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QUANTIFIER ELIMINATION OF THE PRODUCTS OF ORDERED ABELIAN GROUPS

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Abstract. In this paper, we study the theories of lexicographic products of ordered abelian groups.

1. Introduction

Komori [2] and Weispfenning [6] showed that the lexicographic product of $\mathbf{Z}$ and $\mathbf{Q}$ admits quantifier elimination in a language expanding $L_{og} = \{0, +, -, <\}$, where $\mathbf{Z}$ ($\mathbf{Q}$) is the ordered abelian group of integers (of rational numbers). Moreover they recursively axiomatized $\mathrm{Th}(\mathbf{Z} \times \mathbf{Q})$. Extending these, Suzuki [4] showed that for the lexicographic product G of an ordered abelian group H and an ordered divisible abelian group K, if H admits quantifier elimination in a language L expanding L_{og}, then G admits quantifier elimination in $L \cup \{I\}$, where we interpret I as $\{0\} \times K$. Moreover if H is recursively axiomatizable, then so is G. In this paper, we give a simple proof for Suzuki's results. In addition we show the converse of Suzuki's results.

2. Main Results

Let $\mathscr{L}$ be a language. By an *unnested atomic* $\mathscr{L}$*-formula* we mean an atomic formula of one of the following forms: $x = y$, $c = y$, $F(\bar{x}) = y$ or $R(\bar{x})$, where x, y and n-tuple $\bar{x}$ are free variables, c is some constant symbol in $\mathscr{L}$, F is some function symbol in $\mathscr{L}$ and R is some relation symbol in $\mathscr{L}$.

Let L_{og} be the language $\{0, +, -, <\}$ of ordered groups. Let L be the language $L_{og} \cup L_r \cup L_c$, where L_r and L_c are sets of relation and constant symbols, respectively. Let H be an L-structure whose reduct to the language L_{og} is an

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ordered abelian group. Let K be an ordered abelian group and an L_{og}-structure. Let I be a new unary relation symbol. We now give the lexicographic product $G := H \times K$ as an $L \cup \{I\}$-structure by the following interpretation:

(1) $0^G := (0^H, 0^K)$;

(2) $c^G := (c^H, 0^K)$ for each $c \in L_c$;

(3) $+$ and $-$ are defined coordinatewise;

(4) $<$ is the lexicographic order of H and K;

(5) For each n-ary relation symbol $R \in L_r$,

$$R^G := \{(g_1, \dots, g_n) \in G^n \mid (h_1, \dots, h_n) \in R^H\},$$

where h_i is the first coordinate of g_i

(6) $I^G := \{0\} \times K$.

We call this interpretation the *product interpretation* of H and K.

Let s, t and u be terms. Then, the formula $s < t \wedge t < u$ is written as $s < t < u$.

LEMMA 1. *Let $G = H \times K$ be the above structure and $\bar{g} = (g_1, \dots, g_n)$ a tuple of elements from G. For each $i \leq n$, let $g_i = (h_i, k_i)$ with $h_i \in H$ and $k_i \in K$. Let $\bar{h} = (h_1, \dots, h_n)$. Let $\varphi(\bar{x})$ be a quantifier-free L-formula. Then there exists a quantifier-free $L \cup \{I\}$-formula $\varphi^*(\bar{x})$ such that $H \models \varphi(\bar{h})$ if and only if $G \models \varphi^*(\bar{g})$.*

PROOF. Let $\varphi(\bar{x})$ be a quantifier-free L-formula. Then the formula $\varphi(\bar{x})$ is a Boolean combination of the forms $t(\bar{x}) = 0$, $0 < t(\bar{x})$ and $R(t_1(\bar{x}), \dots, t_m(\bar{x}))$, where $t, t_1, \dots, t_m$ are terms and R is an m-ary relation symbol. Let $\varphi^*(\bar{x})$ be the formula obtained from $\varphi(\bar{x})$ by replacing $t(\bar{x}) = 0$ and $0 < t(\bar{x})$ with $I(t(\bar{x}))$ and $0 < t(\bar{x}) \wedge \neg I(t(\bar{x}))$, respectively. Then $H \models \varphi(\bar{h})$ if and only if $G \models \varphi^*(\bar{g})$. □

We give the new structures to show recursive axiomatizability in Theorem 3.

For any model G^* of $\mathrm{Th}(G)$, we consider the structures H^*, K^* such that $K^* := \{g \in G^* \mid g \models I(x)\}$ and $H^* := \{g/{\sim} \mid g \in G^*\}$, where an equivalent relation $\sim$ on G^* by $a \sim b \Leftrightarrow a - b \in K^*$. Then H^* is an ordered abelian group as an L-structure, K^* is an ordered abelian group as an L_{og}-structure. Then we notice that $H \equiv H^*$ and $K \equiv K^*$. Moreover we obtain that $G^* \equiv_{L \cup \{I\}} H^* \times K^*$ by the next lemma.

LEMMA 2. *Suppose that H, K, H^*, K^* are the above structures. Then we obtain that $H \times K \equiv H^* \times K^*$ in the language $L \cup \{I\}$, where $H^* \times K^*$ is the product interpretation of H^* and K^*.*

Proof. It suffices to show that $H \times K \equiv H^* \times K^*$ for any finite language of $L \cup \{I\}$. We fix L' as a finite language of $L \cup \{I\}$ and may assume that L' contains L_{og} and $\{I\}$. According to [1, Corollary 3.3.3], we have to prove the followings:

$$\text{for each } n < \omega, \quad H \times K \approx_n H^* \times K^*.$$

When A, B are the same structures with a finite language, $A \approx_n B$ means that for any n-tuple $(c_1, \dots, c_n)$ in $A \cup B$, there exists partial isomorphism f from A to B such that we find some n-tuple $(d_1, \dots, d_n)$ in $A \cup B$ satisfying the following conditions: for each $i \le n$ if $c_i \in A$ (B, respectively) then let $a_i = c_i$ and $b_i = d_i = f(c_i) \in B$ (let $b_i = c_i$ and $a_i = d_i = f^{-1}(c_i) \in A$, respectively) and $A \models \varphi(a_1, \dots, a_n) \Leftrightarrow B \models \varphi(b_1, \dots, b_n)$ for any unnested atomic formula $\varphi(x_1, \dots, x_n)$.

The unnested atomic L-formulas are of the formulas of the forms $x = y$, $y = c$, $y = 0$, $x_0 + x_1 = y$, $-x = y$, $R(\bar{x})$, $x_0 < x_1$, $I(x)$, where x, y, x_0, x_1 and n-tuple $\bar{x}$ are free variables.

For $n < \omega$, let $(c_1, \dots, c_n)$ be any n-tuple from $(H \times K) \cup (H^* \times K^*)$. When we see it coordinatewisely, we have partial isomorphisms $f : H \to H^*$ and $g : K \to K^*$ satisfying the above condition. We will obtain some n-tuple $(d_1, \dots, d_n)$ as follows: for $i \le n$ if c_i is in $H \times K$ then we split it into $c_i = (h_i, k_i)$ and let $a_i = c_i$ and $b_i = d_i = (h_i^*, k_i^*) = (f(h_i), g(k_i)) \in H^* \times K^*$. If c_i is in $H^* \times K^*$ then we let $b_i = c_i$ and $a_i = d_i = (h_i, k_i) = (f^{-1}(h_i^*), g^{-1}(k_i^*)) \in H \times K$ similarly. Then we have that $H \times K \models \varphi(a_1, \dots, a_n) \Leftrightarrow H^* \times K^* \models \varphi(b_1, \dots, b_n)$ for every unnested atomic L'-formula $\varphi(x_1, \dots, x_n)$.

In the case of "$x_0 + x_1 = y$" we obtain that $a_i + a_j = a_l \Leftrightarrow (h_i, k_i) + (h_j, k_j) = (h_l, k_l) \Leftrightarrow (h_i + h_j = h_l$ and $k_i + k_j = k_l) \Leftrightarrow (f(h_i) + f(h_j) = f(h_l)$ and $g(k_i) + g(k_j) = g(k_l)) \Leftrightarrow (h_i^* + h_j^* = h_l^*$ and $k_i^* + k_j^* = k_l^*) \Leftrightarrow (h_i^*, k_i^*) + (h_j^*, k_j^*) = (h_l^*, k_l^*) \Leftrightarrow b_i + b_j = b_l$.

Moreover we can also argue the other cases similarly. Therefore it holds that $H \times K \approx_n H^* \times K^*$. □

We now give a simple proof for Suzuki's results [4].

Theorem 3. *Let $G = H \times K$ be the above structure. If the ordered abelian group H admits quantifier elimination in L and the ordered abelian group K is divisible, then the ordered abelian group G admits quantifier elimination in $L \cup \{I\}$. Moreover, if H is recursively axiomatizable, then so is G.*

Proof. Let $\exists x \varphi(x, \bar{y})$ be an $L \cup \{I\}$-formula, where $\varphi(x, \bar{y})$ is a quantifier-free $L \cup \{I\}$-formula. We may assume that the formula φ is of the form $\varphi_1 \wedge \cdots \wedge \varphi_j$, where each φ_i is an atomic formula or the negation of an atomic formula. Since $\varphi(x, \bar{y})$ is a quantifier-free $L \cup \{I\}$-formula, the formula $\varphi(x, \bar{y})$ is a Boolean combination of the forms $mx = t(\bar{y})$, $t(\bar{y}) < mx$, $mx < t(\bar{y})$, $I(s(x, \bar{y}))$ and $R(s_1(x, \bar{y}), \dots, s_l(x, \bar{y}))$, where l, m are positive integers, $t, s, s_1, \dots, s_l$ are terms and R is an l-ary relation symbol. Now the formulas $t = s$ and $t < s$ are equivalent to $nt = ns$ and $nt < ns$ for each positive integer n, respectively. Hence, we may assume that the formula $\varphi(x, \bar{y})$ is equivalent to either $t(\bar{y}) < mx < u(\bar{y}) \wedge \psi(x, \bar{y})$ or $mx = s(\bar{y}) \wedge \psi(x, \bar{y})$, where the formula $\psi(x, \bar{y})$ is a finite conjunction of formulas of the forms I, $R(s_1, \dots, s_l)$ or negation of these.

Let the formula $\varphi(x, \bar{y})$ be $t(\bar{y}) < mx < u(\bar{y}) \wedge \psi(x, \bar{y})$. Let $\bar{g} = (g_1, \dots, g_n)$ be a tuple of elements from the ordered abelian group G. For each $i \leq n$, let $g_i = (h_i, k_i)$ with $h_i \in H$ and $k_i \in K$. Let $\bar{h} = (h_1, \dots, h_n)$ and $\bar{k} = (k_1, \dots, k_n)$. Let $\psi^1(x, \bar{y})$ be the formula obtained from $\psi(x, \bar{y})$ by replacing $I(t(x, \bar{y}))$ with $t(x, \bar{y}) = 0$. Let $t^2(\bar{y})$ $(u^2(\bar{y}))$ be the term obtained from $t(\bar{y})$ $(u(\bar{y}))$ by replacing each $c \in L_c$ with 0. Then $G \models \exists x(t(\bar{g}) < mx < u(\bar{g}) \wedge \psi(x, \bar{g}))$ if and only if

(1) $H \models \exists x(t(\bar{h}) < mx < u(\bar{h}) \wedge \psi^1(x, \bar{h}))$,
(2) $H \models \exists x(t(\bar{h}) = mx < u(\bar{h}) \wedge \psi^1(x, \bar{h}))$ and $K \models \exists x(t^2(\bar{k}) < mx)$,
(3) $H \models \exists x(t(\bar{h}) < mx = u(\bar{h}) \wedge \psi^1(x, \bar{h}))$ and $K \models \exists x(mx < u^2(\bar{k}))$, or
(4) $H \models \exists x(t(\bar{h}) = mx = u(\bar{h}) \wedge \psi^1(x, \bar{h}))$ and $K \models \exists x(t^2(\bar{k}) < mx < u^2(\bar{k}))$.

Since the ordered abelian group H admits quantifier elimination in L and the ordered abelian group K is divisible, there exist quantifier-free L-formulas $\theta_1(\bar{y})$, $\theta_2(\bar{y})$, $\theta_3(\bar{y})$ and $\theta_4(\bar{y})$ such that $G \models \exists x(t(\bar{g}) < mx < u(\bar{g}) \wedge \psi(x, \bar{g}))$ if and only if

(1) $H \models \theta_1(\bar{h})$,
(2) $H \models \theta_2(\bar{h})$,
(3) $H \models \theta_3(\bar{h})$, or
(4) $H \models \theta_4(\bar{h}) \wedge t(\bar{h}) = u(\bar{h})$ and $K \models t^2(\bar{k}) < u^2(\bar{k})$.

By Lemma 1, there exist quantifier-free $L \cup \{I\}$-formulas $\theta_1^*(\bar{y})$, $\theta_2^*(\bar{y})$, $\theta_3^*(\bar{y})$ and $\theta_4^*(\bar{y})$ such that $G \models \exists x(t(\bar{g}) < mx < u(\bar{g}) \wedge \psi(x, \bar{g}))$ if and only if

(1) $G \models \theta_1^*(\bar{g})$,
(2) $G \models \theta_2^*(\bar{g})$,
(3) $G \models \theta_3^*(\bar{g})$, or
(4) $G \models \theta_4^*(\bar{g}) \wedge t(\bar{g}) < u(\bar{g}) \wedge I(u(\bar{g}) - t(\bar{g}))$.

Hence, the formula $\exists x(t(\bar{y}) < mx < u(\bar{y}) \wedge \psi(x, \bar{y}))$ is equivalent to a quantifier-free $L \cup \{I\}$-formula.

Similarly, the formula $\exists x(mx = s(\bar{y}) \wedge \psi(x, \bar{y}))$ is equivalent to a quantifier-free $L \cup \{I\}$-formula. It follows that the ordered abelian group G admits quantifier elimination in $L \cup \{I\}$.

Last we show that in the theorem, if H is recursively axiomatizable, so is G.

By lemma 2, for any model G^* of $\mathrm{Th}(G)$ there exist $H^* \models \mathrm{Th}(H)$ and

$K^* \models \mathrm{Th}(K)$ such that G^* is elementarily equivalent to $H^* \times K^*$. Thus we have G is recursively axiomatizable since H is recursively axiomatizable. □

Finally we show the converse of Suzuki's results.

THEOREM 4. *Let $G = H \times K$ be the above structure. If the ordered abelian group G admits quantifier elimination in $L \cup \{I\}$, then the ordered abelian group H admits quantifier elimination in L and the ordered abelian group K is divisible. Moreover if G is recursively axiomatizable, then so is H.*

PROOF. First, we show that the ordered abelian group H admits quantifier elimination in L. Let $\exists x\varphi(x, \bar{y})$ be an L-formula, where $\varphi(x, \bar{y})$ is a quantifier-free L-formula. Since $\varphi(x, \bar{y})$ is a quantifier-free L-formula, the formula $\varphi(x, \bar{y})$ is a Boolean combination of the forms $mx = t(\bar{y})$, $t(\bar{y}) < mx$, $mx < t(\bar{y})$ and $R(s_1(x, \bar{y}), \ldots, s_l(x, \bar{y}))$, where l, m are positive integers, $t, s, s_1, \ldots, s_l$ are terms and R is an l-ary relation symbol.

Let $\varphi^*(x, \bar{y})$ be the formula obtained from $\varphi(x, \bar{y})$ by replacing $mx = t(\bar{y})$, $t(\bar{y}) < mx$ and $mx < t(\bar{y})$ with $I(t(\bar{y}) - mx)$, $t(\bar{y}) < mx \wedge \neg I(t(\bar{y}) - mx)$ and $mx < t(\bar{y}) \wedge \neg I(t(\bar{y}) - mx)$, respectively. Let $\bar{h} = (h_1, \ldots, h_n)$ be a tuple of elements from the ordered abelian group H. Then, we have

$$H \models \exists x\varphi(x, \bar{h}) \Leftrightarrow G \models \exists x\varphi^*(x, (\overline{h, 0})),$$

where $(\overline{h, 0}) := ((h_1, 0), \ldots, (h_n, 0))$. Since the ordered abelian group G admits quantifier elimination in $L \cup \{I\}$, there exists a quantifier-free $L \cup \{I\}$-formula $\psi(\bar{y})$ such that

$$G \models \exists x\varphi^*(x, (\overline{h, 0})) \Leftrightarrow G \models \psi((\overline{h, 0})).$$

Let $\psi'(\bar{y})$ be the formula obtained from $\psi(\bar{y})$ by replacing $I(t(\bar{y}))$ with $t(\bar{y}) = 0$. Then we have

$$G \models \psi((\overline{h, 0})) \Leftrightarrow H \models \psi'(\bar{h}).$$

It follows that the ordered abelian group H admits quantifier elimination in L.

Next, we show that the ordered abelian group K is divisible. Let $a \in K$. Let n be a positive integer. Since the ordered abelian group G admits quantifier elimination in $L \cup \{I\}$, there exists a quantifier-free $L \cup \{I\}$-formula $\theta_n(x)$ such that

$$G \models \exists y((0, a) = ny \wedge I(y)) \leftrightarrow \theta_n((0, a)).$$

We have $G \models \theta_n((0, 0))$. Suppose that $a > 0$. Then we have $G \models \theta_n((0, na))$. Now the formula $\theta_n(x)$ is a Boolean combination of the forms $mx = t$, $t < mx$, $mx < t$, $I(mx + t)$ and $R(m_1 x + s_1, \ldots, m_l x + s_l)$, where $l, m, m_1, \ldots, m_l$ are positive integers, $t, s_1, \dots, s_l$ are terms which do not contain a free variable and R is an l-ary relation symbol. Notice that $t^K = 0, s_1^K = 0, \dots, s_l^K = 0$.

In the case that $G \models m(0, na) = t$, we have $a = 0$, a contradiction.

In the case that $G \models t < m(0, na)$, we have $t^H \le 0$. Hence $G \models t < m(0, a)$.

In the case that $G \models m(0, na) < t$, we have $G \models m(0, a) < t$ by $a > 0$.

In the case that $G \models I(m(0, na) + t)$, we have $t^H = 0$. Hence $G \models I(m(0, a) + t)$.

In the case that $G \models R(m_1(0, na) + s_1, \dots, m_l(0, na) + s_l)$, since R^G depends only on R^H, $G \models R(m_1(0, a) + s_1, \dots, m_l(0, a) + s_l)$.

Hence, if $a > 0$, then $G \models \theta_n((0, a))$. Similarly, if $a < 0$, then $G \models \theta_n((0, a))$. It follows that the ordered abelian group K is divisible.

Last we show that if G is recursively axiomatizable, then so is H. However we can show it like the proof of Theorem 4. □

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