without imposing any structural assumptions on C_1 and C_2. We notice that the sets

$$\left\{ \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \middle| \beta \in \boldsymbol{R} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \middle| \beta \in \boldsymbol{R} \right\} \tag{4}$$

are commutative subgroups of $SL_2(\boldsymbol{R})$. This attracts our attention to the case (1).

We organize this paper as follows. Section 2 is devoted to the proof of basic spectral properties, the self-adjointness of H and Proposition 1.1. In section 3 we locate the absent gaps of $\sigma(H)$, namely, we evaluate the set $\bigcup_{k=1}^{\infty} B_k \cap B_{k+1}$ in an explicit way. To this end we follow the basic argument in [12], that is, we reduce the problem to a system of algebraic equations by using the monodromy matrix in the proof of Lemma 3.1. We execute, in Lemmas 3.2–3.6, rather hard tasks than in [12], because we need the classfication (I)–(IV), while no such classification is used in [12]. We complete the proof of Theorems 1.2 and 1.4 in this section. In section 4 we identify the indices of absent spectral gaps, that is, we show Theorem 1.3. For this purpose we establish a new characterization of the band edges in terms of the rotation number (see Theorem 4.3). We stress that our characterization is completely different from that for regular potentials; in the latter the rotation number is identically equal to $n/2$ on the n-th gap for each $n \in \boldsymbol{N}$ (cf. [3, Proposition 2.1]).

2. Basic Spectral Properties of H

Our first aim in this section is to prove the self-adjointness of H.

Proposition 2.1. *The operator* $H = H(A_1, A_2, \theta_1, \theta_2)$ *is self-adjoint.*

Proof. Using integration by parts, we readily obtain

$$\mathrm{Dom}(H) \subset \mathrm{Dom}(H^*).$$

Let us prove the reverse inclusion. For all $j \in \boldsymbol{Z}$, we define

$$I_j^1 = (2\pi j, 2\pi j + \kappa) \quad \text{and} \quad I_j^2 = (2\pi j + \kappa, 2\pi(j+1)).$$

We introduce the Sobolev space

$$H^2(I_j^k) = \{ y \in L^2(I_j^k) \mid y', y'' \in L^2(I_j^k) \}$$

equipped with the norm

$$\|y\|_{H^2(I_j^k)} = (\|y\|_{L^2(I_j^k)}^2 + \|y'\|_{L^2(I_j^k)}^2 + \|y''\|_{L^2(I_j^k)}^2)^{1/2}.$$

We pick $u \in \mathrm{Dom}(H^*)$. By the definition of the adjoint operator, there exists $v \in L^2(\boldsymbol{R})$ such that

$$(H\varphi, u)_{L^2(\boldsymbol{R})} = (\varphi, v)_{L^2(\boldsymbol{R})} \tag{5}$$

for all $\varphi \in \mathrm{Dom}(H)$. In particular,

$$(-\varphi'', u)_{L^2(I_j^k)} = (\varphi, v)_{L^2(I_j^k)}$$

for any $\varphi \in C_0^\infty(I_j^k)$ and $k = 1, 2$. Thus we get $-u'' = v \in L^2(I_j^k)$ and hence

$$\|u\|_{H^2(I_j^k)}^2 \le C(\|u\|_{L^2(I_j^k)}^2 + \|v\|_{L^2(I_j^k)}^2),$$

where C is a constant independent of j and k (see [11, Theorem 6.26]). Taking the summation with respect to j and k, we infer that

$$\|u\|_{H^2(\boldsymbol{R}\setminus\Gamma)}^2 \le C(\|u\|_{L^2(\boldsymbol{R})}^2 + \|v\|_{L^2(\boldsymbol{R})}^2) < \infty.$$

Next, we show that

$$\begin{pmatrix} u(x+0) \\ u'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} u(x-0) \\ u'(x-0) \end{pmatrix} \quad \text{for } x \in \Gamma_j,\ j = 1, 2. \tag{6}$$

Integrating (5) by parts and using $-u'' = v$ on $\boldsymbol{R}\setminus\Gamma$, we derive

$$\begin{aligned} &-\sum_{j \in \boldsymbol{Z}} ([\varphi'(x)\overline{u(x)}]_{2\pi j+0}^{2\pi j+\kappa-0} + [\varphi'(x)\overline{u(x)}]_{2\pi j+\kappa+0}^{2\pi(j+1)-0}) \\ &\qquad + \sum_{j \in \boldsymbol{Z}} ([\varphi(x)\overline{u'(x)}]_{2\pi j+0}^{2\pi j+\kappa-0} + [\varphi(x)\overline{u'(x)}]_{2\pi j+\kappa+0}^{2\pi(j+1)-0}) = 0 \end{aligned} \tag{7}$$

for all $\varphi \in \mathrm{Dom}(H)$. Now we prepare particular test functions in $\mathrm{Dom}(H)$. We fix $j \in \boldsymbol{Z}$ and take $f, g \in C_0^\infty(2\pi j, 2\pi(j+1))$ which satisfy

$$\begin{aligned} &f(2\pi j + \kappa) = 1, \quad f'(2\pi j + \kappa) = 0, \\ &g(2\pi j + \kappa) = 0, \quad g'(2\pi j + \kappa) = 1. \end{aligned} \tag{8}$$

For $a, b \in \boldsymbol{R}$, we define

$$\begin{pmatrix} p \\ q \end{pmatrix} = e^{i\theta_2} A_2 \begin{pmatrix} a \\ b \end{pmatrix}. \tag{9}$$

and

$$\varphi_j(x) = \begin{cases} af(x) + bg(x) & \text{if } x \in I_j^1, \\ pf(x) + qg(x) & \text{if } x \in I_j^2, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

Then $\varphi_j(x) \in \mathrm{Dom}(H)$. Substituting φ_j for φ in (7), we obtain

$$-b\overline{u(2\pi j+\kappa-0)}+q\overline{u(2\pi j+\kappa+0)}$$
$$+a\overline{u'(2\pi j+\kappa-0)}-p\overline{u'(2\pi j+\kappa+0)}=0. \tag{11}$$

If $a=1$ and $b=0$, then $p=e^{i\theta_2}\cos\alpha_2$ and $q=e^{i\theta_2}\sin\alpha_2$ hold. In this case, (11) implies

$$e^{-i\theta_2}u(2\pi j+\kappa+0)\sin\alpha_2-e^{-i\theta_2}u'(2\pi j+\kappa+0)\cos\alpha_2+u(2\pi j+\kappa-0)=0. \tag{12}$$

If $a=0$ and $b=1$, then $p=-e^{i\theta_2}\sin\alpha_2$ and $q=e^{i\theta_2}\cos\alpha_2$ are valid. In this case we obtain

$$-u(2\pi j+\kappa-0)+e^{-i\theta_2}u(2\pi j+\kappa+0)\cos\alpha_2$$
$$+e^{-i\theta_2}u'(2\pi j+\kappa+0)\sin\alpha_2=0. \tag{13}$$

by (11). Summarizing (12) and (13), we conclude that

$$\begin{pmatrix} u(2\pi j+\kappa+0) \\ u'(2\pi j+\kappa+0) \end{pmatrix}=e^{i\theta_2}\begin{pmatrix} \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_2 & \cos\alpha_2 \end{pmatrix}\begin{pmatrix} u(2\pi j+\kappa-0) \\ u'(2\pi j+\kappa-0) \end{pmatrix} \tag{14}$$

for all $j\in\boldsymbol{Z}$. In a similar way, we can gain

$$\begin{pmatrix} u(2\pi j+0) \\ u'(2\pi j+0) \end{pmatrix}=e^{i\theta_1}\begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{pmatrix}\begin{pmatrix} u(2\pi j-0) \\ u'(2\pi j-0) \end{pmatrix} \tag{15}$$

for all $j\in\boldsymbol{Z}$. Therefore we arrive at (6), and thus $u\in\mathrm{Dom}(H)$. □

Remark 2.2. *The operator* $H=H(A_1,A_2,\theta_1,\theta_2)$ *is also self-adjoint for* $A_1,A_2\in SL_2(\boldsymbol{R})$, *where*

$$SL_2(\boldsymbol{R})=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\middle| a,b,c,d\in\boldsymbol{R}, ad-bc=1\right\}.$$

Next we show Proposition 1.1. We consider the equation

$$\begin{cases} -y''(x,\lambda)=\lambda y(x,\lambda), & x\in\boldsymbol{R}\backslash\Gamma, \\ \begin{pmatrix} y(x+0,\lambda) \\ y'(x+0,\lambda) \end{pmatrix}=e^{i\theta_j}A_j\begin{pmatrix} y(x-0,\lambda) \\ y'(x-0,\lambda) \end{pmatrix} & \text{for } x\in\Gamma_j,\ j=1,2, \end{cases} \tag{16}$$

where λ is a real parameter and the symbol $'$ stands for the differentiation with respect to x. This differential equation has two solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ which are uniquely determined by the initial conditions:

$$y_1(+0,\lambda) = 1, \quad y_1'(+0,\lambda) = 0,$$

and

$$y_2(+0,\lambda) = 0, \quad y_2'(+0,\lambda) = 1$$

respectively. We introduce the discriminant $D(\lambda)$ of the equation (16):

$$D(\lambda) = y_1(2\pi+0,\lambda) + y_2'(2\pi+0,\lambda). \tag{17}$$

The matrix

$$M(\lambda) := \begin{pmatrix} y_1(2\pi+0,\lambda) & y_2(2\pi+0,\lambda) \\ y_1'(2\pi+0,\lambda) & y_2'(2\pi+0,\lambda) \end{pmatrix} \tag{18}$$

is called the monodromy matrix of (16). Since $\det M(\lambda) = e^{2i(\theta_1+\theta_2)}$, the characteristic equation of $M(\lambda)$ is

$$t^2 - D(\lambda)t + e^{2i(\theta_1+\theta_2)} = 0. \tag{19}$$

So, λ is an eigenvalue of H_μ if and only if $e^{i\mu}$ is a root of (19). Thus, the sequence $\{\lambda_j(\mu)\}_{j=1}^{\infty}$ provides all the zeros of $D(\lambda) - (e^{i\mu} + e^{i(2\theta_1+2\theta_2-\mu)})$. We recall (2). The components of the monodromy matrix are directly calculated as follows.

$$\begin{aligned} y_1(2\pi+0,\lambda) = e^{i(\theta_1+\theta_2)}\Big[&(a_1a_2 - b_1b_2)\cos\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ (a_1b_2 + b_1a_2)\sqrt{\lambda}\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ \left(\frac{a_1b_2}{\sqrt{\lambda}} + b_1a_2\sqrt{\lambda}\right)\cos\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda} \\ &+ (-a_1a_2 + b_1b_2\lambda)\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}\Big]. \end{aligned} \tag{20}$$

$$\begin{aligned} y_1'(2\pi+0,\lambda) = e^{i(\theta_1+\theta_2)}\Big[&(a_1b_2 + a_2b_1)\cos\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ (b_1b_2\sqrt{\lambda} - a_1a_2\sqrt{\lambda})\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ \left(\frac{b_1b_2}{\sqrt{\lambda}} - \sqrt{\lambda}a_1a_2\right)\cos\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda} \\ &+ (-a_2b_1 - \lambda a_1b_2)\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}\Big]. \end{aligned} \tag{21}$$

$$
\begin{aligned}
y_2(2\pi+0,\lambda) = e^{i(\theta_1+\theta_2)}\Big[& -(a_1b_2+a_2b_1)\cos\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\
& + \left(\frac{a_1a_2}{\sqrt{\lambda}} - \frac{b_1b_2}{\sqrt{\lambda}}\right)\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\
& + \left(\frac{a_1a_2}{\sqrt{\lambda}} - b_1b_2\sqrt{\lambda}\right)\cos\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda} \\
& + \left(\frac{a_1b_2}{\lambda} + b_1a_2\right)\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}\Big]. \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
y_2'(2\pi+0,\lambda) = e^{i(\theta_1+\theta_2)}\Big[& (a_1a_2-b_1b_2)\cos\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\
& + \left(\frac{b_1a_2}{\sqrt{\lambda}} + \frac{a_1b_2}{\sqrt{\lambda}}\right)\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\
& + \left(\frac{b_1a_2}{\sqrt{\lambda}} + b_2a_1\sqrt{\lambda}\right)\cos\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda} \\
& + \left(\frac{b_1b_2}{\lambda} - a_1a_2\right)\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}\Big]. \qquad (23)
\end{aligned}
$$

Proof of Proposition 1.1. We have only to prove the statements (e), (f), and (g), since the demonstrations of (a), (b), (c), and (d) are similar to those of [9, Theorem XIII.89 (a), (b), and (c)] and of [9, Theorem XIII.90 (a)].

Let us prove (e). By the definition of H_μ and that of $\mathscr{H}_\mu$, we claim that

$$\sigma(H_\mu(A_1, A_2, \theta_1, \theta_2)) = \sigma(H_{\mu-\theta_1-\theta_2}(A_1, A_2, 0, 0)).$$

This combined with (b) and (d) implies the claim (e).

Next we show (f) and (g). Let $\theta_1 = \theta_2 = 0$. It follows from (17), (20), and (23) that

$$\lim_{\lambda\to-\infty} D(\lambda) = \begin{cases} +\infty & \text{if } b_1b_2 > 0, \\ -\infty & \text{if } b_1b_2 < 0. \end{cases}$$

So we arrive at the conclusions (f) and (g) in a similar way to [12, Proposition 1, (d) and (e)]. □

3. Location of the Absent Gaps of H

Henceforth, we assume that $\kappa \neq \pi$ and $\theta_1 + \theta_2 = 0$. In this section we use the notations a_j and b_j instead of $\cos\alpha_j$ and $\sin\alpha_j$ for the sake of simplicity. It is also useful to rewrite the clasification (I), (II), (III), and (IV) as the following equivalent forms, respectively.

(I)′ $a_1b_2 - b_1a_2 \neq 0$, $a_1b_2 + b_1a_2 \neq 0$.
(II)′ $a_1b_2 - b_1a_2 \neq 0$, $a_1b_2 + b_1a_2 = 0$.
(III)′ $a_1b_2 - b_1a_2 = 0$, $a_1b_2 + b_1a_2 = 0$, i.e.,

$$(a_1, a_2, b_1, b_2) = (0, 0, 1, 1), (0, 0, 1, -1), (0, 0, -1, 1), (0, 0, -1, -1).$$

(IV)′ $a_1b_2 - b_1a_2 = 0$, $a_1b_2 + b_1a_2 \neq 0$.

We note that λ is a double eigenvalue of H_0 (respectively, H_π) if and only if $M(\lambda) = I$ (respectively, $M(\lambda) = -I$); cf. [12, Lemma 4].

To handle the case (I) we prove the following lemma.

LEMMA 3.1. *If* $M(\lambda) = \pm I$. $\lambda \neq 0$, $\lambda \neq 1$, $a_2b_2 \neq 0$ *and* $a_1b_2 - b_1a_2 \neq 0$, *then we have* $\sin \tau\sqrt{\lambda} = \sin \kappa\sqrt{\lambda} = 0$ *and* $a_2b_1 + a_1b_2 = 0$.

PROOF. Suppose $M(\lambda) = \pm I$. Since

$$y_1'(2\pi + 0, \lambda) = 0, \quad y_2(2\pi + 0, \lambda) = 0, \quad y_1(2\pi + 0, \lambda) - y_2'(2\pi + 0, \lambda) = 0,$$

we have the following three equalities.

$$(a_1b_2 + a_2b_1) \cos \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} + \left(\frac{b_1b_2}{\sqrt{\lambda}} - \sqrt{\lambda}a_1a_2\right) \cos \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}$$
$$+ (b_1b_2 - a_1a_2)\sqrt{\lambda} \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} - (a_2b_1 + \lambda a_1b_2) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} = 0. \quad (24)$$

$$-(a_1b_2 + a_2b_1) \cos \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} + \left(\frac{a_1a_2}{\sqrt{\lambda}} - b_1b_2\sqrt{\lambda}\right) \cos \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}$$
$$+ \left(\frac{a_1a_2}{\sqrt{\lambda}} - \frac{b_1b_2}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} + \left(\frac{a_1b_2}{\lambda} + b_1a_2\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} = 0. \quad (25)$$

$$\left(\frac{a_1b_2}{\sqrt{\lambda}} + b_1a_2\sqrt{\lambda} - \frac{a_2b_1}{\sqrt{\lambda}} - a_1b_2\sqrt{\lambda}\right) \cos \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}$$
$$+ \left(a_1b_2\sqrt{\lambda} + a_2b_1\sqrt{\lambda} - \frac{a_2b_1}{\sqrt{\lambda}} - \frac{a_1b_2}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda}$$
$$+ b_1b_2\left(\lambda - \frac{1}{\lambda}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} = 0. \quad (26)$$

First of all, we prove that $\sin\tau\sqrt{\lambda}\sin\kappa\sqrt{\lambda}=0$ by contradiction. We assume $\sin\tau\sqrt{\lambda}\sin\kappa\sqrt{\lambda}\neq 0$. We put $x_1=\cot\kappa\sqrt{\lambda}$, $x_2=\cot\tau\sqrt{\lambda}$. We devide (24), (25), and (26) by $\sin\tau\sqrt{\lambda}\sin\kappa\sqrt{\lambda}$ to get

$$(a_1b_2+a_2b_1)x_1x_2+\left(\frac{b_1b_2}{\sqrt{\lambda}}-\sqrt{\lambda}a_1a_2\right)x_1$$

$$+(b_1b_2\sqrt{\lambda}-a_1a_2\sqrt{\lambda})x_2+(-a_2b_1-\lambda a_1b_2)=0, \tag{27}$$

$$-(a_2b_1+a_1b_2)x_1x_2+\left(\frac{a_1a_2}{\sqrt{\lambda}}-b_1b_2\sqrt{\lambda}\right)x_1$$

$$+\left(\frac{a_1a_2}{\sqrt{\lambda}}-\frac{b_1b_2}{\sqrt{\lambda}}\right)x_2+\left(\frac{a_1b_2}{\lambda}+b_1a_2\right)=0, \tag{28}$$

$$\left(\frac{a_1b_2}{\sqrt{\lambda}}+b_1a_2\sqrt{\lambda}-\frac{a_2b_1}{\sqrt{\lambda}}-a_1b_2\sqrt{\lambda}\right)x_1$$

$$+\left(a_1b_2\sqrt{\lambda}+a_2b_1\sqrt{\lambda}-\frac{a_2b_1}{\sqrt{\lambda}}-\frac{a_1b_2}{\sqrt{\lambda}}\right)x_2+b_1b_2\left(\lambda-\frac{1}{\lambda}\right)=0. \tag{29}$$

Calculating $\{(27)+(28)\}\times(a_2b_1+a_1b_2)-(29)\times(b_1b_2-a_1a_2)$, we have

$$2a_2b_2x_1+b_2^2\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right)=0.$$

Since $a_2b_2\neq 0$, we obtain

$$x_1=-\frac{b_2}{2a_2}\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right). \tag{30}$$

Substituting this for (29), we infer that

$$x_2=-\frac{b_2}{2a_2}\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right). \tag{31}$$

Furthermore, we substitute (30) and (31) for (28) to get

$$\frac{b_2^2}{4a_2^2}(a_2b_1-a_1b_2)\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right)^2=a_1b_2-b_1a_2.$$

Because $a_1b_2-b_1a_2\neq 0$, we obtain $x_1^2=x_2^2=-1$. However, this contradicts the fact that $\cot z\neq\pm\sqrt{-1}$ for all $z\in\boldsymbol{C}$. Hence we have $\sin\tau\sqrt{\lambda}\sin\kappa\sqrt{\lambda}=0$, that is,

$$\sin\tau\sqrt{\lambda}=0 \quad\text{or}\quad \sin\kappa\sqrt{\lambda}=0. \tag{32}$$

As the next step, we show $\sin \tau\sqrt{\lambda} = \sin \kappa\sqrt{\lambda} = 0$. Let us discuss the former case of (32). We infer by $y_1'(2\pi+0,\lambda) + y_2(2\pi+0,\lambda) = 0$ and $y_1(2\pi+0,\lambda) - y_2'(2\pi+0,\lambda) = 0$ that

$$(a_1a_2 - b_1b_2)\left(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} = 0, \tag{33}$$

$$(a_1b_2 + b_1a_2)\left(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} = 0. \tag{34}$$

Since $(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2 = 1$, we claim that $a_1a_2 - b_1b_2$ and $a_1b_2 + b_1a_2$ do not vanish simultaneously. This together with (33), (34), and the assumption $\lambda \neq 1$ yields $\sin \kappa\sqrt{\lambda} = 0$. Likewise, the latter of (32) implies $\sin \tau\sqrt{\lambda} = 0$.

Therefore we get $\sin \tau\sqrt{\lambda} = \sin \kappa\sqrt{\lambda} = 0$. Combing this with (12), we conclude that $a_2b_1 + a_1b_2 = 0$. □

In the next two lemmas we discuss $M(0)$ and $M(1)$.

LEMMA 3.2. *If $M(1) = \pm I$, then we have $a_1b_2 + b_1a_2 = 0$.*

PROOF. It follows from $M(1) = \pm I$ and (9) that $a_1b_2 + a_2b_1 = 0$. □

LEMMA 3.3. *Suppose $M(0) = \pm I$. Then we have*

$$b_2 = 0 \quad \textit{and} \quad a_1b_2 + a_2b_1 = 0.$$

PROOF. By $M(0) = \pm I$, (24), (25), and (26), we have

$$(a_1b_2 + a_2b_1) + b_1b_2\tau = 0, \tag{35}$$

$$-(a_1b_2 + a_2b_1) + a_1a_2\tau + a_1a_2\kappa - b_1b_2\kappa + a_1b_2\tau\kappa = 0, \tag{36}$$

$$a_1b_2(\tau - \kappa) - a_2b_1(\tau + \kappa) - b_1b_2\tau\kappa = 0. \tag{37}$$

Since $2\pi - \kappa = \tau$, we find that (37) is equivalent to the equation

$$b_1b_2\kappa^2 - 2(a_1b_2 + b_1b_2\pi)\kappa + 2\pi(a_1b_2 - a_2b_1) = 0. \tag{38}$$

We prove $b_2 = 0$ by contradiction. We assume $b_2 \neq 0$. First of all, we show $b_1 \neq 0$. We also proceed by contradiction. Suppose $b_1 = 0$. Then we infer by

$a_1 \neq 0$ and (35) that $b_2 = 0$ which is a contradiction. Thus we obtain $b_1 \neq 0$. In view of (35) we claim that $a_1 b_2 + a_2 b_1 = -b_1 b_2 \tau \neq 0$. By (35), we have

$$\kappa = 2\pi + \frac{a_1 b_2 + a_2 b_1}{b_1 b_2}. \tag{39}$$

We get $(a_2 b_1 - a_1 b_2)(a_2 b_1 + a_1 b_2) = 0$ by substituting (39) for (38). This means $a_2 b_1 - a_1 b_2 = 0$ because of $a_1 b_2 + a_2 b_1 \neq 0$. Substituting (39) for (36), we derive the equation.

$$a_1^3 b_2^3 + a_1 a_2^2 b_1^2 b_2 + 2\pi a_1^2 b_1 b_2^3 + 2a_1^2 a_2 b_1 b_2^2 + 2\pi b_1^3 b_2^3 + 2a_1 b_1^2 b_2^3 + 2a_2 b_1^3 b_2^2 = 0. \tag{40}$$

Plugging $a_2 b_1 = b_2 a_1$ into (40), we arrive at $a_1 = -\pi b_1/2$. Since $a_1 = -\pi b_1/2$ and $a_2 b_1 - a_1 b_2 = 0$, we have $a_2 = -\pi b_2/2$. Inserting $a_1 = -\pi b_1/2$ and $a_2 = -\pi b_2/2$ into (39), we get $\kappa = \pi$ which is a contradiction. Therefore, the initial supposition is false, that is, we obtain $b_2 = 0$. Furthermore we have $b_1 = 0$ in view of $a_2 \neq 0$ and (35). Thus we get $a_1 b_2 + a_2 b_1 = 0$. Lemma 3.3 is completely proved now. □

Next, we discuss the case where $a_2 b_2 = 0$.

Lemma 3.4. *If* $a_2 b_1 + a_1 b_2 \neq 0$, $a_1 b_2 - b_1 a_2 \neq 0$, *and* $a_2 b_2 = 0$, *then we have* $M(\lambda) \neq \pm I$ *for all* $\lambda \in \boldsymbol{R}$, *i.e.*, $G_j \neq \varnothing$ *for all* $j \in \boldsymbol{N}$.

Proof. We define

$$S = \{\lambda \in \boldsymbol{R} \mid M(\lambda) = I \text{ or } M(\lambda) = -I\}. \tag{41}$$

First of all, we remark Lemma 3.2 and Lemma 3.3 say $0, 1 \notin$ S. We prove $S = \varnothing$ by contradiction. Suppose that $S \neq \varnothing$. Then there exists $\lambda \in \boldsymbol{R} \backslash \{0, 1\}$ for which $\lambda \in S$. Since $a_2 b_2 = 0$, we have

$$b_2 = 0 \quad \text{or} \quad a_2 = 0. \tag{42}$$

Let us discuss the former case of (42). We have $a_2 b_1 \neq 0$ by $b_2 = 0$ and $a_2 b_1 + a_1 b_2 \neq 0$. It follows by (20), (23), and $b_2 = 0$ that

$$0 = y_1(2\pi + 0, \lambda) - y_2'(2\pi + 0, \lambda) = a_2 b_1 \left(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\right) \sin 2\pi\sqrt{\lambda}. \tag{43}$$

Noticing $\lambda \neq 1$ and $a_2 b_1 \neq 0$, we have $\sin 2\pi\sqrt{\lambda} = 0$ from (43). Substituting $\sin 2\pi\sqrt{\lambda} = 0$ for $y_1'(2\pi + 0, \lambda) = 0$, we find a contradiction $\cos 2\pi\sqrt{\lambda} = 0$.

Next we consider the latter case of (42). It follows by $a_2 = 0$ and $a_1 b_2 - b_1 a_2 \neq 0$ that $a_1 b_2 \neq 0$. It follows from

$$y_1(2\pi + 0, \lambda) - y_2'(2\pi + 0, \lambda) = 0,$$

$$y_1'(2\pi + 0, \lambda) + y_2(2\pi + 0, \lambda) = 0$$

and $a_2 = 0$ that

$$a_1 \sin(\kappa - \tau)\sqrt{\lambda} + b_1 \left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} = 0, \tag{44}$$

$$b_1 \sin(\kappa - \tau)\sqrt{\lambda} - a_1 \left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} = 0. \tag{45}$$

So we have

$$(b_1^2 + a_1^2)\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} = 0. \tag{46}$$

We demonstrate $\lambda \neq -1$ by contradiction. Assume that $\lambda = -1$. Then (44) is equivalent to $\sin(\kappa - \tau)i = 0$ owing to $a_1 \neq 0$. Moreover $\sin(\kappa - \tau)i = 0$ is equivalent to $\kappa = \pi$. This is a contradiction for our assumption $\kappa \neq \pi$. Thus we get $\lambda \neq -1$. This combined with (46) yields $\sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} = 0$. Furthermore, we get $\sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0$ in view of (44). It follows by $\sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0$, $a_2 = 0$, (22), and $y_2(2\pi + 0, \lambda) = 0$ that $a_1 b_2 = 0$ which is a contradiction.

Since we have found contradictions in both cases of (42), we conclude $S = \emptyset$. □

We are now in a position to prove Theorem 1.2.

PROOF OF THEOREM 1.2. The assertion immediately follows from Lemmas 3.1–3.4. □

Next, we prove Theorem 1.4.

PROOF OF THEOREM 1.4. By (IV) we have

$$(a_1, b_1) = \pm(a_2, b_2). \tag{47}$$

We recall (41). We note that Lemmas 3.2 and 3.3 imply that $0, 1 \notin S$. Since $a_1 b_2 - b_1 a_2 = 0$, the elements of the monodromy matrix take the following forms.

$$\begin{aligned} y_1(2\pi+0,\lambda) = \Big[&(a_1a_2 - b_1b_2)\cos\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ 2a_1b_2\sqrt{\lambda}\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ a_1b_2\left(\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda}\right)\cos\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda} \\ &+ (-a_1a_2 + b_1b_2\lambda)\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}\Big]. \end{aligned} \tag{48}$$

$$\begin{aligned} y_1'(2\pi+0,\lambda) = \Big[&2a_1b_2\cos\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ (b_1b_2 - a_1a_2)\sqrt{\lambda}\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ \left(\frac{b_1b_2}{\sqrt{\lambda}} - \sqrt{\lambda}a_1a_2\right)\cos\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda} \\ &- a_1b_2(1+\lambda)\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}\Big]. \end{aligned} \tag{49}$$

$$\begin{aligned} y_2(2\pi+0,\lambda) = \Big[&-2a_1b_2\cos\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ \left(\frac{a_1a_2}{\sqrt{\lambda}} - \frac{b_1b_2}{\sqrt{\lambda}}\right)\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ \left(\frac{a_1a_2}{\sqrt{\lambda}} - b_1b_2\sqrt{\lambda}\right)\cos\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda} \\ &+ a_1b_2\left(\frac{1}{\lambda} + 1\right)\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}\Big]. \end{aligned} \tag{50}$$

$$\begin{aligned} y_2'(2\pi+0,\lambda) = \Big[&(a_1a_2 - b_1b_2)\cos\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ \frac{2a_1b_2}{\sqrt{\lambda}}\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda} \\ &+ a_1b_2\left(\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda}\right)\cos\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda} \\ &+ \left(\frac{b_1b_2}{\lambda} - a_1a_2\right)\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}\Big]. \end{aligned} \tag{51}$$

Suppose $\lambda \in S$. Because $y_1(2\pi+0,\lambda) - y_2'(2\pi+0,\lambda) = 0$, we have

$$2a_1b_2\left(\sqrt{\lambda}-\frac{1}{\sqrt{\lambda}}\right)\sin\kappa\sqrt{\lambda}\cos\tau\sqrt{\lambda}+b_1b_2\left(\lambda-\frac{1}{\lambda}\right)\sin\kappa\sqrt{\lambda}\sin\tau\sqrt{\lambda}=0,$$

that is,

$$\left\{2a_1\cos\tau\sqrt{\lambda}+b_1\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right)\sin\tau\sqrt{\lambda}\right\}\sin\kappa\sqrt{\lambda}=0 \tag{52}$$

on account of $b_2 \neq 0$ and $\lambda \neq 1$.

Next we show $\sin\kappa\sqrt{\lambda} \neq 0$ by contradiction. We suppose $\sin\kappa\sqrt{\lambda} = 0$. Substituting this for $y_1'(2\pi+0,\lambda) + y_2(2\pi+0,\lambda) = 0$, we observe that

$$(b_1b_2+a_1a_2)\left(\frac{1}{\sqrt{\lambda}}-\sqrt{\lambda}\right)\sin\tau\sqrt{\lambda}=0. \tag{53}$$

Let us prove $\sin\tau\sqrt{\lambda} \neq 0$. Seeking a contradiction, we assume that $\sin\tau\sqrt{\lambda} = 0$. Then we have $a_1\cos\kappa\sqrt{\lambda} = 0$ from (53) and $b_2 \neq 0$. By $\sin\kappa\sqrt{\lambda} = 0$, it turns out that $a_1 = 0$ which is a contradiction with $a_1b_2 - a_2b_1 = 0$ and $a_1b_2 + a_2b_1 \neq 0$. This means $\sin\tau\sqrt{\lambda} \neq 0$. Though we have $b_1b_2 + a_1a_2 = 0$ from $\sin\tau\sqrt{\lambda} \neq 0$, $\lambda \neq 1$, and (53), this is a contradiction. In fact, $b_1b_2 + a_1a_2 = 0$ and our assumption $a_1b_2 - a_2b_1 = 0$ reduce to $a_2 = 0$. This is why $\sin\kappa\sqrt{\lambda} \neq 0$.

It follows from (52), $a_1 \neq 0$, and $\sin\kappa\sqrt{\lambda} \neq 0$ that

$$\cos\tau\sqrt{\lambda}=-\frac{b_1}{2a_1}\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right)\sin\tau\sqrt{\lambda}. \tag{54}$$

This yields $\sin\tau\sqrt{\lambda} \neq 0$. Substituting (54) for $y_2(2\pi+0,\lambda) = 0$, we have

$$0=(b_1b_2+a_1a_2)\cos\kappa\sqrt{\lambda}+\frac{b_2}{2a_1}\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right)\sin\kappa\sqrt{\lambda}. \tag{55}$$

By (47) this equation reduces to

$$\cos\kappa\sqrt{\lambda}=-\frac{b_1}{2a_1}\left(\sqrt{\lambda}+\frac{1}{\sqrt{\lambda}}\right)\sin\kappa\sqrt{\lambda}. \tag{56}$$

We get $\sin(\kappa-\tau)\sqrt{\lambda} = 0$ by (54) and (56). Thus, there exists $k \in \boldsymbol{N}$ satisfying $\lambda = \eta_k$. This implies the inclusion

$$S \subset \left\{\eta_j \,\middle|\, \cos\kappa\sqrt{\eta_j}=-\frac{b_1}{2a_1}\left(\sqrt{\eta_j}+\frac{1}{\sqrt{\eta_j}}\right)\sin\kappa\sqrt{\eta_j},\, j\in\boldsymbol{N}\right\}.$$

Conversely, we assume that there exists $j \in \boldsymbol{N}$ for which $\lambda = \eta_j$ satisfies (56). Plugging $\lambda = (\pi j/(\kappa-\tau))^2$ into (56), we obtain (54). Substituting (54) and (56) for (48)–(51), we observe $M(\lambda) = I$ or $M(\lambda) = -I$. So we get

$$S \supset \left\{ \eta_j \,|\, \cos \kappa\sqrt{\eta_j} = -\frac{b_1}{2a_1}\left(\sqrt{\eta_j} + \frac{1}{\sqrt{\eta_j}}\right) \sin \kappa\sqrt{\eta_j}, j \in \boldsymbol{N} \right\}.$$

Therefore we conclude (3). □

We discuss the case (III) in the following lemma.

Lemma 3.5. *Assume that* (III) *holds. If* $\kappa/\pi \notin \boldsymbol{Q}$, *then* $S = \{1\}$.
If $\kappa/\pi = m/n$, $(n,m) \in \boldsymbol{N}^2$, $\gcd(n,m) = 1$, $m \notin 2\boldsymbol{N}$, *then*

$$S = \{1\} \cup \{n^2 j^2 \,|\, j \in \boldsymbol{N}\}.$$

If $\kappa/\pi = m/n$, $(n,m) \in \boldsymbol{N}^2$, $\gcd(n,m) = 1$, $m \in 2\boldsymbol{N}$, *then*

$$S = \{1\} \cup \left\{ \frac{n^2 j^2}{4} \,\middle|\, j \in \boldsymbol{N} \right\}.$$

Proof. Suppose (III), i.e., $a_1 = a_2 = 0$. Then $b_2 \neq 0$. This implies $0 \notin S$ due to Lemma 3.3. Since $a_1 = a_2 = 0$, we have $M(I) = -b_1 b_2 I$. Taking notice of $b_1 b_2 = \pm 1$, we conclude $1 \in S$. Finally, we discuss $M(\lambda)$ for $\lambda \notin \{0, 1\}$. Since $a_1 = a_2 = 0$, we get

$$y_1(2\pi + 0, \lambda) = b_1 b_2(-\cos \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} + \lambda \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}), \tag{57}$$

$$y_1'(2\pi + 0, \lambda) = b_1 b_2\left(\sqrt{\lambda} \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} \cos \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}\right), \tag{58}$$

$$y_2(2\pi + 0, \lambda) = b_1 b_2\left(\frac{1}{\sqrt{\lambda}} \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} - \sqrt{\lambda} \cos \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}\right),$$

$$y_2'(2\pi + 0, \lambda) = b_1 b_2\left(-\cos \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} + \frac{1}{\lambda} \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}\right). \tag{59}$$

Assume that $M(\lambda) = \pm I$. Since $y_1(2\pi + 0, \lambda) - y_2'(2\pi + 0, \lambda) = 0$, we have

$$0 = b_1 b_2\left(\lambda - \frac{1}{\lambda}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}.$$

Since $y_1'(2\pi + 0, -1) = 0$ is equivalent to $\kappa = \pi$, we get $\lambda \neq -1$. Since $\lambda \neq \pm 1$ and $b_1 b_2 = \pm 1$, we see $\sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} = 0$. This together with $y_1'(2\pi + 0, \lambda) = 0$ implies $\sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0$.

Conversely, if $\sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0$ holds, then it follows that

$$M(\lambda) = I \quad \text{or} \quad M(\lambda) = -I.$$

Summarizing the above discussion, we have

$$S\backslash\{1\} = \{\lambda \in \boldsymbol{R}\backslash\{0,1\} \mid \sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0\}.$$

In a similar way to [12, Lemma 6], we acquire the conclusion. □

We put an end to this section by proving the following lemma.

Lemma 3.6. *Assume that* (II) *is valid. Then we have the following statements* (1)–(3).

(1) *If* $\kappa/\pi \notin \boldsymbol{Q}$, *then* $S = \{1\}$.

(2) *If* $(m,n) \in \boldsymbol{N}^2$, $\kappa/\pi = m/n$, $\gcd(n,m) = 1$, *and* $m \notin 2\boldsymbol{N}$, *then*

$$S = \{1\} \cup \{n^2 j^2 \mid j \in \boldsymbol{N}\}.$$

(3) *If* $(m,n) \in \boldsymbol{N}^2$, $\kappa/\pi = m/n$, $\gcd(n,m) = 1$, *and* $m \in 2\boldsymbol{N}$, *then*

$$S = \{1\} \cup \left\{ \frac{n^2 j^2}{4} \,\middle|\, j \in \boldsymbol{N} \right\}.$$

Proof. Using Lemma 3.3 and $b_2 \neq 0$, we have $M(0) \neq \pm I$. This implies $0 \notin S$.

Next, we discuss $M(1)$. By (II) we have $a_1 a_2 - b_1 b_2 = \pm 1$. It follows from (20)–(23) that $M(1) = (a_1 a_2 - b_1 b_2) I$. So we get $1 \in \mathrm{S}$.

Next we discuss $M(\lambda)$ for $\lambda \notin \{0,1\}$. Suppose $\lambda \notin \{0,1\}$ and $M(\lambda) = \pm I$. Since $y_1(2\pi + 0, \lambda) - y_2'(2\pi + 0, \lambda) = 0$, $b_2 \neq 0$, $a_1 b_2 + b_1 a_2 = 0$, and $\lambda \neq 1$, we have

$$\sin \tau\sqrt{\lambda} \left\{ 2a_1 \cos \kappa\sqrt{\lambda} - b_1 \left(\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda} \right) \sin \kappa\sqrt{\lambda} \right\} = 0. \tag{60}$$

Let us show that

$$2a_1 \cos \kappa\sqrt{\lambda} - b_1 \left(\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda} \right) \sin \kappa\sqrt{\lambda} \neq 0. \tag{61}$$

Seeking a contradiction, we assume that

$$2a_1 \cos \kappa\sqrt{\lambda} - b_1 \left(\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda} \right) \sin \kappa\sqrt{\lambda} = 0.$$

Since $a_1 \neq 0$, we have

$$\cos \kappa\sqrt{\lambda} = \frac{b_1}{2a_1}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda}. \tag{62}$$

This means $\sin \kappa\sqrt{\lambda} \neq 0$. Substituting (62) and $a_1 b_2 + b_1 a_2 = 0$ for each element of the monodromy matrix, we have

$$\begin{aligned} y_1(2\pi + 0, \lambda) = \Bigg[&(a_1 a_2 - b_1 b_2)\frac{b_1}{2a_1}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} \\ &+ a_1 b_2\left(\frac{1}{\sqrt{\lambda}} - \sqrt{\lambda}\right)\frac{b_1}{2a_1}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} \\ &+ (-a_1 a_2 + b_1 b_2 \lambda) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}\Bigg], \end{aligned} \tag{63}$$

$$\begin{aligned} y_1'(2\pi + 0, \lambda) = \Bigg[&(b_1 b_2 - a_1 a_2)\sqrt{\lambda} \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} \\ &+ \left(\frac{b_1 b_2}{\sqrt{\lambda}} - \sqrt{\lambda} a_1 a_2\right)\frac{b_1}{2a_1}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} \\ &+ a_1 b_2 (1 - \lambda) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}\Bigg], \end{aligned} \tag{64}$$

$$\begin{aligned} y_2(2\pi + 0, \lambda) = \Bigg[&(a_1 a_2 - b_1 b_2)\frac{1}{\sqrt{\lambda}} \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} \\ &+ \left(\frac{a_1 a_2}{\sqrt{\lambda}} - b_1 b_2\sqrt{\lambda}\right)\frac{b_1}{2a_1}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} \\ &+ a_1 b_2\left(\frac{1}{\lambda} - 1\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}\Bigg], \end{aligned} \tag{65}$$

$$\begin{aligned} y_2'(2\pi + 0, \lambda) = \Bigg[&(a_1 a_2 - b_1 b_2)\frac{b_1}{2a_1}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda} \\ &+ a_1 b_2\left(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\right)\frac{b_1}{2a_1}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda} \\ &+ \left(\frac{b_1 b_2}{\lambda} - a_1 a_2\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}\Bigg]. \end{aligned} \tag{66}$$

It follows from $y_1'(2\pi + 0, \lambda) + y_2(2\pi + 0, \lambda) = 0$ that

$$0 = (b_1 b_2 - a_1 a_2)\left(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \cos \tau\sqrt{\lambda}$$
$$+ \left\{(b_1 b_2 + a_1 a_2)\left(\frac{1}{\sqrt{\lambda}} - \sqrt{\lambda}\right)\right\} \frac{b_1}{2a_1}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}$$
$$+ a_1 b_2\left(\frac{1}{\lambda} - \lambda\right) \sin \kappa\sqrt{\lambda} \sin \tau\sqrt{\lambda}.$$

Because $\sin \kappa\sqrt{\lambda} \neq 0$, this equation reduces to

$$(b_1 b_2 - a_1 a_2) \cos \tau\sqrt{\lambda} = \left\{\frac{b_1}{2a_1}(b_1 b_2 + a_1 a_2) + a_1 b_2\right\}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \sin \tau\sqrt{\lambda}. \quad (67)$$

So we have $\sin \tau\sqrt{\lambda} \neq 0$. Likewise, the equation

$$y_1'(2\pi + 0, \lambda) - y_2(2\pi + 0, \lambda) = 0$$

reduces to

$$0 = (b_1 b_2 - a_1 a_2)\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right) \cos \tau\sqrt{\lambda}$$
$$+ (b_1 b_2 - a_1 a_2)\frac{b_1}{2a_1}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^2 \sin \tau\sqrt{\lambda}$$
$$+ a_1 b_2\left\{-\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^2 + 4\right\} \sin \tau\sqrt{\lambda}. \quad (68)$$

Substituting (67) for (68), we have

$$\left\{\frac{b_1}{2a_1}(b_1 b_2 + a_1 a_2) + a_1 b_2\right\}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^2 \sin \tau\sqrt{\lambda}$$
$$+ \frac{b_1}{2a_1}(b_1 b_2 - a_1 a_2)\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^2 \sin \tau\sqrt{\lambda}$$
$$- a_1 b_2\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^2 \sin \tau\sqrt{\lambda} + 4a_1 b_2 \sin \tau\sqrt{\lambda} = 0$$

and therefore

$$\frac{b_1^2}{4a_1^2}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^2 = -1. \quad (69)$$

Considering (62) and (69), we find a contradiction $\cot^2 \kappa\sqrt{\lambda} = -1$. Therefore it turns out that (61) holds.

Using (60), we get $\sin \tau\sqrt{\lambda} = 0$. Furthermore it follows from $a_1 b_2 + b_1 a_2 = 0$, (22), and $y_2(2\pi + 0, \lambda) = 0$ that $\sin \kappa\sqrt{\lambda} = 0$. Thus we have

$$S\backslash\{1\} \subset \{\lambda \in \boldsymbol{R}\backslash\{0,1\} \mid \sin \kappa\sqrt{\lambda} = \sin \tau\sqrt{\lambda} = 0\}.$$

It is easy to show the reverse inclusion. Therefore we complete the proof. □

4. Indices of the Absent Gaps of H

In this section we demonstrate Theorem 1.3. To this end we introduce the Prüfer transform of a solution to (16); see [2, Chapter 8]. Let $\theta_1 = \theta_2 = 0$ and let y be a solution of (16). By (r, ω) we denote the polar coordinates of (y, y'):

$$y = r \sin \omega, \quad y' = r \cos \omega.$$

Then $\omega = \omega(x, \lambda)$ verifies the equations

$$\frac{d}{dx}\omega(x,\lambda) = \cos^2 \omega(x,\lambda) + \lambda \sin^2 \omega(x,\lambda), \quad x \in \boldsymbol{R}\backslash\Gamma, \tag{70}$$

$$\omega(x+0,\lambda) = \omega(x-0,\lambda) - \alpha_j, \quad \text{for } x \in \Gamma_j,\ j = 1,2. \tag{71}$$

The initial condition

$$\omega(+0,\lambda) = \omega_0 \tag{72}$$

uniquely determines the solution to the above equations. The function ω is called the Prüfer transform of y. We define the rotation number of (16) as

$$\rho(\lambda) = \lim_{t\to\infty} \frac{\omega(2\pi t + 0, \lambda) - \omega(+0,\lambda)}{2\pi t}. \tag{73}$$

In a similar way to the proof of [6, Theorem 2.1], we see that the limit exists and is independent of the initial value ω_0. Furthermore, the function $\rho(\lambda)$ is non-decreasing on $\boldsymbol{R}$. Henceforth we discuss the cases (III) and (II). Since $\alpha_1 + \alpha_2 \equiv 0$ in each case and since the discriminant $D(\lambda)$ is π-antiperiodic with respect to α_2, it suffices to consider the case where

$$\alpha_1 = \pi - \alpha_2. \tag{74}$$

First, we prove the following claim.

Lemma 4.1. *Suppose* (74). *Let* $\omega(x, \lambda, \omega_0)$ *be the solution of* (70)–(72) *with*

$$0 \le \omega_0 < \pi.$$

We fix $\gamma \in [0, \pi)$. *Then we have*

$$\lim_{\lambda \to -\infty} \omega(2\pi + 0, \lambda, \omega_0) = \begin{cases} \alpha_2 - \pi & \text{if } \alpha_2 \in (-\pi, 0), \\ \alpha_2 - 2\pi & \text{if } \alpha_2 \in (0, \pi), \end{cases}$$

where the limit is uniform with respect to $\omega_0 \in [0, \gamma]$.

Proof. In order to prove this lemma, we recall a basic fact on the Prüfer transform from [2, Chapter 8, Theorem 2.1]. Let $c < d$. For $\beta \in [0, \pi)$, let $\theta = \theta(x, \lambda, c, \beta)$ be the solution to the equations

$$\frac{d}{dx}\theta = \cos^2 \theta + \lambda \sin^2 \theta \quad \text{on } \boldsymbol{R}, \tag{75}$$

$$\theta|_{x=c} = \beta.$$

Then it holds that

$$\lim_{\lambda \to -\infty} \theta(d, \lambda, c, \beta) = 0. \tag{76}$$

We fix $\omega_0 \in [0, \gamma]$. By (76) we have

$$\lim_{\lambda \to -\infty} \omega(\kappa - 0, \lambda, \omega_0) = 0$$

and thus

$$\lim_{\lambda \to -\infty} \omega(\kappa + 0, \lambda, \omega_0) = -\alpha_2. \tag{77}$$

First we discuss the case where $\alpha_2 \in (0, \pi)$. We pick δ such that $-\pi < -\alpha_2 < \delta < 0$. It follows by (77) that there exists $\lambda_0 = \lambda_0(\omega_0) \in \boldsymbol{R}$ such that

$$-\pi < \omega(\kappa + 0, \lambda, \omega_0) < \delta \quad \text{for } \lambda \le \lambda_0.$$

This combined with the comparison theorem [2, Chapter 8] implies

$$\theta(2\pi, \lambda, \kappa, -\pi) < \omega(2\pi - 0, \lambda, \omega_0) < \theta(2\pi, \lambda, \kappa, \delta) \quad \text{for } \lambda \le \lambda_0.$$

Since the right side of (75) is π-periodic in θ and since (76) holds, we have

$$\lim_{\lambda \to -\infty} \theta(2\pi, \lambda, \kappa, -\pi) = \lim_{\lambda \to -\infty} \theta(2\pi, \lambda, \kappa, \delta) = -\pi.$$

Thus we get

$$\lim_{\lambda \to -\infty} \omega(2\pi - 0, \lambda, \omega_0) = -\pi$$

and hence

$$\lim_{\lambda \to -\infty} \omega(2\pi + 0, \lambda, \omega_0) = \alpha_2 - 2\pi.$$

By the comparison theorem and $\omega_0 \in [0, \gamma]$, we have

$$\omega(2\pi + 0, \lambda, 0) \le \omega(2\pi + 0, \lambda, \omega_0) \le \omega(2\pi + 0, \lambda, \gamma).$$

Thus we get the assertion for $\alpha_2 \in (0, \pi)$. Likewise we get the conclusion for $\alpha_2 \in (-\pi, 0)$. □

Lemma 4.2. *For all* $\alpha_2 \in (-\pi, 0) \cup (0, \pi)$, *we have*

$$\lim_{\lambda \to -\infty} \rho(\lambda) = -1.$$

Proof. First we discuss the case where $\alpha_2 \in (0, \pi)$. Pick γ such that $\alpha_2 < \gamma < \pi$. By Lemma 4.1 there exists $\lambda_0 \in \boldsymbol{R}$ such that

$$-2\pi \le \omega(2\pi + 0, \lambda, \omega_0) \le \gamma - 2\pi$$

for $\lambda \le \lambda_0$ and $0 \le \omega_0 \le \gamma$. Since the right side of (70) is π-periodic with respect to ω, we have

$$-4\pi \le \omega(4\pi + 0, \lambda, \omega_0) \le \gamma - 4\pi$$

for $\lambda \le \lambda_0$ and $0 \le \omega_0 \le \gamma$. By induction we get

$$-2n\pi \le \omega(2n\pi + 0, \lambda, \omega_0) \le -2n\pi + \gamma$$

for $n \in \boldsymbol{N}$, $\lambda \le \lambda_0$, and $0 \le \omega_0 \le \gamma$. Thus

$$\rho(\lambda) = -1$$

for $\lambda \le \lambda_0$. Therefore we obtain the assertion for $\alpha_2 \in (0, \pi)$. In a similar way we get the claim for $\alpha_2 \in (-\pi, 0)$. □

We can now characterize the endpoints of the n-th gap as follows. For $n \in \boldsymbol{N}$, we put $B_n = [a_n, b_n]$.

Theorem 4.3. *For* $n \in \boldsymbol{N}$, *we have*

$$b_n = \min\left\{\lambda \in \boldsymbol{R} \mid \rho(\lambda) = -1 + \frac{n}{2}\right\},$$

$$a_n = \max\left\{\lambda \in \boldsymbol{R} \mid \rho(\lambda) = -1 + \frac{n-1}{2}\right\}.$$

PROOF. Using Lemma 4.2 and following the lines of the proof of [3, Proposition 2.1], we get the claim. □

We are now ready to complete the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. First, we discuss the case (III). We have only to consider the case where $\alpha_1 = \alpha_2 = \pi/2$. In this case, (71) reduces to

$$\omega(\kappa + 0, \lambda) = \omega(\kappa - 0, \lambda) - \frac{\pi}{2}, \tag{78}$$

$$\omega(2\pi + 0, \lambda) = \omega(2\pi - 0, \lambda) - \frac{\pi}{2}. \tag{79}$$

If $\lambda = 1$, (70) is equivalent to $\omega' = 1$. This together with (78) and (79) implies $\omega(2\pi t + 0, 1, 0) = \pi t$. So, we get $\rho(1) = 1/2$. On the other hand, it follows from Lemma 3.5 that $1 \in \mathrm{S}$. This means that a gap disappears at $\{1\}$. Since $\rho(1) = 1/2$ and $1 \in \mathrm{S}$, we conclude that the third gap is absent by Theorem 4.3.

Suppose $\kappa/\pi = m/n$, $(n, m) \in \boldsymbol{N}^2$, $\gcd(n, m) = 1$ and $m \notin 2\boldsymbol{N}$. Due to Lemma 3.5, we have $S = \{1\} \cup \{n^2 j^2 \mid j \in \boldsymbol{N}\}$. Put $\lambda = n^2 j^2$. The equation (70) takes the form

$$\omega'(x, n^2 j^2) = 1 + (n^2 j^2 - 1) \sin^2 \omega(x, n^2 j^2). \tag{80}$$

The general solution of this equation is

$$\arctan\left(\frac{\tan(njx + v)}{nj}\right),$$

where v is a constant. Adopting $\omega_0 = 0$ in the initial condition (72) and using the jumps (78) and (79), we have $\omega(2\pi t + 0, n^2 j^2, 0) = (2nj - 1)\pi t$. Thus $\rho(\lambda) = (2nj - 1)/2$. By Theorem 4.3, we conclude that the $(2nj + 1)$-st gap vanishes at the point $\{n^2 j^2\}$.

Next we consider the case where $\kappa/\pi = m/n$, $(n, m) \in \boldsymbol{N}^2$, $\gcd(n, m) = 1$ and $m \in 2\boldsymbol{N}$. By a similar argument as above we see that

$$\omega(2\pi t + 0, n^2 j^2/4, 0) = (nj - 1)\pi t$$

and that the $(nj + 1)$-st gap disappears at the point $\{n^2 j^2/4\}$. Thus we have the assertion of Theorem 1.3 in the case (III).

Likewise, we get the claim in the case (II). □

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