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ON THE BRANCHING THEOREM OF THE PAIR $(F_4, Spin(9))$

By

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Introduction

Let G be a compact connected Lie group and K be a closed subgroup. A finite dimensional complex irreducible representation $V^G(\lambda)$ of G with highest weight λ is decomposed into a direct sum of irreducible representations $V^K(\mu)$ of K with highest weight μ

$$V^G(\lambda) = \sum_{\mu} m(\lambda, \mu) V^K(\mu).$$

Let $V = G \times_K V^K(\mu)$ be the irreducible complex homogeneous vector bundle on $M = G/K$. By a theorem of Peter and Weyl, the space of sections $\Gamma(V)$ of V is a unitary direct sum of finite dimensional representations of G. By the Frobenius reciprocity theorem, the multiplicity of a complex irreducible representation $V^G(\lambda)$ in $\Gamma(V)$ conincides with the coefficient $m(\lambda, \mu)$.

Branching theorem of the pair $(F_4, Spin(9))$ was studied first by Lepowsky ([1], [2]). His result is not sufficient to decompose the space of sections $\Gamma(V)$, for the main interest of Lepowsky's work is in those pairs (λ, μ) with $m(\lambda, \mu) = 1$ (see also [3]). In the previous paper [4], the author carried the Lepowsky's calculation forward for the purpose of giving the decomposition of the space of complex p-form on the Cayley projective plane $P^2(\mathbf{Ca})$. Actually we obtained the decomposition for $p \leq 5$ and applied them to calculate the spectra of Laplacian Δ^p acting on p-forms of $P^2(\mathbf{Ca})$.

In [5], F. Sato studied the stability of branching coefficient. Roughly speaking, the branching coefficnent $m(\lambda, \mu)$ satisfies $m(\lambda, \mu) = m(\lambda + \lambda_0, \mu)$ if λ_0 is a spherical representation of (G, K) and λ is sufficiently large.

In this note we will prove the following

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Theorem 1. *Let* $\lambda = \sum_{i=1}^{4} a_i\varepsilon_i = \sum_{i=1}^{4} u_i\lambda_i$ *be a dominant integral weight of* F_4 *and* $\mu = \sum_{i=1}^{4} b_i\varepsilon_i$ *be a dominant integral weight of* $Spin(9)$. *If* $a_1 - a_3 = u_1 + u_2 + u_3 + u_4 \geq b_1 + b_2 + b_3 + b_4 + 2$ *then we have* $m(\lambda,\mu) = m(\lambda + \lambda_4, \mu)$.

Theorem 2. *Let* $\lambda = \sum_{i=1}^{4} a_i\varepsilon_i = \sum_{i=1}^{4} u_i\lambda_i$ *be a dominant integral weight of* F_4 *and* $\mu = \sum_{i=1}^{4} b_i\varepsilon_i$ *be a dominant integral weight of* $Spin(9)$. *If* $a_2 + a_4 = u_1 + u_2 + u_3 \geq b_1 + b_2 + b_3 + b_4 + 3$ *then the coefficient* $m(\lambda,\mu)$ *is equal to* 0.

Using the above results we will give tables of branching coefficents and calculate the spectra of Laplacian Δ^p on the Cayley projective plane acting on p-forms $(p = 6, 7, 8)$.

1. Root and Weight System of F_4 and $Spin(9)$

Let T be a maximal torus of $Spin(9)$. We denote by $\mathfrak{f}_4$, $\mathfrak{so}(9)$ and $\mathfrak{t}$ the Lie algebras of F_4, $Spin(9)$ and T respectively. The complexification $\mathfrak{t}^{\mathbf{C}}$ of $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{f}_4^{\mathbf{C}}$ and $\mathfrak{so}(9)^{\mathbf{C}}$. Under a suitable choise of an orthonormal base $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ of $\mathfrak{t}^{\mathbf{C}}$, the set $D(F_4)$ [resp. $D(Spin(9))$] of dominant weights of F_4 [resp. $Spin(9)$] with respect to the lexicographic order $>$ in $\mathfrak{t}^{\mathbf{C}}$ defined by

$$\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \varepsilon_4 > 0.$$

are

$$D(F_4) = \left\{ \sum_{i=1}^{4} a_i\varepsilon_i \mid a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0, a_1 \geq a_2 + a_3 + a_4 \right\},$$

$$D(Spin(9)) = \left\{ \sum_{i=1}^{4} b_i\varepsilon_i \mid b_1 \geq b_2 \geq b_3 \geq |b_4|, b_1 \geq b_2 + b_3 + b_4 \right\}.$$

A weight $\sum_{i=1}^{4} x_i\varepsilon_i$ is an integral weight of $Spin(9)$ if and only if

$$x_1 - x_2, x_2 - x_3, x_3 - x_4, 2x_4 \in \mathbf{Z}.$$

The set of fundamental weights of F_4 is

$$\lambda_1 = \varepsilon_1 + \varepsilon_2, \quad \lambda_2 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad \lambda_3 = (3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2, \quad \lambda_4 = \varepsilon_1,$$

and the set of fundamental weights of $Spin(9)$ is

$$\mu_1 = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2, \quad \mu_2 = \varepsilon_1 + \varepsilon_2, \quad \mu_3 = (3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2, \quad \mu_4 = \varepsilon_1.$$

Half the sum of positive roots of F_4 is

$$\delta = (11/2)\varepsilon_1 + (5/2)\varepsilon_2 + (3/2)\varepsilon_3 + (1/2)\varepsilon_4.$$

2. Proof of Theorem

In our previous paper ([4]) we proved the following

THEOREM 3. *Let* $\lambda = \sum_{i=1}^4 a_i\varepsilon_i$ *[resp.* $\mu = \sum_{i=1}^4 b_i\varepsilon_i$*] be a dominant integral weight of* F_4 *[resp.* $Spin(9)$*]. The coefficient* $m(\lambda,\mu)$ *of* $V^{Spin(9)}(\mu)$ *in* $V^{F_4}(\lambda)$ *is given by*

$$m(\lambda,\mu) = N_{\lambda+\delta}(\mu+\delta) + N_{(\lambda+\delta)^*}(\mu+\delta) - N_{(\lambda+\delta)^{**}}(\mu+\delta) \tag{1}$$

where

$$\lambda+\delta = (a_1+11/2)\varepsilon_1 + (a_2+5/2)\varepsilon_2 + (a_3+3/2)\varepsilon_3 + (a_4+1/2)\varepsilon_4$$

$$(\lambda+\delta)^* = \frac{a_1+a_2+a_3-a_4+9}{2}\varepsilon_1 + \frac{a_1+a_2-a_3+a_4+7}{2}\varepsilon_2 + \frac{a_1-a_2+a_3+a_4+5}{2}\varepsilon_3 + \frac{a_1-a_2-a_3-a_4+1}{2}\varepsilon_4$$

$$(\lambda+\delta)^{**} = \frac{a_1+a_2+a_3+a_4+10}{2}\varepsilon_1 + \frac{a_1+a_2-a_3-a_4+6}{2}\varepsilon_2 + \frac{a_1-a_2+a_3-a_4+4}{2}\varepsilon_3 + \frac{a_1-a_2-a_3+a_4+2}{2}\varepsilon_4.$$

and $N_\nu(\mu+\delta)$*, for a dominant integral weight* $\nu = \sum_{i=1}^4 x_i\varepsilon_i$ *of* $Spin(9)$*, is the number of integral quadruples*

$$i = (i_1,i_2,i_3,i_4) \in ([1,x_1-x_2]\times[1,x_2-x_3]\times[1,x_3-x_4]\times[1,2x_4])\cap \mathbf{Z}^4$$

satisfying

$$b_1+b_2+b_3+b_4+i_1+i_2-i_3-i_4-x_1-x_2+x_3+x_4+8>0 \tag{2.1}$$

$$b_1+b_2-b_3-b_4+i_1-i_2+i_3-i_4-x_1+x_2-x_3+x_4+6>0 \tag{2.2}$$

$$b_1-b_2+b_3-b_4+i_1-i_2-i_3+i_4-x_1+x_2+x_3-x_4+4>0 \tag{2.3}$$

$$-b_1-b_2-b_3-b_4-i_1-i_2-i_3-i_4+x_1+x_2+x_3+x_4-6\geq 0 \tag{2.4}$$

$$-b_1-b_2+b_3+b_4-i_1-i_2+i_3+i_4+x_1+x_2-x_3-x_4-6\geq 0 \tag{2.5}$$

$$-b_1+b_2-b_3+b_4-i_1+i_2-i_3+i_4+x_1-x_2+x_3-x_4-4\geq 0 \tag{2.6}$$

$$-b_1+b_2+b_3-b_4-i_1+i_2+i_3-i_4+x_1-x_2-x_3+x_4-2\geq 0 \tag{2.7}$$

$$\sum_{l=1}^4 x_l + \sum_{l=1}^4 b_l + \sum_{l=1}^4 i_l \equiv 0 \pmod 2. \tag{2.8}$$

For an integral weights $\nu = \sum_{i=1}^{4} x_i \varepsilon_i$, put

$$I_\nu = ([1, x_1 - x_2] \times [1, x_2 - x_3] \times [1, x_3 - x_4] \times [1, 2x_4]) \cap \mathbf{Z}^4$$

and denote by $P_\nu(\mu + \delta)$ the set of quatdruples $i \in I_\nu$ satisfying (2.1)–(2.8).

PROOF OF THEOREM 1. If we put $\nu = \sum_{i=1}^{4} x_i \varepsilon_i$, $x_1 - x_2, \dots, 2x_4$ are as in the following table

ν	$x_1 - x_2$	$x_2 - x_3$	$x_3 - x_4$	$2x_4$
$\lambda + \delta$	$a_1 - a_2 + 3$	$a_2 - a_3 + 1$	$a_3 - a_4 + 1$	$2a_4 + 1$
$(\lambda + \delta)^*$	$a_3 - a_4 + 1$	$a_2 - a_3 + 1$	$a_3 + a_4 + 2$	$a_1 - a_2 - a_3 - a_4 + 1$
$(\lambda + \delta)^{**}$	$a_3 + a_4 + 2$	$a_2 - a_3 + 1$	$a_3 - a_4 + 1$	$a_1 - a_2 - a_3 + a_4 + 2$

We put $\nu = \lambda + \delta = \sum_{i=1}^{4} x_i \varepsilon_i$ and $\bar{\nu} = \lambda + \lambda_4 + \delta = \sum_{i=1}^{4} \bar{x}_i \varepsilon_i$. An integral quadruple $(1, i_2, i_3, i_4) \in I_{\bar{\nu}}$ does not satisfy (2.1). It is easily verified that the mapping

$$I_\nu \to I_{\bar{\nu}}; \quad (i_1, i_2, i_3, i_4) \mapsto (i_1 + 1, i_2, i_3, i_4)$$

induces a bijection $P_\nu(\mu + \delta) \to P_{\bar{\nu}}(\mu + \delta)$. Namely we have $N_\nu(\mu + \delta) = N_{\bar{\nu}}(\mu + \delta)$.

We put $\nu = (\lambda + \delta)^* = \sum_{i=1}^{4} x_i \varepsilon_i$ and $\bar{\nu} = (\lambda + \lambda_4 + \delta)^* = \sum_{i=1}^{4} \bar{x}_i \varepsilon_i$. In this case, we have $I_{\bar{\nu}} \supset I_\nu$. Any integral quadruple $i \in I_{\bar{\nu}} \backslash I_\nu$, which is of the form $i = (i_1, i_2, i_3, 2x_4 + 1)$, does not satisfy (2.1). It is easily verified that the mapping

$$I_\nu \to I_{\bar{\nu}}; \quad (i_1, i_2, i_3, i_4) \mapsto (i_1, i_2, i_3, i_4)$$

ineduces a bijection $Q_\nu(\mu + \delta) \to Q_{\bar{\nu}}(\mu + \delta)$. Namely we have $N_\nu(\mu + \delta) = N_{\bar{\nu}}(\mu + \delta)$.

Similary we have $N_{(\lambda+\delta)^{**}}(\mu + \delta) = N_{(\lambda+\lambda_4+\delta)^{**}}(\mu + \delta)$. q.e.d

LEMMA 4. *Let* $\lambda = \sum_{i=1}^{4} a_i \varepsilon_i = \sum_{i=1}^{4} u_i \lambda_i \in D(F_4)$ *and* $\mu = \sum_{i=1}^{4} b_i \varepsilon_i \in D(Spin(9))$.

(1) *If* $a_2 - a_4 \geq b_1 + b_2 + b_3 + b_4 + 4$ *then* $N_{\lambda+\delta}(\mu + \delta) = N_{(\lambda+\delta)^{**}}(\mu + \delta) = 0$.

(2) *If* $a_2 + a_4 \geq b_1 + b_2 + b_3 + b_4 + 3$ *then* $N_{(\lambda+\delta)^*}(\mu + \delta) = 0$.

PROOF. Let $i = (i_1, i_2, i_3, i_4)$ be an element of I_ν. If we assume that $x_2 - x_4 \geq b_1 + b_2 + b_3 + b_4 + 6$, then (2.1) does not hold.

If $\nu = \lambda + \delta$ or $\nu = (\lambda + \delta)^{**}$ then $x_2 - x_4 \geq a_2 - a_4 + 2$ and if $\nu = (\lambda + \delta)^{*}$ then $x_2 - x_4 = a_3 + a_4 + 3$. Thus we obtain the Lemma. q.e.d

Proof of Theorem 2. From Lemma 4 we have $N_{(\lambda+\delta)^*}(\mu + \delta) = 0$. We put $\nu = \lambda + \delta$ and $\bar{\nu} = (\lambda + \delta)^{**}$.

Let $\nu = \sum_{i=1}^{4} x_i \varepsilon_i = \lambda + \delta$ and let $i = (i_1, i_2, i_3, i_4)$ be an integral quadruple contained in I_ν. If we assume that $i_1 \leq a_1 - a_2 - a_3 - a_4 + 1$, then (2.1) does not hold. Thus $P_\nu(\mu + \delta)$ is the set of integral quadruples $i = (i_1, i_2, i_3, i_4)$ satisfying

$$\begin{cases} \sum_{l=1}^{4}(a_l + b_l + i_l) \equiv 0 \pmod 2, \\ a_1 - a_2 - a_3 - a_4 + 2 \leq i_1 \leq a_1 - a_2 + 3, \\ 1 \leq i_2 \leq a_2 - a_3 + 1, \quad 1 \leq i_3 \leq a_3 - a_4 + 1, \quad 1 \leq i_4 \leq 2a_4 + 1, \\ -a_1 - a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4 + i_1 + i_2 - i_3 - i_4 + 2 > 0, \\ -a_1 + a_2 - a_3 + a_4 + b_1 + b_2 - b_3 - b_4 + i_1 - i_2 + i_3 - i_4 + 2 > 0, \\ -a_1 + a_2 + a_3 - a_4 + b_1 - b_2 + b_3 - b_4 + i_1 - i_2 - i_3 + i_4 + 2 > 0, \\ a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - b_3 - b_4 - i_1 - i_2 - i_3 - i_4 + 4 \geq 0, \\ a_1 + a_2 - a_3 - a_4 - b_1 - b_2 + b_3 + b_4 - i_1 - i_2 + i_3 + i_4 \geq 0, \\ a_1 - a_2 + a_3 - a_4 - b_1 + b_2 - b_3 + b_4 - i_1 + i_2 - i_3 + i_4 \geq 0, \\ a_1 - a_2 - a_3 + a_4 - b_1 + b_2 + b_3 - b_4 - i_1 + i_2 + i_3 - i_4 \geq 0. \end{cases}$$

Let $\bar{\nu} = \sum_{i=1}^{4} \bar{x}_i \varepsilon_i = \lambda + \delta$ and let $i = (i_1, i_2, i_3, i_4)$ be an integral quadruple contained in $I_{\bar{\nu}}$. If we assume that $i_4 \geq 2a_4 + 1$, then (2.1) does not holds. Thus $P_{\bar{\nu}}(\mu + \delta)$ is the set of integral quadruples $i = (i_1, i_2, i_3, i_4)$ satisfying

$$\begin{cases} \sum_{l=1}^{4}(a_l + b_l + i_l) \equiv 0 \pmod 2, \\ 1 \leq i_1 \leq a_3 + a_4 + 2, \quad 1 \leq i_2 \leq a_2 - a_3 + 1, \\ 1 \leq i_3 \leq a_3 - a_4 + 1, \quad 1 \leq i_4 \leq 2a_4 + 1, \\ -2a_2 + b_1 + b_2 + b_3 + b_4 + i_1 + i_2 - i_3 - i_4 + 3 > 0, \\ -2a_3 + b_1 + b_2 - b_3 - b_4 + i_1 - i_2 + i_3 - i_4 + 3 > 0, \\ -2a_4 + b_1 - b_2 + b_3 - b_4 + i_1 - i_2 - i_3 + i_4 + 3 > 0, \\ 2a_1 - b_1 - b_2 - b_3 - b_4 - i_1 - i_2 - i_3 - i_4 + 5 \geq 0, \\ 2a_2 - b_1 - b_2 + b_3 + b_4 - i_1 - i_2 + i_3 + i_4 - 1 \geq 0, \\ 2a_3 - b_1 + b_2 - b_3 + b_4 - i_1 + i_2 - i_3 + i_4 - 1 \geq 0, \\ 2a_4 - b_1 + b_2 + b_3 - b_4 - i_1 + i_2 + i_3 - i_4 - 1 \geq 0. \end{cases}$$

It is easily verified that the mapping

$$\mathbf{Z}^4 \to \mathbf{Z}^4; \quad (i_1, i_2, i_3, i_4) \mapsto (i_1 + a_1 - a_2 - a_3 - a_4 + 1, i_2, i_3, i_4)$$

induces a bijection $P_{\bar{\nu}}(\mu + \delta) \to P_\nu(\mu + \delta)$. Thus we have $N_{\bar{\nu}}(\mu + \delta) = N_\nu(\mu + \delta)$ and the Theorem was obtained. q.e.d

3. Tables of Branching Coefficient

As an application of our theorems 1 and 2, we give tables of branching coefficients $m(\lambda,\mu)$ for dominant integral weight μ of $Spin(9)$ which appears as a summand of the p-th exterior power $\bigwedge^p(T_o^*(P^2(\mathbf{Ca}))^{\mathbf{C}}$ of the complexfied cotangnet space of $P^2(\mathbf{Ca})$ (cf. [4, p. 373], for the irreducible decomposition of $\bigwedge^p(T_o^*(P^2(\mathbf{Ca}))^{\mathbf{C}})$.

In following tables, highest weights λ of F_4 are expressed by its coefficents with respect to the fundamental weights $\lambda_1,\dots,\lambda_4$.

Table 1: $\mu = 0$

λ	$m(\lambda,\mu)$
$(0,0,0,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 2: $\mu = \mu_1$

λ	$m(\lambda,\mu)$
$(0,0,0,n)$	$1\ \ (n \geq 1)$
$(0,0,1,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 3: $\mu = \mu_2$

λ	$m(\lambda,\mu)$
$(1,0,0,n)$	$1\ \ (n \geq 0)$
$(0,0,1,n)$	$1\ \ (n \geq 0)$
$(0,1,0,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 4: $\mu = \mu_4$

λ	$m(\lambda,\mu)$
$(0,0,0,n)$	$1\ \ (n \geq 1)$
$(1,0,0,n)$	$1\ \ (n \geq 0)$
$(0,0,1,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 5: $\mu = 2\mu_1$

λ	$m(\lambda,\mu)$
$(0,0,0,n)$	$1\ \ (n \geq 2)$
$(0,0,1,n)$	$1\ \ (n \geq 1)$
$(0,0,2,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 6: $\mu = \mu_3$

λ	$m(\lambda,\mu)$
$(1,0,0,n)$	$1\ \ (n \geq 1)$
$(0,0,1,n)$	$1\ \ (n \geq 0)$
$(0,1,0,n)$	$1\ \ (n \geq 0)$
$(1,0,1,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 7: $\mu = 3\mu_1$

λ	$m(\lambda,\mu)$
$(0,0,0,n)$	$1\ \ (n \geq 3)$
$(0,0,1,n)$	$1\ \ (n \geq 2)$
$(0,0,2,n)$	$1\ \ (n \geq 1)$
$(0,0,3,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 8: $\mu = 4\mu_1$

λ	$m(\lambda,\mu)$
$(0,0,0,n)$	$1\ \ (n \geq 4)$
$(0,0,1,n)$	$1\ \ (n \geq 3)$
$(0,0,2,n)$	$1\ \ (n \geq 2)$
$(0,0,3,n)$	$1\ \ (n \geq 1)$
$(0,0,4,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 9: $\mu = 2\mu_4$

λ	$m(\lambda,\mu)$
$(0,0,0,n)$	$1\ \ (n \geq 2)$
$(1,0,0,n)$	$1\ \ (n \geq 1)$
$(0,0,1,n)$	$1\ \ (n \geq 1)$
$(2,0,0,n)$	$1\ \ (n \geq 0)$
$(1,0,1,n)$	$1\ \ (n \geq 0)$
$(0,0,2,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 10: $\mu = \mu_1 + \mu_2$

λ	$m(\lambda,\mu)$
$(1,0,0,n)$	$1\ \ (n \geq 1)$
$(0,0,1,n)$	$1\ \ (n \geq 1)$
$(0,1,0,n)$	$1\ \ (n \geq 1)$
$(1,0,1,n)$	$1\ \ (n \geq 0)$
$(0,0,2,n)$	$1\ \ (n \geq 0)$
$(0,1,1,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 11: $\mu = 2\mu_2$

λ	$m(\lambda,\mu)$
$(2,0,0,n)$	$1\ \ (n \geq 0)$
$(1,0,1,n)$	$1\ \ (n \geq 0)$
$(0,0,2,n)$	$1\ \ (n \geq 0)$
$(1,1,0,n)$	$1\ \ (n \geq 0)$
$(0,1,1,n)$	$1\ \ (n \geq 0)$
$(0,2,0,n)$	$1\ \ (n \geq 0)$
otherwise	0

Table 12: $\mu = \mu_1 + \mu_4$

λ	$m(\lambda, \mu)$
$(0, 0, 0, n)$	$1 \quad (n \geq 2)$
$(1, 0, 0, n)$	$1 \quad (n \geq 1)$
$(0, 0, 1, n)$	$2 \quad (n \geq 1)$
	$1 \quad (n = 0)$
$(0, 1, 0, n)$	$1 \quad (n \geq 0)$
$(1, 0, 1, n)$	$1 \quad (n \geq 0)$
$(0, 0, 2, n)$	$1 \quad (n \geq 0)$
otherwise	0

Table 13: $\mu = \mu_2 + \mu_4$

λ	$m(\lambda, \mu)$
$(1, 0, 0, n)$	$1 \quad (n \geq 1)$
$(0, 0, 1, n)$	$1 \quad (n \geq 1)$
$(0, 1, 0, n)$	$2 \quad (n \geq 1)$
	$1 \quad (n = 0)$
$(2, 0, 0, n)$	$1 \quad (n \geq 0)$
$(1, 0, 1, n)$	$2 \quad (n \geq 0)$
$(0, 0, 2, n)$	$1 \quad (n \geq 0)$
$(1, 1, 0, n)$	$1 \quad (n \geq 0)$
$(0, 1, 1, n)$	$1 \quad (n \geq 0)$
otherwise	0

Table 14: $\mu = 2\mu_1 + \mu_2$

λ	$m(\lambda, \mu)$
$(1, 0, 0, n)$	$1 \quad (n \geq 2)$
$(0, 0, 1, n)$	$1 \quad (n \geq 2)$
$(0, 1, 0, n)$	$1 \quad (n \geq 2)$
$(1, 0, 1, n)$	$1 \quad (n \geq 1)$
$(0, 0, 2, n)$	$1 \quad (n \geq 1)$
$(0, 1, 1, n)$	$1 \quad (n \geq 1)$
$(1, 0, 2, n)$	$1 \quad (n \geq 0)$
$(0, 0, 3, n)$	$1 \quad (n \geq 0)$
$(0, 1, 2, n)$	$1 \quad (n \geq 0)$
otherwise	0

Table 15: $\mu = 3\mu_4$

λ	$m(\lambda, \mu)$
$(0, 0, 0, n)$	$1 \quad (n \geq 3)$
$(1, 0, 0, n)$	$1 \quad (n \geq 2)$
$(0, 0, 1, n)$	$1 \quad (n \geq 2)$
$(2, 0, 0, n)$	$1 \quad (n \geq 1)$
$(1, 0, 1, n)$	$1 \quad (n \geq 1)$
$(0, 0, 2, n)$	$1 \quad (n \geq 1)$
$(3, 0, 0, n)$	$1 \quad (n \geq 0)$
$(2, 0, 1, n)$	$1 \quad (n \geq 0)$
$(1, 0, 2, n)$	$1 \quad (n \geq 0)$
$(0, 0, 3, n)$	$1 \quad (n \geq 0)$
otherwise	0

Table 16: $\mu = 2\mu_1 + \mu_4$

λ	$m(\lambda, \mu)$
$(0,0,0,n)$	$1 \ (n \geq 3)$
$(1,0,0,n)$	$1 \ (n \geq 2)$
$(0,0,1,n)$	$2 \ (n \geq 2)$
	$1 \ (n = 1)$
$(0,1,0,n)$	$1 \ (n \geq 1)$
$(1,0,1,n)$	$1 \ (n \geq 1)$
$(0,0,2,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0,1,1,n)$	$1 \ (n \geq 0)$
$(1,0,2,n)$	$1 \ (n \geq 0)$
$(0,0,3,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 18: $\mu = \mu_1 + \mu_3$

λ	$m(\lambda, \mu)$
$(1,0,0,n)$	$1 \ (n \geq 2)$
$(0,0,1,n)$	$1 \ (n \geq 1)$
$(0,1,0,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(1,0,1,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0,0,2,n)$	$1 \ (n \geq 0)$
$(1,1,0,n)$	$1 \ (n \geq 0)$
$(0,1,1,n)$	$1 \ (n \geq 0)$
$(1,0,2,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 17: $\mu = \mu_3 + \mu_4$

λ	$m(\lambda, \mu)$
$(1,0,0,n)$	$1 \ (n \geq 2)$
$(0,0,1,n)$	$1 \ (n \geq 1)$
$(0,1,0,n)$	$1 \ (n \geq 1)$
$(2,0,0,n)$	$1 \ (n \geq 1)$
$(1,0,1,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0,0,2,n)$	$1 \ (n \geq 0)$
$(1,1,0,n)$	$1 \ (n \geq 0)$
$(0,1,1,n)$	$1 \ (n \geq 0)$
$(2,0,1,n)$	$1 \ (n \geq 0)$
$(1,0,2,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 19: $\mu = \mu_1 + 2\mu_4$

λ	$m(\lambda, \mu)$
$(0,0,0,n)$	$1 \ (n \geq 3)$
$(1,0,0,n)$	$1 \ (n \geq 2)$
$(0,0,1,n)$	$2 \ (n \geq 2)$
	$1 \ (n = 1)$
$(0,1,0,n)$	$1 \ (n \geq 1)$
$(2,0,0,n)$	$1 \ (n \geq 1)$
$(1,0,1,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0,0,2,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(1,1,0,n)$	$1 \ (n \geq 0)$
$(0,1,1,n)$	$1 \ (n \geq 0)$
$(2,0,1,n)$	$1 \ (n \geq 0)$
$(1,0,2,n)$	$1 \ (n \geq 0)$
$(0,0,3,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 20: $\mu = \mu_2 + 2\mu_4$

λ	$m(\lambda, \mu)$
$(1,0,0,n)$	$1 \ (n \geq 2)$
$(0,0,1,n)$	$1 \ (n \geq 2)$
$(0,1,0,n)$	$2 \ (n \geq 2)$
	$1 \ (n = 1)$
$(2,0,0,n)$	$1 \ (n \geq 1)$
$(1,0,1,n)$	$2 \ (n \geq 1)$
$(0,0,2,n)$	$1 \ (n \geq 1)$
$(1,1,0,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(3,0,0,n)$	$1 \ (n \geq 0)$
$(0,1,1,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(2,0,1,n)$	$2 \ (n \geq 0)$
$(1,0,2,n)$	$2 \ (n \geq 0)$
$(2,1,0,n)$	$1 \ (n \geq 0)$
$(0,0,3,n)$	$1 \ (n \geq 0)$
$(1,1,1,n)$	$1 \ (n \geq 0)$
$(0,1,2,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 21: $\mu = 2\mu_3$

λ	$m(\lambda, \mu)$
$(2,0,0,n)$	$1 \ (n \geq 2)$
$(1,0,1,n)$	$1 \ (n \geq 1)$
$(0,0,2,n)$	$1 \ (n \geq 0)$
$(1,1,0,n)$	$1 \ (n \geq 1)$
$(0,1,1,n)$	$1 \ (n \geq 0)$
$(2,0,1,n)$	$1 \ (n \geq 1)$
$(1,0,2,n)$	$1 \ (n \geq 0)$
$(0,2,0,n)$	$1 \ (n \geq 0)$
$(1,1,1,n)$	$1 \ (n \geq 0)$
$(2,0,2,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 22: $\mu = \mu_2 + \mu_3$

λ	$m(\lambda, \mu)$
$(0,1,0,n)$	$1 \ (n \geq 1)$
$(2,0,0,n)$	$1 \ (n \geq 1)$
$(1,0,1,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0,0,2,n)$	$1 \ (n \geq 0)$
$(1,1,0,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0,1,1,n)$	$2 \ (n \geq 0)$
$(2,0,1,n)$	$1 \ (n \geq 0)$
$(1,0,2,n)$	$1 \ (n \geq 0)$
$(0,2,0,n)$	$1 \ (n \geq 0)$
$(1,1,1,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 23: $\mu = 2\mu_1 + \mu_3$

λ	$m(\lambda, \mu)$
$(1,0,0,n)$	$1 \ (n \geq 3)$
$(0,0,1,n)$	$1 \ (n \geq 2)$
$(0,1,0,n)$	$2 \ (n \geq 2)$
	$1 \ (n = 1)$
$(1,0,1,n)$	$2 \ (n \geq 2)$
	$1 \ (n = 1)$
$(0,0,2,n)$	$1 \ (n \geq 1)$
$(1,1,0,n)$	$1 \ (n \geq 1)$
$(0,1,1,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(1,0,2,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0,0,3,n)$	$1 \ (n \geq 0)$
$(1,1,1,n)$	$1 \ (n \geq 0)$
$(0,1,2,n)$	$1 \ (n \geq 0)$
$(1,0,3,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 24: $\mu = \mu_1 + \mu_2 + \mu_4$

λ	$m(\lambda, \mu)$
$(1, 0, 0, n)$	$1 \ (n \geq 2)$
$(0, 0, 1, n)$	$1 \ (n \geq 2)$
$(0, 1, 0, n)$	$2 \ (n \geq 2)$
	$1 \ (n = 1)$
$(2, 0, 0, n)$	$1 \ (n \geq 1)$
$(1, 0, 1, n)$	$3 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0, 0, 2, n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(1, 1, 0, n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0, 1, 1, n)$	$3 \ (n \geq 1)$
	$2 \ (n = 0)$
$(2, 0, 1, n)$	$1 \ (n \geq 0)$
$(1, 0, 2, n)$	$2 \ (n \geq 0)$
$(0, 2, 0, n)$	$1 \ (n \geq 0)$
$(0, 0, 3, n)$	$1 \ (n \geq 0)$
$(1, 1, 1, n)$	$1 \ (n \geq 0)$
$(0, 1, 2, n)$	$1 \ (n \geq 0)$
otherwise	0

Table 25: $\mu = \mu_1 + 2\mu_2$

λ	$m(\lambda, \mu)$
$(2, 0, 0, n)$	$1 \ (n \geq 1)$
$(1, 0, 1, n)$	$1 \ (n \geq 1)$
$(0, 0, 2, n)$	$1 \ (n \geq 1)$
$(1, 1, 0, n)$	$1 \ (n \geq 1)$
$(0, 1, 1, n)$	$1 \ (n \geq 1)$
$(2, 0, 1, n)$	$1 \ (n \geq 0)$
$(1, 0, 2, n)$	$1 \ (n \geq 0)$
$(0, 2, 0, n)$	$1 \ (n \geq 1)$
$(0, 0, 3, n)$	$1 \ (n \geq 0)$
$(1, 1, 1, n)$	$1 \ (n \geq 0)$
$(0, 1, 2, n)$	$1 \ (n \geq 0)$
$(0, 2, 1, n)$	$1 \ (n \geq 0)$
otherwise	0

Table 26: $\mu = 3\mu_1 + \mu_4$

λ	$m(\lambda,\mu)$
$(0,0,0,n)$	$1 \ (n \geq 4)$
$(1,0,0,n)$	$1 \ (n \geq 3)$
$(0,0,1,n)$	$2 \ (n \geq 3)$
	$1 \ (n = 2)$
$(0,1,0,n)$	$1 \ (n \geq 2)$
$(1,0,1,n)$	$1 \ (n \geq 2)$
$(0,0,2,n)$	$2 \ (n \geq 2)$
	$1 \ (n = 1)$
$(0,1,1,n)$	$1 \ (n \geq 1)$
$(1,0,2,n)$	$1 \ (n \geq 1)$
$(0,0,3,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0,1,2,n)$	$1 \ (n \geq 0)$
$(1,0,3,n)$	$1 \ (n \geq 0)$
$(0,0,4,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 27: $\mu = 2\mu_1 + 2\mu_4$

λ	$m(\lambda,\mu)$
$(0,0,0,n)$	$1 \ (n \geq 4)$
$(1,0,0,n)$	$1 \ (n \geq 3)$
$(0,0,1,n)$	$2 \ (n \geq 3)$
	$1 \ (n = 2)$
$(0,1,0,n)$	$1 \ (n \geq 2)$
$(2,0,0,n)$	$1 \ (n \geq 2)$
$(1,0,1,n)$	$2 \ (n \geq 2)$
	$1 \ (n = 1)$
$(0,0,2,n)$	$3 \ (n \geq 2)$
	$2 \ (n = 1)$
	$1 \ (n = 0)$
$(1,1,0,n)$	$1 \ (n \geq 1)$
$(0,1,1,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(2,0,1,n)$	$1 \ (n \geq 1)$
$(1,0,2,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0,2,0,n)$	$1 \ (n \geq 0)$
$(0,0,3,n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(1,1,1,n)$	$1 \ (n \geq 0)$
$(0,1,2,n)$	$1 \ (n \geq 0)$
$(2,0,2,n)$	$1 \ (n \geq 0)$
$(1,0,3,n)$	$1 \ (n \geq 0)$
$(0,0,4,n)$	$1 \ (n \geq 0)$
otherwise	0

Table 28: $\mu = \mu_1 + \mu_3 + \mu_4$

λ	$m(\lambda, \mu)$
$(1, 0, 0, n)$	$1 \ (n \geq 3)$
$(0, 0, 1, n)$	$1 \ (n \geq 2)$
$(0, 1, 0, n)$	$2 \ (n \geq 2)$
	$1 \ (n = 1)$
$(2, 0, 0, n)$	$1 \ (n \geq 2)$
$(1, 0, 1, n)$	$3 \ (n \geq 2)$
	$2 \ (n = 1)$
$(0, 0, 2, n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(1, 1, 0, n)$	$3 \ (n \geq 1)$
	$1 \ (n = 0)$
$(0, 1, 1, n)$	$3 \ (n \geq 1)$
	$2 \ (n = 0)$
$(2, 0, 1, n)$	$2 \ (n \geq 1)$
	$1 \ (n = 0)$
$(1, 0, 2, n)$	$3 \ (n \geq 1)$
	$2 \ (n = 0)$
$(0, 2, 0, n)$	$1 \ (n \geq 0)$
$(2, 1, 0, n)$	$1 \ (n \geq 0)$
$(0, 0, 3, n)$	$1 \ (n \geq 0)$
$(1, 1, 1, n)$	$2 \ (n \geq 0)$
$(0, 1, 2, n)$	$1 \ (n \geq 0)$
$(2, 0, 2, n)$	$1 \ (n \geq 0)$
$(1, 0, 3, n)$	$1 \ (n \geq 0)$
otherwise	0

Table 29: Spectra of Laplacian Δ^6 on $P^2(\mathbf{Ca})$

eigenvalue	multiplicity of $V^{F_4}(\lambda + n\lambda_4)$					
	n					λ
	$n=0$	$n=1$	$n=2$	$n=3$	$n \geq 4$	
n^2+11n	0	0	0	1	$\leftarrow$	0
$n^2+14n+24$	2	5	9	$\leftarrow$	$\leftarrow$	λ_3
$n^2+17n+54$	3	7	$\leftarrow$	$\leftarrow$	$\leftarrow$	$2\lambda_3$
$n^2+20n+90$	4	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$3\lambda_3$
$n^2+15n+36$	3	8	11	$\leftarrow$	$\leftarrow$	λ_2
$n^2+18n+68$	5	8	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_2+\lambda_3$
$n^2+21n+106$	3	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_2+3\lambda_3$
$n^2+13n+18$	1	3	7	8	$\leftarrow$	λ_1
$n^2+16n+46$	4	10	11	$\leftarrow$	$\leftarrow$	$\lambda_1+\lambda_3$
$n^2+19n+80$	6	7	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_1+2\lambda_3$
$n^2+22n+120$	1	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_1+3\lambda_3$
$n^2+17n+60$	3	5	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_1+\lambda_2$
$n^2+20n+96$	2	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_1+\lambda_2+\lambda_3$
$n^2+15n+40$	0	2	$\leftarrow$	$\leftarrow$	$\leftarrow$	$2\lambda_1$
$n^2+18n+72$	3	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$2\lambda_1+\lambda_3$
$n^2+19n+88$	1	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$2\lambda_1+\lambda_2$
$n^2+17n+66$	1	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$3\lambda_1$

Table 30: Spectra of Laplacian Δ^7 on $P^2(\mathbf{Ca})$

eigenvalue	multiplicity of $V^{F_4}(\lambda + n\lambda_4)$						
	n						λ
	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n \geq 5$	
n^2+11n	0	1	2	$\leftarrow$	$\leftarrow$	$\leftarrow$	0
$n^2+14n+24$	2	5	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	λ_3
$n^2+17n+54$	3	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$2\lambda_3$
$n^2+15n+36$	2	4	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	λ_2
$n^2+18n+68$	2	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_2+\lambda_3$
$n^2+13n+18$	1	3	4	$\leftarrow$	$\leftarrow$	$\leftarrow$	λ_1
$n^2+16n+46$	4	5	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_1+\lambda_3$
$n^2+19n+80$	1	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_1+2\lambda_3$
$n^2+17n+60$	2	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\lambda_1+\lambda_2$
$n^2+15n+40$	1	2	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$2\lambda_1$
$n^2+18n+72$	1	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$2\lambda_1+\lambda_3$

Table 31: Spectra of Laplacian Δ^8 on $P^2(\mathbf{Ca})$

eigenvalue	multiplicity of $V^{F_4}(\lambda + n\lambda_4)$						
	n						λ
	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n \geq 5$	
n^2+11n	1	2	4	6	8	←	0
$n^2+14n+24$	2	6	10	12	←	←	λ_3
$n^2+17n+54$	8	12	14	←	←	←	$2\lambda_3$
$n^2+20n+90$	4	6	←	←	←	←	$3\lambda_3$
$n^2+23n+132$	2	←	←	←	←	←	$4\lambda_3$
$n^2+15n+36$	2	6	8	←	←	←	λ_2
$n^2+18n+68$	8	10	←	←	←	←	$\lambda_2+\lambda_3$
$n^2+21n+106$	2	←	←	←	←	←	$\lambda_2+2\lambda_3$
$n^2+19n+84$	4	←	←	←	←	←	$2\lambda_2$
$n^2+13n+18$	0	2	4	6	←	←	λ_1
$n^2+16n+46$	6	12	14	←	←	←	$\lambda_1+\lambda_3$
$n^2+19n+80$	6	8	←	←	←	←	$\lambda_1+2\lambda_3$
$n^2+22n+120$	2	←	←	←	←	←	$\lambda_1+3\lambda_3$
$n^2+17n+60$	4	8	←	←	←	←	$\lambda_1+\lambda_2$
$n^2+20n+96$	4	←	←	←	←	←	$\lambda_1+\lambda_2+\lambda_3$
$n^2+15n+40$	2	4	6	←	←	←	$2\lambda_1$
$n^2+18n+72$	2	4	←	←	←	←	$2\lambda_1+\lambda_3$
$n^2+21n+110$	2	←	←	←	←	←	$2\lambda_1+2\lambda_3$

4. The Spectra of Laplacian

In the previous paper ([4]), we calculated the spectra of Laplacian acting on p-forms on the Cayley projective plane for $p \leq 5$. The set of eigenvalues of Δ^p is given as follows

$$\left\{\langle \lambda + 2\delta, \lambda \rangle \mid \lambda \in D(G), \dim_{\mathbf{C}} \mathrm{Hom}_{Spin(9)}\left(\bigwedge^p T_o(\mathbf{Ca}P^2)^{\mathbf{C}}, V^{F_4}(\lambda)\right) \neq 0\right\}.$$

We can find all complex irreducible representation $V^{Spin(9)}(\mu)$ with

$$\mathrm{Hom}_{Spin(9)}\left(\bigwedge^p T_o(\mathbf{Ca}P^2)^{\mathbf{C}}, V^{F_4}(\lambda)\right) \neq 0$$

by using Table 1–28. In this section we calculate the eigenvalue of the Laplacian Δ^p for $p = 6, 7, 8$.

In the table below, we mean by leftarrow (←) that the multiciplicity coincides with that given in the left entry.

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