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ON A RESULT OF FLAMMENKAMP-LUCA CONCERNING NONCOTOTIENT SEQUENCE

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Abstract. Let $\varphi(n)$ be the *Euler totient* function of n. A positive integer m is called a *noncototient* if the equation $n - \varphi(n) = m$ has no solution in positive integers n. The sequence $(2^k p)_{k=1}^{\infty}$ which is noncototient for some prime p will be called as Sierpiński's sequence. In this paper we prove some interesting properties of the Sierpiński sequence given in the Theorem 1, 2, 3.

1. Introduction

In 1959 Sierpiński ([6], pp. 200–201) asked whether there exist infinitely many natural numbers m such that $m \neq n - \varphi(n)$, (see also, Erdős [2] and B36 in [4]). Using Riesel's result ([5]), that all numbers of the form $2^k p_0 - 1$ with prime $p_0 = 509203$ are composite for $k = 1, 2, \dots$, Browkin and Schinzel [1] proved that all numbers $2^k p_0$ can not be presented in the form $n - \varphi(n)$. It is a positive answer to the question posed by Sierpiński.

Hence there is Sierpiński's sequence with $p_0 = 509203$.

In the paper [3] Flammenkamp and Luca proved the following sufficient condition for the sequence $(2^k p)_{k \geq 1}$ to be noncototient.

Let p be a positive integer satisfying the following four conditions:

(i) p is an odd prime

(ii) p is not a Mersenne prime

(iii) the number $2^k p - 1$ is composite for all integers $k \geq 1$

(iv) $2p$ is a noncototient.

Then the sequence $(2^k p)_{k \geq 1}$ is a noncototient, so is the Sierpiński sequence.

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In this connection we prove, in the Theorem 1 that the above conditions are also neccessary. Moreover in the Theorem 2 we prove that there are infinitely many primes p for which the conditions (ii) and (iii) are fulfilled. Further in the Theorem 3 we prove that $2p$ is of the form $n-\varphi(n)$ if and only if there are different odd primes p_j where $j=1,2,\dots,r;\ r\geq 1$ such that $p=p_1p_2\cdots p_r-\frac{1}{2}(p_1-1)(p_2-1)\cdots(p_r-1)$.

2. The Results

THEOREM 1. *Let p be an odd prime. The sequence $(2^kp)_{k=1}^{\infty}$ is the Sierpiński sequence if and only if:*

1^0. *$2p$ is a noncototient*

2^0. *p is not a Mersenne prime*

3^0. *2^kp-1 is composite for every positive integer $k\geq 1$.*

THEOREM 2. *There are infinitely many primes p in the arithmetical progression: $m\prod_{j=1}^{6}q_j+p_0$, where $p_0=509203$, $\prod_{j=1}^{6}q_j=3\times 5\times 7\times 13\times 17\times 241$ such that:*

1^0. *2^kp-1 is composite for every positive integer $k\geq 1$*

2^0. *p is not a Mersenne prime.*

THEOREM 3. *The number $2p$, where p is an odd prime is of the form $n-\varphi(n)$ if and only if there are different odd primes p_j, where $j=1,2,\dots,r;\ r\geq 1$ such that $p=p_1p_2\cdots p_r-\frac{1}{2}(p_1-1)(p_2-1)\cdots(p_r-1)$.*

3. Proof of Theorem 1

The sufficiency of conditions we prove by induction with respect to k. Suppose that the conditions 1^0–3^0 are satisfied. Then we see that the first step of inductive process follows by the assumption 1^0. Now, we assume that the number $2^{k-1}p$ is a noncototient and suppose that 2^kp is a cototient. Hence, for some natural number n_k we have

$$2^kp=n_k-\varphi(n_k). \tag{3.1}$$

Since $\varphi(n_k)\equiv 0 \pmod 4$ or $\varphi(n_k)\equiv 2 \pmod 4$ then from (3.1) we have $n_k\equiv 0 \pmod 4$ or $n_k=2q^{\alpha}$, $\alpha\geq 1$, where q is odd prime, respectively. If $n_k\equiv 0 \pmod 4$ then $\frac{\varphi(n_k)}{2}=\varphi\left(\frac{n_k}{2}\right)$ and by (3.1) it follows that

$$2^{k-1}p = \frac{n_k}{2} - \varphi\left(\frac{n_k}{2}\right), \tag{3.2}$$

and we get a contradiction with inductive assumption.

In the second case (3.1) implies

$$2^k p = 2q^\alpha - \varphi(2q^\alpha) = q^{\alpha-1}(q+1). \tag{3.3}$$

If $\alpha = 1$ then (3.3) implies $2^k p - 1 = q$, contrary to condition 3^0. Hence, $\alpha > 1$ and from (3.3) we obtain

$$q^{\alpha-1} | p. \tag{3.4}$$

From (3.4) we have $\alpha = 2$ and $q = p$ and (3.3) implies $2^k - 1 = p$. Again we obtain a contradiction with condition 2^0.

Now, we can prove the necessity of these conditions. Suppose that the sequence $(2^k p)_{k \geq 1}$ is noncototient for every positive integer k. Then we see that the condition 1^0 follows immediately for $k = 1$.

It remains to prove that the conditions 2^0 and 3^0 are satisfied. We prove this fact by contraposition. Indeed suppose that for some natural $k > 1$ we have $2^k - 1 = p$ or $2^k p - 1 = q$, where p and q are odd primes. Let $p = 2^k - 1$. Then taking $n = 2p^2$ we get

$$n - \varphi(n) = 2p^2 - \varphi(2p^2) = 2p^2 - p(p-1) = p(p+1). \tag{3.5}$$

Since $p = 2^k - 1$ then (3.5) implies $n - \varphi(n) = 2^k p$.

The case $2^k p - 1 = q$ is considered similarly. Taking $n = 2q$ we get

$$n - \varphi(n) = 2q - \varphi(2q) = 2q - (q-1) = q + 1 = 2^k p$$

and the proof of the theorem 1 is complete. ■

4. Proof of the Theorem 2

In the proof of the Theorem 2 we use of the following Lemma:

LEMMA 1. *Let $p_0 = 509203$. Then we have*

$$p_0 \equiv 2^{a_j} \pmod{q_j} \tag{4.1}$$

$$2^k p_0 \equiv 1 \pmod{q_j}, \tag{4.2}$$

where

$$\langle qj, a_j \rangle = \{\langle 3,0 \rangle, \langle 5,3 \rangle, \langle 7,1 \rangle, \langle 13,5 \rangle, \langle 17,1 \rangle, \langle 241,21 \rangle\} \tag{4.3}$$

for $j = 1, 2, \ldots, 6$ *and every integer* k *satisfies of the congruences*

$$k \equiv -a_j \pmod{m_j}, \tag{4.4}$$

for $m_j = 2, 4, 3, 12, 8, 24$ *and* $j = 1, 2, \ldots, 6$ *respectively.*

The proof of Lemma is given in the paper [1]. For the proof of the Theorem 2 consider the following arithmetical progression:

$$m \prod_{j=1}^{6} q_j + p_0, \quad \prod_{j=1}^{6} q_j = 3 \times 5 \times 7 \times 13 \times 17 \times 241. \tag{4.5}$$

By (4.2) it follows that $\left(p_0, \prod_{j=1}^{6} q_j\right) = 1$ and consequently Dirichlet's theorem on arithmetical progression implies that there are infinitely many primes p contained in the progression (4.5). Let $p = m \prod_{j=1}^{6} q_j + p_0$ be one of such primes. Then we have

$$2^k p - 1 = 2^k \left(m \prod_{j=1}^{6} q_j + p_0\right) - 1 = 2^k m \prod_{j=1}^{6} q_j + 2^k p_0 - 1. \tag{4.6}$$

From (4.6) and (4.2) we obtain

$$2^k p - 1 \equiv 0 \pmod{q_j},$$

hence, all numbers $2^k p - 1$ are composite.

For the proof of the second part of the theorem 2 suppose that there is a prime number p in the arithmetical progression $m \prod_{j=1}^{6} q_j + p_0$ that is a Mersenne prime. Hence for some prime k we have

$$p = m \prod_{j=1}^{6} q_j + p_0 = 2^k - 1. \tag{4.7}$$

From (4.7) we get

$$q_j \mid 2^k - 1 - p_0, \quad \text{for some } q_j = 3, 5, 7, 13, 17, 241. \tag{4.8}$$

By (4.8) it follows that

$$q_j \mid p_0(2^k - 1 - p_0) = 2^k p_0 - p_0(p_0 + 1) + 1 - 1. \tag{4.9}$$

Since $q_j \mid 2^k p_0 - 1$, from (4.2) then by (4.9) it follows that

$$q_j \mid p_0(p_0+1) - 1. \tag{4.10}$$

Using computer calculation we get the following factorization into primes

$$p_0(p_0+1) - 1 = 509203 \times 509204 - 1 = 59 \times 71 \times 809 \times 76511. \tag{4.11}$$

From (4.11) it follows that none of $q_j = 3, 5, 7, 13, 17, 241$ satisfies the relation (4.10).

The proof of the theorem 2 is complete. ■

5. Proof of the Theorem 3

Suppose that for some natural number n the number $2p$ has presantation in the form

$$2p = n - \varphi(n) \tag{5.1}$$

Let $n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where p_j are different odd primes for $j = 1, 2, \ldots, r$; $r \geq 1$ then $\varphi(n) = 2^{\alpha-1} p_1^{\alpha_1 - 1} \cdots p_r^{\alpha_r - 1}(p_1 - 1) \cdots (p_r - 1)$ and by (5.1) it follows that

$$2p = 2^{\alpha-1} p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1}(2p_1 \cdots p_r - (p_1 - 1) \cdots (p_r - 1)). \tag{5.2}$$

If $\alpha \geq 2$ then (5.2) is impossible. Hence, $\alpha = 1$ and by (5.2) follows that $\alpha_j = 1$ for $j = 1, 2, \ldots, r$ and consequently (5.2) implies that

$$p = p_1 p_2 \cdots p_r - \frac{1}{2}(p_1 - 1)(p_2 - 1) \cdots (p_r - 1). \tag{5.3}$$

Conversely, assume that (5.3) is satisfied. Then putting $n = 2p_1 \cdots p_r$ we have $\varphi(n) = (p_1 - 1) \cdots (p_r - 1)$ and we see that (5.3) implies $2p = n - \varphi(n)$. The proof is complete. ■

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