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SUBSPACES OF THE SORGENFREY LINE AND THEIR PRODUCTS

By

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Abstract. In this article we study the products of subspaces of the Sorgenfrey line $\mathscr{S}$. Using an idea by D. K. Burke and J. T. Moore [2] we prove in particular the following:

Let X_i, $i = 1, \dots, n$, $n \geq 1$, be subspaces of $\mathscr{S}$, where each X_i is uncountable. Then $X_1 \times \cdots \times X_n \times \mathscr{Q}$ can be embedded in $\mathscr{S}^{n+1}$ but can not be embedded in $\mathscr{S}^n$, where $\mathscr{Q}$ is the space of rational numbers with the natural topology.

This statement strengthens [2, Theorem 2.1].

1 Introduction

All spaces considered here are assumed to be completely regular. Recall (see for example [4]) that the Sorgenfrey line $\mathscr{S}$ is the real line $\mathscr{R}$ with the topology whose base is the family $\{[a,b) : a, b \in \mathscr{R}$ with $a < b\}$. It is well known that $\mathscr{S}$ is a first-countable, hereditarily Lindelöf, hereditarily separable, Baire space such that the product $\mathscr{S}^2$ is not normal. The space $\mathscr{S}$ has different nice properties (see for example [1], [2], [3], [8]). In particular, D. K. Burke and J. T. Moore proved the following [2, Theorem 2.1].

If $X_0, \dots, X_n$, $n \geq 1$, are uncountable subspaces of $\mathscr{S}$ then the product $X_0 \times \cdots \times X_n$ can not be embedded in $\mathscr{S}^n$.

This result shows that

(a) for any uncountable subspace X of $\mathscr{S}$, X^n is homeomorphic to X^m iff $n = m$ where n, m are positive integers;

(b) for a subspace X of $\mathscr{S}$ if the subspace X^n of $\mathscr{S}^n$ can be embedded in $\mathscr{S}^{n-1}$ then X is countable.

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Using an idea of their proof we shall prove the following.

Define $\mathscr{S}^{-1} = \{\varnothing\}$ and $\mathscr{S}^0 = \mathscr{Q}$, where $\mathscr{Q}$ is the space of rational numbers with the natural topology. Put also $q(m,n,p) = m+1$ if $n,m > 0$, and $q(m,n,p) = m$ otherwise, where m, n, p are integers ≥ 0.

Theorem 1.1. *Let $\mathscr{F}$ be a finite family of non-empty subsets of $\mathscr{S}$ which are either uncountable, or homeomorphic to $\mathscr{Q}$ or discrete. Let also m be the number of uncountable elements of $\mathscr{F}$, n the number of elements of $\mathscr{F}$ homeomorphic to $\mathscr{Q}$, p the number of discrete elements of $\mathscr{F}$ and $1 \leq n+m+p$. Then the product $\prod \mathscr{F}$ of all elements of $\mathscr{F}$ can be embedded in $\mathscr{S}^q$ and can not be embedded in $\mathscr{S}^{q-1}$, where $q = q(m,n,p)$.*

Observe that Theorem 1.1 strengthens the mentioned above [2, Theorem 2.1] because any uncountable subspace of $\mathscr{S}$ contains a copy of $\mathscr{Q}$ as we will see in Lemma 2.2.

Note also that any subspace of $\mathscr{S}$ is either uncountable, or countable with at least one limit point, or discrete (and of course countable).

The next result is not complete as we wanted.

Theorem 1.2. *Let $\mathscr{F}$ be any finite family of non-empty subsets of $\mathscr{S}$. Let m be the number of uncountable elements of $\mathscr{F}$, n the number of countable elements of $\mathscr{F}$ with at least one limit point, p the number of discrete elements of $\mathscr{F}$ and $1 \leq n+m+p$. If $m \leq 2$ then the product $\prod \mathscr{F}$ of all elements of $\mathscr{F}$ can be embedded in $\mathscr{S}^q$ and can not be embedded in $\mathscr{S}^{q-1}$, where $q = q(m,n,p)$.*

In particular,

Theorem 1.3. (i) *Let X_1 and X_2 be subspaces of $\mathscr{S}$. Then $X_1 \times X_2$ can be embedded in $\mathscr{S}$ iff X_1, X_2 are both countable or one of them is discrete.*

(ii) *Let X_i, $i = 1,2,3$, be subspaces of $\mathscr{S}$. Then $X_1 \times X_2 \times X_3$ can be embedded in $\mathscr{S}$ iff all X_i, $i = 1,2,3$, are countable or two of them are discrete. $X_1 \times X_2 \times X_3$ can be embedded in $\mathscr{S}^2$ iff at least two of X_i, $i = 1,2,3$, are countable, or one of them is discrete.*

Problem 1.1. Can one remove the condition $m \leq 2$ in Theorem 1.2?

A positive answer on this question would also evidently strengthen Theorem 1.1.

REMARK 1.1. There is an analog of Theorem 1.2 for the space $\mathscr{R}$ of real numbers with the natural topology. Really, define $\mathscr{R}^{-1} = \{\emptyset\}$ and $\mathscr{R}^0 = \mathscr{P}$, where $\mathscr{P}$ is the space of irrational numbers with the natural topology. Note that any subspace of $\mathscr{R}$ is either one-dimensional (and so contains an interval), or zero-dimensional with at least one limit point, or discrete. Using in particular Brouwer theorem about the invariance of internal points and the theorem about the universality of $\mathscr{P}$ for zero-dimensional spaces with countable bases one can prove the following:

Let $\mathscr{F}$ be any finite family of non-empty subsets of $\mathscr{R}$. Let m be the number of one-dimensional elements of $\mathscr{F}$, n the number of zero-dimensional elements of $\mathscr{F}$ with at least one limit point, p the number of discrete elements of $\mathscr{F}$ and $1 \leq n + m + p$. Then the product $\prod \mathscr{F}$ of all elements of $\mathscr{F}$ can be embedded in $\mathscr{R}^q$ and can not be embedded in $\mathscr{R}^{q-1}$, where $q = q(m, n, p)$.

2 Preliminaries

A subset $A \subset \mathscr{R}$ with the topology induced from $\mathscr{S}$ will be denoted by $A_{\mathscr{S}}$. The notation $X \approx Y$ means that the spaces X and Y are homeomorphic. Our terminology follows [4].

We will continue with some properties of subspaces of $\mathscr{S}$.

Countable subspaces properties:

(1) Every countable subspace of $\mathscr{S}^k$, $k \geq 1$, has a countable base (readily);

(2) Every countable space with a countable base can be embedded in $\mathscr{Q}$ (see for example [6, Theorem 2, page 296]);

(3) Every countable space with a countable base and which has no isolated points is homeomorphic to $\mathscr{Q}$ (see for example [7, Theorem 1.9.6]);

LEMMA 2.1. (i) $\mathscr{Q} \approx \mathscr{Q}_{\mathscr{S}}$;

(ii) *For every open non-empty subspace U of $\mathscr{Q}$, we have $U \approx \mathscr{Q}$;*

(iii) *If $\mathscr{Q} = Q_1 \cup \cdots \cup Q_n$, $n \geq 1$, then there is an index m and a subspace P of Q_m such that $P \approx \mathscr{Q}$;*

(iv) *If $X_1, \ldots, X_n$, $n \geq 1$, are countable subspaces of $\mathscr{S}$ then $X_1 \times \cdots \times X_n$ can be embedded in $\mathscr{Q}$ (and hence in $\mathscr{Q}_{\mathscr{S}}$ and in $\mathscr{S}$).*

PROOF. Observe that the points (i) and (ii) are simple corollaries of the properties (1) and (3). The point (iv) is a corollary of the properties (1) and (2). In order to prove the point (iii) it is enough to show that if $\mathscr{Q} = A \cup B$ then either A contains a subspace $C \approx \mathscr{Q}$ or there is an open interval $(a, b) \subset \mathscr{R}$ such that $(a, b) \cap \mathscr{Q} \subset B$. Really, on the first step consider the system v_1 of open intervals

$(n, n+1)$, $n \in \mathscr{Z}$. Either there is an element E of v_1 disjoint from A and we have done by the point (ii) or we can choose from each interval of the system v_1 a point from A. Denote the chosen set by A_1. On the second step consider the system v_2 of open intervals $\left(a, a+\frac{1}{2^1}\right)$, $\left(a+\frac{1}{2^1}, b\right)$, $(a,b) \in v_1$. Either there is an element E of v_2 disjoint from A and we have done by the point (ii) or we can choose from each interval of the system v_2 a point from A. Denote the chosen set by A_2. Continue by this way we either will find an open interval disjoint from A or construct a countable sequence $A_1, A_2, \dots$ of subsets of A. Observe that the system v_{i+1} consists of the open intervals $\left(a, a+\frac{1}{2^i}\right)$, $\left(a+\frac{1}{2^i}, b\right)$, $(a,b) \in v_i$. Denote $C = \bigcup_{i=1}^{\infty} A_i$. Observe that the set $C \subset A$ is countable and without isolated points. So $C \approx \mathscr{Q}$ by the property (3). The lemma is proved.

Uncountable subspaces properties:

(4) Every uncountable subspace A of $\mathscr{S}^k$, $k \geq 1$, has the weight $wA > \aleph_0$ (readily);

(5) For every uncountable subspace A of $\mathscr{S}$ there is a subspace $B \subset A$ such that each open non-empty subspace of B is uncountable (see for example [8, Lemma 6.1]);

(6) Every uncountable subspace A of $\mathscr{S}$ contains an infinite, closed in $\mathscr{S}$, discrete subspace. So A is not compact ([5, Corollary 1]).

LEMMA 2.2. *Every uncountable subspace A of $\mathscr{S}$ contains a subspace homeomorphic to $\mathscr{Q}$.*

PROOF. By property (5) there is a subspace B of A such that each open non-empty subspace of B is uncountable. We will construct a subspace of B which is homeomorphic to $\mathscr{Q}$. Consider the open cover v_1 of $\mathscr{S}$ consisting of half-open intervals $[n, n+1)$, $n \in \mathscr{Z}$. From each element E of v_1 such that $E \cap B \neq \varnothing$ choose a point from B. Denote the chosen set by B_1. For every $i \geq 1$ consider the open cover v_{i+1} of $\mathscr{S}$ consisting of half-open intervals $\left[a, a+\frac{1}{2^i}\right)$, $\left(\left[a+\frac{1}{2^i}, b\right)\right.$, $(a,b) \in v_i$. From each element E of v_{i+1} such that $E \cap B \neq \varnothing$ choose a point from $B \backslash (B_1 \cup \cdots B_i)$. Denote the chosen set by B_{i+1}. Construct the sequence of countable disjoint subsets $B_1, B_2, \dots$ of B. Denote $C = \bigcup_{i=1}^{\infty} B_i$. Observe that C is countable and has no isolated points. So the subspace C of A is homeomorphic to $\mathscr{Q}$ by the properties (1) and (3). The lemma is proved.

REMARK 2.1. Observe that every subset of $\mathscr{S}$ is either uncountable (and hence containing according to Lemma 2.2 a lot of limit points), or countable with at least one limit point, or discrete.

It is convenient to follow some notations and facts from [2]. An element $x \in \mathcal{S}^n$ is viewed as a finite sequence $x = (x_i)_{i \le n}$. For $0 \le k \le n$, $x \in \mathcal{S}^n$ and $V \subset \mathcal{S}^n$ let

$$\delta_k^n(V, x) = \{y \in V : |\{i \le n : x_i \ne y_i\}| = k\}.$$

This will be used when V is a basic open nbd of x of the form $B_n[x, \varepsilon) = \prod_{i \le n}[x_i, x_i + \varepsilon)$ for $\varepsilon > 0$. Observe that for such V, $\{\delta_k^n(V, x) : 0 \le k \le n\}$ is a partition of V such that $\bigcup_{i=k}^n \delta_i^n(V, x)$ is open in $\mathcal{S}^n$ for any $k \le n$. In addition, for $1 \le k \le n$, $\delta_k^n(V, x)$ is the topological sum of finitely many subspaces of $\mathcal{S}^k$ and so it can be embedded in $\mathcal{S}^k$ (observe also that $\delta_0^n(V, x) = \{x\}$).

3 Products of Subspaces of $\mathcal{S}$

We continue with a statement whose proof follows the base step of induction from [2, Theorem 2.1].

Theorem 3.1. *Let B be an uncountable subspace of $\mathcal{S}$ and for each $b \in B$ let $A(b)$ be a subspace of $\mathcal{S}$ with a limit point $p(b)$. Then the subspace $C = \bigcup_{b \in B}(A(b) \times \{b\})$ of $\mathcal{S}^2$ can not be embedded in $\mathcal{S}$.*

Proof. Assume that there is an embedding $f : C \to \mathcal{S}$ of C into $\mathcal{S}$. Then the mapping $g = f \times id : C \times \mathcal{S} \to \mathcal{S}^2$ is also an embedding. Define

$$E = \bigcup_{b \in B} A(b) \times \{(b, -b)\} \subset C \times \mathcal{S} \subset \mathcal{S}^3.$$

Observe that E is the topological sum of subspaces $E(b) = A(b) \times \{(b, -b)\} \approx A(b)$, $b \in B$, of $\mathcal{S}^3$, each of which embeds in $\mathcal{S}^2$ by g. Let $F = g(E) \subset \mathcal{S}^2$. Observe that F is the topological sum of $F(b) = g(E(b)) \approx A(b)$, $b \in B$. For each $b \in B$, put $x(b) = g(\{p(b)\} \times \{(b, -b)\}) \in F(b)$ (observe that this point is a limit point for $F(b)$) and choose $\varepsilon(b) > 0$ such that $V(b) = B_2[x(b), \varepsilon(b))$ is disjoint from $F(b*)$ for all $b* \ne b$, $b* \in B$.

Recall that the space $\mathcal{S} \times \mathcal{R}$ is hereditarily Lindelöf. For $j = 1, 2$, let σ_j denote the topology on the product $Z_1 \times Z_2$, where $Z_j = \mathcal{S}$ and $Z_i = \mathcal{R}$ for $i \ne j$. These two spaces are of course homeomorphic and hereditarily Lindelöf. Observe that for every $j = 1, 2$, the hereditarily Lindelöf topology σ_j tells us that $(int_{\sigma_j} V(b)) \cap F(b) = \varnothing$ for all but at most countably many $b \in B$. So, we can find $b \in B$ such that $F(b)$ is disjoint from the union $(int_{\sigma_1} V(b)) \cup (int_{\sigma_2} V(b))$. Observe also that

$$V(b) \backslash ((int_{\sigma_1} V(b)) \cup (int_{\sigma_2} V(b))) = \delta_0^2(V(b), x(b)).$$

But $x(b) \in V(b) \cap F(b) \subset \delta_0^2(V(b), x(b)) = x(b)$. So the point $x(b) = V(b) \cap F(b)$ is an open subset of $F(b)$. This is a contradiction because $x(b)$ is a limit point of $F(b)$. The theorem is proved.

Corollary 3.1. *Let B be an uncountable subspace of $\mathscr{S}$ and A a subspace of $\mathscr{S}$ with a limit point p. Then the subspace $C = A \times B$ of $\mathscr{S}^2$ can not be embedded in $\mathscr{S}$. Moreover, there is an uncountable subset E of B such that for each point $q \in \{p\} \times E$, every open nbd of q in $A \times E$ can not be embedded in $\mathscr{S}$.*

Proof. Observe that any open nbd of p in A has p as a limit point. Apply now the property (5).

Corollary 3.2. *Let B be an uncountable subspace of $\mathscr{S}$ and A a subspace of $\mathscr{S}$ homeomorphic to $\mathscr{Q}$. Then the subspace $C = A \times B$ of $\mathscr{S}^2$ can not be embedded in $\mathscr{S}$. Moreover, if every open non-empty subspace of B is uncountable then no open non-empty subspace of $A \times B$ can be embedded in $\mathscr{S}$. In general, there is a subspace E of B such that no open non-empty subspace of $A \times E$ can be embedded in $\mathscr{S}$.*

Proof. Lemma 2.1 (ii) together with the property (5) and Corollary 3.1 prove the statement.

Proposition 3.1. *Let A be a discrete subspace of $\mathscr{S}$ and B a subspace of $\mathscr{S}$. Then $A \times B$ can be embedded in $\mathscr{S}$.*

Proof. Observe first that A is countable. Recall that for any $n \in \mathscr{Z}$, $[n, n+1)_{\mathscr{S}} \approx \mathscr{S}$. Note now that $\mathscr{S}$ is the topological sum of $[n, n+1)_{\mathscr{S}}$, $n \in \mathscr{Z}$, which is homeomorphic to $\mathscr{S} \times \mathscr{Z}$. From this fact the statement follows.

Proof of Theorem 1.3 (i). By Remark 2.1 there is a decomposition of the class of all subspaces of $\mathscr{S}$ in the three disjoint subclasses. According to that there are six different types of products. Now Lemma 2.1, Corollary 3.1 and Proposition 3.1 prove the statement.

Let $p_i : \mathscr{S}^2 \to \mathscr{S}$, $i = 1, 2$, be the projections of $\mathscr{S}^2$ onto i-th factor or the restrictions of these projections on certain subsets of $\mathscr{S}^2$. We continue with a couple of examples following Proposition 3.1.

Example 3.1. Let $A = \left(\{0\} \cup \left\{\frac{1}{i} : i = 1, 2, \ldots\right\}\right) \times \mathscr{S} \subset \mathscr{S}^2$. Recall that A can not be embedded in $\mathscr{S}$ by Corollary 3.1. But A is the union $A_1 \cup A_2$ of two subspaces such that each A_i can be embedded in $\mathscr{S}$. In fact, put $A_1 = \{0\} \times \mathscr{S}$ (a closed subspace of A) and $A_2 = \{\frac{1}{i} : i = 1, 2, \dots\} \times S$ (an open subspace of A). (Observe that $\mathscr{Q} \times \mathscr{S}$ can not be written as a finite union of subspaces which can be embedded in $\mathscr{S}$ as we will see in Lemma 4.1.)

Example 3.2. Fix an embedding of $\mathscr{Q} = \{q_1, q_2, \dots\}$ into $\mathscr{S}$. Define

$$A = \bigcup_{n=1}^{\infty} ([n, n+1) \times \{q_n\}) \subset \mathscr{S}^2.$$

Observe that A is the topological sum of the subspaces $[n, n+1) \times \{q_n\}$, $n = 1, 2, \dots$ where each term $[n, n+1) \times \{q_n\}$ is homeomorphic to $\mathscr{S}$. So $A \approx \mathscr{S}$. But $p_1(A) = \mathscr{S}$ and $p_2(A) = \mathscr{Q}$. Moreover, for every point $q \in \mathscr{Q}$ we have $p_2^{-1}q \approx \mathscr{S}$. This example shows that the uncountability of B in Theorem 3.1 is extremely essential. Compare also this example with Corollary 3.2.

We have more example concerning Theorem 3.1.

Example 3.3. Let A be any uncountable subspace of $\mathscr{S}$. Then the subspace $B = \{(a, -a) : a \in A\}$ of $\mathscr{S}^2$, being non-Lindelöf, can not be embedded in $\mathscr{S}$. Observe that $p_1(B) = A$ and $p_2(B) = -A = \{-a : a \in A\}$. Moreover, $|p_1^{-1}(a)| = |p_2^{-1}(-a)| = 1$ for any $a \in A$. A generalization of this example: Let E be a subspace of $\mathscr{S}^2$ which contains the graph of a strictly decreasing function from $F \subset \mathscr{S}$ to $\mathscr{S}$, where F is an uncountable subset of $\mathscr{S}$. Then E can not be embedded in $\mathscr{S}$.

Theorem 1.3 (i) arises the following

Problem 3.1. Determine what subsets of $\mathscr{S}^2$ can be embedded in $\mathscr{S}$.

The proof of the following statement follows also the idea of the proof from [2, Theorem 2.1].

Theorem 3.2. *Let B be an uncountable subspace of $\mathscr{S}$ and for each $b \in B$ let $A(b)$ be a subspace of $\mathscr{S}^n$, $n \geq 2$, such that no open non-empty subspace of $A(b)$ can be embedded in $\mathscr{S}^{n-1}$. Then the subspace $C = \bigcup_{b \in B} (A(b) \times \{b\})$ of $\mathscr{S}^{n+1}$ can not be embedded in $\mathscr{S}^n$.*

Proof. Assume that there is an embedding $f : C \to \mathscr{S}^n$ of C into $\mathscr{S}^n$. Then the mapping $g = f \times id : C \times \mathscr{S} \to \mathscr{S}^{n+1}$ is also an embedding. Define

$$E = \bigcup_{b \in B} A(b) \times \{(b, -b)\} \subset C \times \mathscr{S} \subset \mathscr{S}^{n+2}.$$

Observe that E is the topological sum of subspaces $E(b) = A(b) \times \{(b, -b)\} \approx A(b)$, $b \in B$, of $\mathscr{S}^{n+2}$, where each $E(b)$ can be embedded in $\mathscr{S}^{n+1}$ by g. Let $F = g(E) \subset \mathscr{S}^{n+1}$. Observe that F is the topological sum of $F(b) = g(E(b)) \approx A(b)$, $b \in B$. For each $b \in B$, pick a point $x(b) \in F(b)$ and choose $\varepsilon(b) > 0$ such that $V(b) = B_{n+1}[x(b), \varepsilon(b))$ is disjoint from $F(b*)$ for all $b* \neq b$, $b* \in B$.

Recall that for any $n \in \mathscr{N}$ the space $\mathscr{S} \times \mathscr{R}^n$ is hereditarily Lindelöf. For $j = 1, \dots, n+1$, let σ_j denote the topology on the product $\prod_{i=1}^{n+1} Z_i$ where $Z_j = \mathscr{S}$ and $Z_i = \mathscr{R}$ for $i \neq j$. These $(n+1)$ spaces are of course pairwise homeomorphic and hereditarily Lindelöf. Observe that for every $j = 1, \dots, n+1$, the hereditarily Lindelöf topology σ_j tells us that $(int_{\sigma_j} V(b)) \cap F(b) = \varnothing$ for all but at most countably many $b \in B$. So, we can find $b \in B$ such that $F(b)$ is disjoint from the union $\bigcup_{i=1}^{n+1} (int_{\sigma_i} V(b))$. Observe also that

$$V(b) \backslash (\bigcup_{i=1}^{n+1} (int_{\sigma_i} V(b))) \subset \bigcup_{i=0}^{n-1} \delta_i^{n+1}(V(b), x(b)).$$

So

$$(*) \qquad x(b) \in V(b) \cap F(b) \subset \bigcup_{i=0}^{n-1} \delta_i^{n+1}(V(b), x(b)).$$

Now, for this b, pick the largest $k < n$ such that $F(b) \cap \delta_k^{n+1}(V(b), x(b)) \neq \varnothing$. Since

$$F(b) \cap \bigcup_{i=k}^{n+1} \delta_i^{n+1}(V(b), x(b)) = F(b) \cap \delta_k^{n+1}(V(b), x(b))$$

is open in $F(b)$ we see that

$$W = g^{-1}[F(b) \cap \delta_k^{n+1}(V(b), x(b))] \approx F(b) \cap \delta_k^{n+1}(V(b), x(b))$$

is open in $g^{-1}[F(b)] = E(b)$. Recall that W can not be embedded in $\mathscr{S}^{n-1}$ by assumption. In the same time the space $F(b) \cap \delta_k^{n+1}(V(b), x(b))$, which is homeomorphic to W, can be embedded in $\mathscr{S}^{n-1}$ by the construction (recall that $k < n$). This is a contradiction. The theorem is proved.

Corollary 3.3. *Let X_i, $i = 1, \dots, n$, $n \geq 2$, be subspaces of $\mathscr{S}$ such that $X_1 \approx \mathscr{Q}$ and for every X_i, $i \geq 2$, each open non-empty subspace of X_i is uncountable. Then $X_1 \times \cdots \times X_n$ can not be embedded in $\mathscr{S}^{n-1}$.*

Proof. Apply an obvious induction. The basis of the induction is Corollary 3.2.

Corollary 3.4. *Let X_i, $i = 1, \dots, n$, $n \geq 2$, be subspaces of $\mathscr{S}$ such that one of them is homeomorphic to $\mathscr{Q}$ and the others are uncountable. Then $X_1 \times \cdots \times X_n$ can not be embedded in $\mathscr{S}^{n-1}$.*

Proof. Apply the property (5) and Corollary 3.3.

Corollary 3.5 ([2, Theorem 2.1]). *Let X_i, $i = 1, \dots, n$, $n \geq 2$, be uncountable subspaces of $\mathcal{S}$. Then $X_1 \times \cdots \times X_n$ can not be embedded in $\mathcal{S}^{n-1}$.*

Proof. Apply Corollary 3.4 and Lemma 2.2.

Proof of Theorem 1.1. Lemma 2.1, Proposition 3.1 and Corollary 3.4 prove the statement.

Theorems 1.1 arises

Problem 3.2. Determine what subsets of $\mathcal{S}^n$ can be embedded in $\mathcal{S}^k$ for $1 \leq k < n$.

Some examples of subsets of $\mathcal{S}^n$ concerning Problem 3.2:

Example 3.4. Recall that $\mathcal{S} \approx ((0,1))_{\mathcal{S}} \approx ([0,1))_{\mathcal{S}} \approx X = (\{0\} \cup \bigcup_{i=1}^{\infty}(a_i, b_i))_{\mathcal{S}}$, where $0 < b_{i+1} < a_i < b_i$ for every i and $a_i \to 0$. Using this fact it is easy to establish that

(i) The subspace

$$A = ([0,1) \times \{0\} \times \{0\}) \cup (\{0\} \times [0,1) \times \{0\}) \cup (\{0\} \times \{0\} \times [0,1))$$

of $\mathcal{S}^3$ is homeomorphic to $\mathcal{S}$. Really, $A = A_1 \cup A_2 \cup A_3$, where

$$A_k = ([0,1))_{\mathcal{S}} = \left(\{0\} \cup \bigcup_{i=1}^{\infty}\left[\frac{1}{i+1}, \frac{1}{i}\right)\right)_{\mathcal{S}}, \quad k = 1,2,3.$$

For each $k = 1,2,3$ define a mapping $f_k : A_k \to X$ as follows. Put $f_k(0) = 0$, and for each $i \geq 1$ let $f_k|_{([1/(i+1),1/i))_{\mathcal{S}}}$ be any homemorphism between $\left(\left[\frac{1}{i+1}, \frac{1}{i}\right)\right)_{\mathcal{S}}$ and $(a_{3(i-1)+k}, b_{3(i-1)+k})_{\mathcal{S}}$. Put $f(x) = f_k(x)$ for any point $x \in A_k$. The mapping f is a homeomorphism between A and $X \approx \mathcal{S}$. Observe also that

$$(**) \qquad A = \bigcup_{i=0}^{1} \delta_i^3(V, (0,0,0)),$$

where $V = B_3[(0,0,0),1)$.

(ii) The subspace

$$B = (\{0\} \times [0,1) \times [0,1)) \cup ([0,1) \times \{0\} \times \{0\})$$

of $\mathcal{S}^3$ can be embedded in $\mathcal{S}^2$ but can not (readily) be embedded in $\mathcal{S}$.

Now we are ready to prove two statements necessary for Theorems 1.2 and 1.3 (ii).

Theorem 3.3. *Let B be an uncountable subspace of $\mathscr{S}$ and for each $b \in B$ let $A(b)$ be a subspace of $\mathscr{S}^2$ with a point $p(b)$ such that no open nbd of $p(b)$ in $A(b)$ can be embedded in $\mathscr{S}$. Then the subspace $C = \bigcup_{b \in B}(A(b) \times \{b\})$ of $\mathscr{S}^3$ can not be embedded in $\mathscr{S}^2$.*

Proof. Follow the proof of Theorem 3.2 but the points $x(b)$ let us pick up as in the proof of Theorem 3.1. Use then the inclusion $(*)$ from the proof of Theorem 3.2 and the equality $(**)$ from Example 3.4 (i).

Corollary 3.6. *Let B_1, B_2 be uncountable subspaces of $\mathscr{S}$ and A a subspace of $\mathscr{S}$ with a limit point p. Then the subspace $C = A \times B_1 \times B_2$ of $\mathscr{S}^3$ can not be embedded in $\mathscr{S}^2$. Moreover, there are uncountable subsets E_1, E_2 of B_1, B_2 respectively such that for each point $q \in \{p\} \times E_1 \times E_2$, no open nbd of q in $A \times E_1 \times E_2$ can be embedded in $\mathscr{S}^2$.*

Proof. Observe that any open nbd of p in A has p as a limit point. Apply now the property (5) and Corollary 3.1.

Proof of Theorem 1.3 (ii). Let us again use the decomposition from Remark 2.1 of the class of all subspaces of $\mathscr{S}$ in the three disjoint subclasses. According to that there are ten different types of products. Lemma 2.1, Corollary 3.6 and Proposition 3.1 prove the statement.

Proof of Theorem 1.2. Lemma 2.1, Proposition 3.1, Corollary 3.1 and Corollary 3.6 prove the statement.

A positive answer to the next question would give a positive answer to Problem 1.1.

Question 3.1. Let $n \geq 4$, $x \in \mathscr{S}^n$ and $V = B_n[x, \varepsilon)$. Can the set $\bigcup_{i=0}^{n-2} \delta_i^n(V, x)$ be embedded in $\mathscr{S}^{n-2}$?

Recall that for $n = 2, 3$ this is right.

Now in order to get a complete picture it is time to make some obvious comments concerning infinite products of subspaces of the Sorgenfrey line.

Denote by $\mathscr{D}$ the discrete two points space.

Proposition 3.2. *Let X be an uncountable space with $wX = \aleph_0$. Then X can not be embedded in $\mathscr{S}^n$ for any $n \in \mathscr{N}$. In particular, the Cantor space $\mathscr{C} = \mathscr{D}^{\aleph_0}$ and any its uncountable subspace can not be embedded in $\mathscr{S}^n$ for any $n \in \mathscr{N}$.*

Proof. Recall from (4) that any uncountable subspace A of $\mathscr{S}^n$, $n \geq 1$, has $wA > \aleph_0$.

Observe that from Proposition 3.2 we have also that the Cantor space can not be embedded in any countable union of subspaces of $\mathscr{S}^k$ for each $k \geq 1$.

Proposition 3.3. *Let τ, v be two infinite cardinals and $\tau < v$. Then $\mathscr{D}^v$ can not be embedded in $\mathscr{S}^\tau$.*

Proof. Really, assume that there is an embedding $f : \mathscr{D}^v \to \mathscr{S}^\tau$. Then $f(\mathscr{D}^v) \approx \mathscr{D}^v$ is compact and $w(f(\mathscr{D}^v)) = w(\mathscr{D}^v) = v$ ([E, p. 84]). By the property (6) there are countable subspaces Y_α, $\alpha \in \tau$, of $\mathscr{S}$ such that $f(\mathscr{D}^v) \subset \prod_{\alpha \in \tau} Y_\alpha$. Recall that by Lemma 2.1 each Y_α, $\alpha \in \tau$, has a countable base. Hence, $w(\prod_{\alpha \in \tau} Y_\alpha) \leq \tau < v$ (see for example [4, Theorem 2.3.23]). This is a contradiction.

Proposition 3.4. *Let τ be an infinite cardinal $\geq c$. Then $\mathscr{S}^\tau$ can be embedded in $\mathscr{D}^\tau$.*

Proof. Observe that $w(\mathscr{S}) = c$. So $\mathscr{S}$ can be embedded in $\mathscr{D}^c$ and hence $\mathscr{S}^\tau$ can be embedded in $(\mathscr{D}^c)^\tau \approx \mathscr{D}^\tau$.

4 Unions of Subspaces of $\mathscr{S}^k$ and Their Products

Recall that *two arrows space*, shortly *TAS*, (see for example [4, Exercise 3.10.C]) defined by Alexandroff and Urysohn, is the union $X = C_0 \cup C_1 \subset \mathscr{R}^2$, where $C_0 = \{(x,0) : 0 < x \leq 1\}$ and $C_1 = \{(x,1) : 0 \leq x < 1\}$, and the topology on X generated by the base consisting of sets of the form

$$\left\{(x,i) \in X : x_0 - \frac{1}{k} < x < x_0 \text{ and } i = 0,1\right\} \cup \{(x_0,0)\},$$

where $0 < x_0 \leq 1$ and $k = 1,2,\ldots$, and of sets of the form

$$\left\{(x,i) \in X : x_0 < x < x_0 + \frac{1}{k} \text{ and } i = 0,1\right\} \cup \{(x_0,1)\},$$

where $0 \leq x_0 < 1$ and $k = 1,2,\ldots$

It is easy to see that the TAS is compact and $|TAS| = c$. So by the property (6) the TAS can not be embedded in $\mathscr{S}^k$ for any $k \geq 1$. Observe that the TAS is the union of two copies of Sorgenfrey line. This motivates the following.

Define two sequences of classes of topological spaces as follows.

$$\mathscr{M}_k^{fin} = \{\text{unions of finitely many subspaces of } \mathscr{S}^k\} \quad \text{and}$$

$$\mathscr{M}_k = \{\text{unions of countably many subspaces of } \mathscr{S}^k\}, \quad \text{where } k \geq 1.$$

Put also $\mathscr{M}_\infty = \{\text{unions of countably many subspaces of } \mathscr{S}, \mathscr{S}^2, \mathscr{S}^3, \dots\}$. We start with obvious remarks about these classes.

Proposition 4.1. (a) $TAS \in \mathscr{M}_1^{fin}$;

(b) *Any space X from $\mathscr{M}_1^{fin}$ (or $\mathscr{M}_1$) is hereditarily Lindelöf and hereditarily separable*;

(c) $\mathscr{M}_k^{fin} \subset \mathscr{M}_k \subset \mathscr{M}_\infty$ *for any* $k \geq 1$;

(d) *If* $X \in \mathscr{M}_k^{fin}(\mathscr{M}_k)$ *and* $Y \in \mathscr{M}_m^{fin}(\mathscr{M}_m)$ *then* $X \times Y \in \mathscr{M}_{k+m}^{fin}(\mathscr{M}_{k+m})$.

The following lemma is one more corollary of Theorem 3.1.

Lemma 4.1. *Let B be an uncountable subspace of $\mathscr{S}$ and for each $b \in B$ let $A(b)$ be a subspace of $\mathscr{S}$. Let also $C = \bigcup_{b \in B}(A(b) \times \{b\})$.*

(a) *If for every $b \in B$ we have $A(b) \approx \mathscr{Q}$ and $C = \bigcup_{i=1}^n Y_i$ for some $n \geq 1$ then there is $k \leq n$ such that Y_k can not be embedded in $\mathscr{S}$*;

(b) *If for every $b \in B$ we have $A(b)$ is uncountable and $C = \bigcup_{i=1}^\infty Y_i$ then there is $k \geq 1$ such that Y_k can not be embedded in $\mathscr{S}$.*

Proof. (a) For each $b \in B$ by Lemma 2.1 (iii) there are $i(b) \leq n$ and subspace $E(b)$ of $A(b)$ such that $E(b) \times \{b\} \subset Y_{i(b)}$ and $E(b) \approx \mathscr{Q}$. Since B is uncountable then there are $k \leq n$ and an uncountable subspace B_1 of B such that for each $b \in B_1$ we have $i(b) = k$. By Theorem 3.1, $\bigcup_{b \in B_1}(E(b) \times \{b\}) \subset Y_{i(b)}$ can not be embedded in $\mathscr{S}$.

(b) This point is proved in the same manner as (a).

By Lemma 4.1 we have readily

Theorem 4.1. (a) *Let $X \in \mathscr{M}_1^{fin}$ and X be uncountable. Then $X \times \mathscr{Q} \notin \mathscr{M}_1^{fin}$ but $X \times \mathscr{Q} \in \mathscr{M}_1$.*

(b) *Let $X, Y \in \mathscr{M}_1$ and X, Y be uncountable. Then $X \times Y \notin \mathscr{M}_1$ but $X \times Y \in \mathscr{M}_2$.*

What could be done else? Well, I think that it could be interesting to look what theorems from the previous section are valid for the *TAS*.

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