TSUKUBA J. MATH.
Vol. 30 No. 1 (2006), 131–135

A SPACE OF MINIMAL TORI WITH ONE END AND CYCLIC SYMMETRY

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Abstract. We will explicitly give the defining equation of the moduli space of symmetric minimal tori with one end.

1. Introduction

The 2-dimensional manifold M of a complete minimal surface $X : M \to \mathbf{R}^3$ of finite total curvature is conformally equivalent to a compact Riemann surface $\overline{M}$ with finite number of points $\{p_1, \dots, p_r\}$ removed ([3]). The removed points are called the ends of the surface. The Weierstrass representation (g, ϕ) of X is a pair consisting of a meromorphic function g on M and a holomorphic one-form ϕ on M such that

$$X(p) = \mathrm{Re} \int^p (\Phi_1, \Phi_2, \Phi_3),$$

$$\Phi_1 = \frac{1}{2}(1 - g^2)\phi, \quad \Phi_2 = \frac{\sqrt{-1}}{2}(1 + g^2)\phi, \quad \Phi_3 = g\phi.$$

The one-forms Φ_1, Φ_2, and Φ_3 are holomorphic on M and extends meromorphically to $\overline{M}$. They have poles at the ends of highest order greater than or equal to 2 and have no real periods on M. The one-form $g^2\phi$ must be nonzero holomorphic one form at a pole of g on M. The meromorphic function g is the stereographic projection of the Gauss map of X. The total curvature of X is -4π times the degree of g.

When the total curvature of X is $-4\pi m$ and X has a unique end $(m = 1, 2, \dots)$, the branch order of g at the end is equal to or smaller than $m - 1$. We will assume that g is maximally branched at the end, too. Since the Enneper

Partly supported by the Grant-in-Aid for Young Scientists (B), The Ministry of Education, Culture, Sports, Science and Technology, Japan.
Received October 8, 2004.
Revised July 11, 2005.

surface of total curvature -4π has this property, we will call this surface a minimal surface of *Enneper type*. The Chen-Gackstatter surface of genus one with total curvature -8π has a unique end of Enneper type. But it is not a minimal surface of Enneper type.

In the case of minimal surfaces of Enneper type with total curvature $-4\pi m$, we may assume that the Gauss map g has a unique pole at the end since the degree of g is m and the order of pole at the end is m. Then the meromorphic one-form ϕ has a unique pole at the end or a holomorphic one-form on $\overline{M}$. The degree of meromorphic or holomorphic one-form on a compact Riemann surface of genus s is $2s-2$ by the Riemann-Roch theorem ([1]). Hence, in the first case, $\overline{M}$ is the Riemann sphere $\mathbf{C}P^1 = \mathbf{C} \cup \{\infty\}$. In the second case, $\overline{M}$ is a conformal torus. Although the classification in the case of genus zero is merely elementary algebraic exercise, that in the case of genus one is more involved since the moduli space of conformal tori with one point removed is parameterized by $\mathbf{C}\backslash\{1,-1\}$.

We will add the assumption that $\overline{M}$ is a conformal torus and that $X(M)$ has symmetry R such that $X(M)$ is invariant under a rotation of 90° around a vertical axis followed by a reflection about a horizontal plane ([2]). This paper is devoted to the arising moduli space, where two minimal surfaces are identified if they are congruent up to orientation-preserving isometries of $\mathbf{R}^3$. The moduli space is filtered by the moduli space of the surfaces with total curvature $-4\pi(2n+3)$ which contains a $2n+1$ dimensional smooth subset $(n = 0, 1, 2, \dots)$.

The author would like to express particular gratitude to the referee, who subsequently improved the original manuscript.

2. The Theorem

We will consider a conformal torus as a two-sheeted holomorphic branched covering of a Riemann sphere $\mathbf{C}P^1 = \mathbf{C} \cup \{\infty\}$ ([1]). Then any conformal torus is written as

$$M_a := \{(w, z) \in \mathbf{C} \times \mathbf{C} \mid w^2 = (z-a)(z^2-1)\} \cup \{(\infty, \infty)\}, \tag{2.1}$$

where $a \in \mathbf{C}\backslash\{-1, 1\}$. The map $z : \overline{M}_a \to \mathbf{C}P^1$ is the holomorphic branched covering map. The points

$$P_0 = z^{-1}(a), \quad P_1 = z^{-1}(1), \quad P_2 = z^{-1}(-1), \quad P_3 = z^{-1}(\infty) = (\infty, \infty)$$

are the branch points of z with branch number 1.

The functions z and w are meromorphic on $\overline{M}_a$ whose divisors are

$$(z) = 2 \cdot P_0 - 2 \cdot P_3, \quad (w) = \sum_{i=0}^{2} 1 \cdot P_i - 3 \cdot P_3.$$

The one-form dz/w is holomorphic on $\overline{M}_a$. The complex vector space of meromorphic functions with a unique pole at P_3 is spanned by the meromorphic functions z^i and $z^i w$ which are linearly independent over $\mathbf{C}$ $(i = 0, 1, \dots)$.

Let $M_a := \overline{M}_a \backslash \{P_3\}$. If (g, ϕ) is the Weierstrass representation of a minimal surface $X : M_a \to \mathbf{R}^3$ of Enneper type such that g has a unique pole at P_3, then $g = Q(z)$ or $Q(z)w$ and $\phi = A\,dz/w$, where $Q(z)$ is a polynomial of z and $A \in \mathbf{C}\backslash\{0\}$. The symmetry R induces a conformal automorphism J of $\overline{M}_a$ such that $J(P_3) = P_3$ and $R \circ X = X \circ J$ ([2]).

Lemma 2.1. *The conformal automorphism J is holomorphic.*

Proof. Let $p \in M_a$ be a fixed point of J. Then $(R \circ X)(p) = (X \circ J)(p) = X(p)$. Since $X(p)$ is a fixed point of R, the fixed point of J on M_a is nonempty and finite. Hence J preserves the orientation of M_a. □

Hence $z \circ J = Bz + C$ $(B \neq 0)$. Since the holomorphic automorphism of $\overline{M}_a$ preserves the branch point set $\{P_0, P_1, P_2, P_3\}$ of z, we have $a = 0$ and $z \circ J = -z$. Thus $w \circ J = iw$.

Let γ be a closed curve in M_0 such that $z(\gamma)$ is a loop winding once around -1 and 0 and leaves 1 in the non-bounded component of $\mathbf{C}\backslash z(\gamma)$, $R_n(z) = \int_\gamma z^n w\,dz \in \mathbf{R}\backslash\{0\}$, $S := \int_\gamma dz/w \in \mathbf{R}\backslash\{0\}$, and $F_n(c) = \sum_{j=0}^n R_{4j} c_j^2 + 2\sum_{0 \le j < k \le n} R_{2j+2k} c_j c_k$.

Theorem 2.2. *The moduli space can be identified with $\bigcup_{n=0}^\infty \mathscr{M}_n$, where $\mathscr{M}_n$ is the moduli space of the surfaces with total curvature $-4\pi(2n+3)$ defined by*

$$\mathscr{M}_n := \{(A, c) \in \mathbf{R} \times \mathbf{C}^{n+1} \mid A > 0, c_n \neq 0, S - F_n(c) = 0\},$$

which contains a real $(2n+1)$-dimensional smooth subset $(n = 0, 1, 2, \dots)$.

Proof. Let N is the Gauss map and $g = \pi \circ N$, where π is the stereographic projection. Since R is orientation-reversing, $N \circ J = -R \circ N$ ([2], (2.1)). Hence $g \circ J = \pi \circ I \circ R \circ N$ by the symmetry of the surface, where $I(x) = -x$ for $x \in \mathbf{R}^3$. Hence $g \circ J = ig$. Thus $g = Q(z)w$, where $Q(-z) = Q(z)$. Hence the total curvature of the surfaces are $-4\pi(2n+3)$ $(n = 0, 1, 2, \dots)$.

The one-form Φ_i does not have residue at the end since it has a unique pole $(i = 1, 2, 3)$. The first homology $H^1(M_a)$ of M_a is generated by $\{\gamma, J\gamma\}$. Since $\Phi_3 = Q(z)\,dz$, Φ_3 is exact. Let $g = \sum_{j=0}^{n} c_j z^{2j} w$ and $\phi = A\,dz/w$, where c_j, $A \in \mathbf{C}$ $(j = 0, 1, 2, \dots, n)$. Since $J^*\phi = -\phi$ and $g \circ J = ig$, we have

$$\mathrm{Re}\int_{\gamma}(1-g^2)\phi = -\mathrm{Re}\int_{J(\gamma)}\sqrt{-1}(1+g^2)\phi$$

$$\mathrm{Re}\int_{\gamma}\sqrt{-1}(1+g^2)\phi = \mathrm{Re}\int_{J(\gamma)}(1-g^2)\phi.$$

Thus the period condition is reduced to

$$\overline{\int_{\gamma}\phi} - \int_{\gamma} g^2\phi = 0,$$

that is

$$S\bar{A} - AF_n(c) = 0 \quad (n = 0, 1, 2, \dots).$$

Hence the set $\bigcup_{n=0}^{\infty}\mathcal{N}_n$, $\mathcal{N}_n := \{(A, c_0, \dots, c_n) \in \mathbf{C}^{n+2} \mid A \neq 0, c_n \neq 0, S\bar{A} - AF_n(c) = 0\}$ contains all the surfaces up to rigid motion in $\mathbf{R}^3$. The action of the group of rotation around vertical axis with angle t induces the action of the group $S^1 = \{e^{ti} \mid t \in \mathbf{R}\}$ on $\mathcal{N}_n$ as $e^{ti} \cdot (A, c) := (Ae^{-ti}e^{ti}c)$, where $t \in \mathbf{R}$, $e^{ti}c := (e^{ti}c_0, e^{ti}c_1, \dots, e^{ti}c_n)$, and $(A, c) \in \mathcal{N}_n$. Hence the moduli space can be identified with $\bigcup_{n=0}^{\infty}\mathcal{M}_n$. Each $\mathcal{M}_n$ has an element $\pm p_n := (A, 0, \dots, 0, \pm\sqrt{S/R_{4n}})$ $(A > 0)$. Since $(\partial F_n/\partial c_n)(p_n) = 2R_{4n}\sqrt{S/R_{4n}} \neq 0$, $\mathcal{M}_n$ contains $(2n+1)$ dimensional smooth subset around $\pm p_n$. □

COROLLARY 2.3. *There exists a unique complete minimal torus of Enneper type with total curvature -12π having symmetry R up to homotheties and isometries of $\mathbf{R}^3$.*

PROOF. The solution of $S - F_0(c) = 0$ is $\pm\sqrt{S/R_0}$. The surface corresponding to $(g, \phi) = (-\sqrt{S/R_0}w, A\,dz/w)$ $(A > 0)$ is the reflection of the surface corresponding to $(g, \phi) = (\sqrt{S/R_0}w, A\,dz/w)$ about the horizontal plane. Since A is the parameter of homotheties, the corollary holds. □

REMARK 2.4. Figure 1 shows the surface in Corollary 2.3. The left of Figure 2 shows the surfaces cut by the plane $\{x_1 = 0\}$ and the right shows the surfaces cut by the plane $\{x_3 = 0\}$.

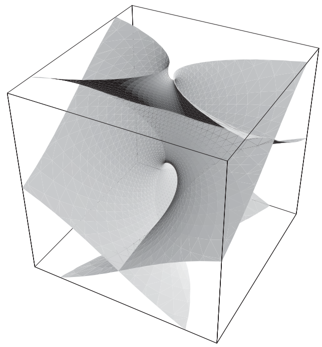

Figure 1. The surface of genus one with one end of total curvature -12π.

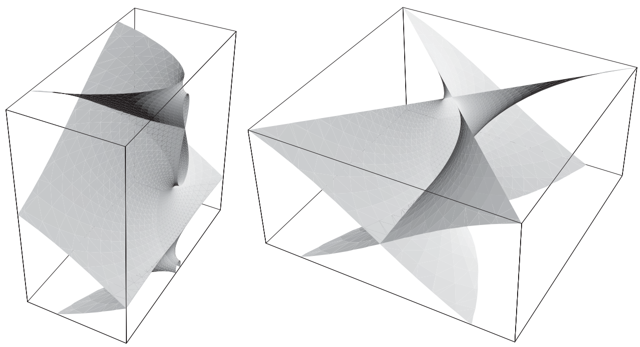

Figure 2. The surfaces cut by $\{x_1 = 0\}$ (left) and by $\{x_3 = 0\}$ (right).

References

[1] Farkas, H. M. and Kra, I., Riemann surfaces, Graduate Texts in Mathematics, Vol. 71, Second edition, Springer-Verlag, New York, 1992.

[2] Hoffman, D. and Meeks, W. H., III, Embedded minimal surfaces of finite topology, Ann. of Math. (2) **131** no. 1 (1990), 1–34.

[3] López, F. J. and Martín, F., Complete minimal surfaces in $\mathbf{R}^3$, Publ. Mat **43** no. 2 (1999), 341–449.

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