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GENERALIZED TATE COHOMOLOGY

By

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Abstract. We consider two classes of left R-modules, $\mathscr{P}$ and $\mathscr{C}$, such that $\mathscr{P} \subset \mathscr{C}$. If the module M has a $\mathscr{P}$-resolution and a $\mathscr{C}$-resolution then for any module N and $n \geq 0$ we define generalized Tate cohomology modules $\widehat{Ext}^n_{\mathscr{C},\mathscr{P}}(M,N)$ and show that we get a long exact sequence connecting these modules and the modules $Ext^n_{\mathscr{C}}(M,N)$ and $Ext^n_{\mathscr{P}}(M,N)$. When $\mathscr{C}$ is the class of Gorenstein projective modules, $\mathscr{P}$ is the class of projective modules and when M has a complete resolution we show that the modules $\widehat{Ext}^n_{\mathscr{C},\mathscr{P}}(M,N)$ for $n \geq 1$ are the usual Tate cohomology modules and prove that our exact sequence gives an exact sequence provided by Avramov and Martsinkovsky. Then we show that there is a dual result. We also prove that over Gorenstein rings Tate cohomology $\widehat{Ext}^n_R(M,N)$ can be computed using either a complete resolution of M or a complete injective resolution of N. And so, using our dual result, we obtain Avramov and Martsinkovsky's exact sequence under hypotheses different from theirs.

1. Introduction

We consider two classes of left R-modules $\mathscr{P}$, $\mathscr{C}$ such that $Proj \subset \mathscr{P} \subset \mathscr{C}$, where $Proj$ is the class of projective modules. Let M be a left R-module. Let $\mathbf{P.}$ be a deleted $\mathscr{P}$-resolution of M, $\mathbf{C.}$ a deleted $\mathscr{C}$-resolution of M (see Section 2 for definitions), let $u : \mathbf{P.} \to \mathbf{C.}$ be a chain map induced by Id_M, and $M(u)$ the associated mapping cone. We define the generalized Tate cohomology module $\widehat{Ext}^n_{\mathscr{C},\mathscr{P}}(M,N)$ by the equality $\widehat{Ext}^n_{\mathscr{C},\mathscr{P}}(M,N) = H^{n+1}(Hom(M(u),N))$, for any $n \geq 0$ and any left R-module N. We show that $\widehat{Ext}^n_{\mathscr{C},\mathscr{P}}(M,-)$ is well-defined. We

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also show that there is an exact sequence connecting these modules and the modules $Ext^n_{\mathscr{C}}(M,N)$ and $Ext^n_{\mathscr{P}}(M,N)$:

$$0 \to Ext^1_{\mathscr{C}}(M,N) \to Ext^1_{\mathscr{P}}(M,N) \to \widehat{Ext}^1_{\mathscr{C},\mathscr{P}}(M,N) \to \cdots \tag{1}$$

We prove (Proposition 1) that when we apply this procedure to $\mathscr{C} = Gor\,Proj$, $\mathscr{P} = Proj$, over a left noetherian ring R, for an R-module M with $Gor\,proj\,dim\,M = g < \infty$, the modules $\widehat{Ext}^n_{\mathscr{C},\mathscr{P}}(M,N)$ are the usual Tate cohomology modules for any $n \geq 1$. In this case our exact sequence (1) becomes L. L. Avramov and A. Martsinkovsky's exact sequence ([1], th. 7.1):

$$0 \to Ext^1_{\mathscr{G}}(M,N) \to Ext^1_R(M,N) \to \widehat{Ext}^1_R(M,N) \to \cdots$$

$$\to Ext^g_{\mathscr{G}}(M,N) \to Ext^g_R(M,N) \to \widehat{Ext}^g_R(M,N) \to 0$$

Our proof works in a more general case, for any module M of finite Gorenstein projective dimension, whether finitely generated or not.

There is also a dual result (Theorem 1). If $Gor\,inj\,dim\,N = d < \infty$ then the dth cosyzygy H of an injective resolution of N is a Gorenstein injective module. So there exists an exact sequence $\mathscr{E} : \cdots \to E_1 \to E_0 \to E_{-1} \to E_{-2} \to \cdots$ of injective modules such that $Hom(I,\mathscr{E})$ is exact for any injective left R-module I and $H = Ker(E_0 \to E_{-1})$. We call such sequence a complete injective resolution of N. We show that a complete injective resolution of N is unique up to homotopy. For each left R-module M and for each $n \in \boldsymbol{Z}$ let $\overline{Ext}^n_R(M,N) \stackrel{\text{def}}{=} H^n(Hom(M,\mathscr{E}))$. A dual argument of the proof of Proposition 1 shows the existence of an exact sequence $0 \to Ext^1_{\mathscr{G}\mathscr{I}}(M,N) \to Ext^1_R(M,N) \to \overline{Ext}^1_R(M,N) \to Ext^2_{\mathscr{G}\mathscr{I}}(M,N) \to \cdots \to Ext^d_{\mathscr{G}\mathscr{I}}(M,N) \to Ext^d_R(M,N) \to \overline{Ext}^d_R(M,N) \to 0$ where $Ext^i_{\mathscr{G}\mathscr{I}}(M,N)$ are the right derived functors of $Hom(M,N)$, computed using a right Gorenstein injective resolution of N. If $Gor\,proj\,dim\,M < \infty$ then $Ext^i_{\mathscr{G}}(M,N) \simeq Ext^i_{\mathscr{G}\mathscr{I}}(M,N)$, for all $i \geq 0$ ([4], Theorem 3.6). So in this case we obtain an exact sequence

$$0 \to Ext^1_{\mathscr{G}}(M,N) \to Ext^1_R(M,N) \to \overline{Ext}^1_R(M,N) \to \cdots$$

We prove (Theorem 2) that over Gorenstein rings we have $\overline{Ext}^n_R(M,N) \simeq \widehat{Ext}^n_R(M,N)$ for all left R-modules M, N, for any $n \in \boldsymbol{Z}$. Thus, over Gorenstein rings there is a new way of computing the Tate cohomology.

2. Preliminaries

Let R be an associative ring with 1 and let $\mathscr{P}$ be a class of left R-modules.

DEFINITION 1 [3]. *For a left R-module M a morphism $\phi : P \to M$ where $P \in \mathscr{P}$ is a $\mathscr{P}$-precover of M if $Hom(P', P) \to Hom(P', M) \to 0$ is exact for any $P' \in \mathscr{P}$.*

DEFINITION 2. *A $\mathscr{P}$-resolution of a left R-module M is a complex $\mathbf{P} : \cdots \to P_1 \to P_0 \to M \to 0$ (not necessarily exact) with each $P_i \in \mathscr{P}$ and such that for any $P' \in \mathscr{P}$ the complex $Hom(P', \mathbf{P})$ is exact.*

Throughout the paper we refer to the complex $\mathbf{P.} : \cdots \to P_1 \to P_0 \to 0$ as a *deleted $\mathscr{P}$ resolution* of M.

We note that a complex $\mathbf{P}$ as in Definition 2 is a $\mathscr{P}$-resolution if and only if $P_0 \to M$, $P_1 \to Ker(P_0 \to M)$ and $P_i \to Ker(P_{i-1} \to P_{i-2})$ for $i \geq 2$ are $\mathscr{P}$-precovers. If $\mathscr{P}$ contains all the projective left R-modules then any $\mathscr{P}$-precover is a surjective map and therefore any $\mathscr{P}$-resolution is an exact complex.

A $\mathscr{P}$-resolution of a left R-module M is unique up to homotopy ([3], pg. 169) and so it can be used to compute derived functors.

DEFINITION 3. *Let M be a left R-module that has a $\mathscr{P}$-resolution $\mathbf{P} : \cdots \to P_1 \to P_0 \to M \to 0$. Then $Ext^n_{\mathscr{P}}(M, N) = H^n(Hom(\mathbf{P.}, N))$ for any left R-module N and any $n \geq 0$, where $\mathbf{P.}$ is the deleted resolution.*

We prove the existence of the exact sequence (1).

Let $\mathscr{P}$, $\mathscr{C}$ be two classes of left R-modules such that $Proj \subset \mathscr{P} \subset \mathscr{C}$ where $Proj$ is the class of projective modules. Let M be a left R-module that has both a $\mathscr{P}$-resolution $\mathbf{P} : \cdots \to P_1 \to P_0 \to M \to 0$ and a $\mathscr{C}$-resolution $\mathbf{C} : \cdots \to C_1 \to C_0 \to M \to 0$.

$P_i \in \mathscr{P} \subset \mathscr{C}$ so $Hom(P_i, \mathbf{C})$ is an exact complex for any $i \geq 0$. It follows that there are morphisms $P_i \to C_i$ making

$$\begin{array}{ccccccccc}
\mathbf{P} : \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \| & & \\
\mathbf{C} : \cdots & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

into a commutative diagram.

Let $u : \mathbf{P} \to \mathbf{C}$, $u = (u_i)_{i \geq 0}$ be such a chain map induced by Id_M and let $\overline{M(u)}$ be the associated mapping cone. Since $0 \to \mathbf{C} \to \overline{M(u)} \to \mathbf{P}[1] \to 0$ is exact and both $\mathbf{P}$ and $\mathbf{C}$ are exact complexes, the exactness of $\overline{M(u)}$ follows. $\overline{M(u)}$ has the exact subcomplex $0 \to M \xrightarrow{Id} M \to 0$. Forming the quotient, we get an exact complex, $M(u)$, which is the mapping cone of the chain map $u : \mathbf{P.} \to \mathbf{C.}$ ($\mathbf{P.}$ and $\mathbf{C.}$ being the deleted $\mathscr{P}$, $\mathscr{C}$-resolutions). The sequence $0 \to \mathbf{C.} \to M(u) \to \mathbf{P.}[1] \to 0$ is split exact in each degree, so for any left R-module N we have an exact sequence of complexes $0 \to Hom(\mathbf{P.}[1], N) \to Hom(M(u), N) \to Hom(\mathbf{C.}, N) \to 0$ and therefore an associated cohomology exact sequence: $\cdots \to H^n(Hom(M(u), N)) \to H^n(Hom(\mathbf{C.}, N)) \to H^{n+1}(Hom(\mathbf{P.}[1], N)) \to H^{n+1}(Hom(M(u), N)) \to H^{n+1}(Hom(\mathbf{C.}, N)) \to \cdots$ Since $M(u)$ is exact and the functor $Hom(-, N)$ is left exact, it follows that $H^0(Hom(M(u), N)) = H^1(Hom(M(u), N)) = 0$. We have $H^0(Hom(\mathbf{C.}, N)) \simeq Hom(M, N)$ and $H^1(Hom(\mathbf{P.}[1], N)) \simeq Hom(M, N)$. So, the long exact sequence above is: $0 \to Hom(M, N) \to Hom(M, N) \to 0 \to H^1(Hom(\mathbf{C.}, N)) \to H^2(Hom(\mathbf{P.}[1], N)) \to H^2(Hom(M(u), N)) \to \cdots$ After factoring out the exact sequence $0 \to Hom(M, N) \xrightarrow{\sim} Hom(M, N) \to 0$ we obtain the exact sequence (1):

$$0 \to Ext^1_{\mathscr{C}}(M, N) \to Ext^1_{\mathscr{P}}(M, N) \to \widehat{Ext}^1_{\mathscr{C},\mathscr{P}}(M, N) \to \cdots$$

We prove that the generalized Tate cohomology $\widehat{Ext}_{\mathscr{C},\mathscr{P}}(M, -)$ is well defined.

Let $\mathscr{P}$, $\mathscr{C}$ be two classes of left R-modules such that $\mathscr{P} \subset \mathscr{C}$.

Let $\mathbf{P}$, $\mathbf{P}'$ be two $\mathscr{P}$-resolutions of M and let $\mathbf{C}$, $\mathbf{C}'$ be two $\mathscr{C}$-resolutions of M.

$$\mathbf{P} : \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0, \quad \mathbf{P}' : \cdots \xrightarrow{f_2'} P_1' \xrightarrow{f_1'} P_0' \xrightarrow{f_0'} M \to 0$$

$$\mathbf{C} : \cdots \xrightarrow{g_2} C_1 \xrightarrow{g_1} C_0 \xrightarrow{g_0} M \to 0, \quad \mathbf{C}' : \cdots \xrightarrow{g_2'} C_1' \xrightarrow{g_1'} C_0' \xrightarrow{g_0'} M \to 0$$

There exist maps of complexes $u : \mathbf{P} \to \mathbf{C}$ and $v : \mathbf{P}' \to \mathbf{C}'$, both induced by Id_M. $\overline{M(u)} : \cdots \to C_3 \oplus P_2 \xrightarrow{\delta_3} C_2 \oplus P_1 \xrightarrow{\delta_2} C_1 \oplus P_0 \xrightarrow{\delta_1} C_0 \oplus M \xrightarrow{\delta_0} M \to 0$ and $\overline{M(v)} : \cdots \to C_3' \oplus P_2' \xrightarrow{\delta_3'} C_2' \oplus P_1' \xrightarrow{\delta_2'} C_1' \oplus P_0' \xrightarrow{\delta_1'} C_0' \oplus M \xrightarrow{\delta_0'} M \to 0$ (with $\delta_n(x, y) = (g_n(x) + u_{n-1}(y), -f_{n-1}(y))$ for $n \geq 1$, $\delta_0(x, y) = g_0(x) + y$, $\delta_n'(x, y) = (g_n'(x) + v_{n-1}(y), -f_{n-1}'(y))$ for $n \geq 1$, $\delta_0'(x, y) = g_0'(x) + y$) are the associated mapping cones.

$M(u) : \cdots \to C_3 \oplus P_2 \xrightarrow{\delta_3} C_2 \oplus P_1 \xrightarrow{\delta_2} C_1 \oplus P_0 \xrightarrow{\overline{\delta_1}} C_0 \to 0$ (with $\overline{\delta_1}(x, y) = g_1(x) + u_0(y)$) and $M(v) : \cdots \to C_3' \oplus P_2' \xrightarrow{\delta_3'} C_2' \oplus P_1' \xrightarrow{\delta_2'} C_1' \oplus P_0' \xrightarrow{\overline{\delta_1'}} C_0' \to 0$ (with $\overline{\delta_1'}(x, y) = g_1'(x) + v_0(y)$) are the mapping cones of $u : \mathbf{P.} \to \mathbf{C.}$ and $v : \mathbf{P}'. \to \mathbf{C}'.$.

Since the exact sequence of complexes $0 \to \mathbf{C} \to \overline{M(u)} \to \mathbf{P}[1] \to 0$ is split exact in each degree, for each ${}_RF$ we have an exact sequence: $0 \to Hom(F, \mathbf{C}) \to Hom(F, \overline{M(u)}) \to Hom(F, \mathbf{P}[1]) \to 0$. If $F \in \mathscr{P} \subset \mathscr{C}$ then both complexes $Hom(F, \mathbf{C})$ and $Hom(F, \mathbf{P}[1])$ are exact, so the exactness of $Hom(F, \overline{M(u)})$ follows.

Each $P_i \in \mathscr{P}$, so by the above, the complex $Hom(P_i, \overline{M(u)})$ is exact.

Let $\overline{M}$ denote the complex $0 \to M \xrightarrow{Id} M \to 0$. The exact sequence of complexes $0 \to \overline{M} \to \overline{M(u)} \to M(u) \to 0$ is split exact in each degree. Consequently the sequence $0 \to Hom(P_i, \overline{M}) \to Hom(P_i, \overline{M(u)}) \to Hom(P_i, M(u)) \to 0$ is exact for any $i \geq 0$. Since both $Hom(P_i, \overline{M(u)})$ and $Hom(P_i, \overline{M})$ are exact complexes, it follows that

$$(2) \qquad Hom(P_i, M(u)) \text{ is an exact complex,}$$

for any $i \geq 0$.

The identity map Id_M induces maps of complexes $h : \mathbf{P.} \to \mathbf{P'.}$ and $k : \mathbf{C.} \to \mathbf{C'.}$.

Both $v \circ h : \mathbf{P.} \to \mathbf{C'.}$ and $k \circ u : \mathbf{P.} \to \mathbf{C'.}$ are maps of complexes induced by Id_M, so $v \circ h$ and $k \circ u$ are homotopic. Hence there exists $s_i \in Hom(P_i, C'_{i+1})$, $i \geq 0$ such that $v_0 \circ h_0 - k_0 \circ u_0 = g'_1 \circ s_0$ and $v_n \circ h_n - k_n \circ u_n = g'_{n+1} \circ s_n + s_{n-1} \circ f_n$ for any $n \geq 1$.

Then $\omega : M(u) \to M(v)$ defined by $\bar{\omega} : C_0 \to C'_0$, $\bar{\omega} = k_0$, $\omega_n : C_{n+1} \oplus P_n \to C'_{n+1} \oplus P'_n$, $\omega_n(x, y) = (k_{n+1}(x) - s_n(y), h_n(y))$ for any $n \geq 0$, is a map of complexes.

The identity map Id_M also induces maps of complexes $l : \mathbf{P'.} \to \mathbf{P.}$, $t : \mathbf{C'.} \to \mathbf{C.}$. Then $t \circ v : \mathbf{P'.} \to \mathbf{C.}$ and $u \circ l : \mathbf{P'.} \to \mathbf{C.}$ are homotopic.

So we have a map of complexes $\psi : M(v) \to M(u)$ where $\psi_n : C'_{n+1} \oplus P'_n \to C_{n+1} \oplus P_n$ is defined by $\psi_n(x, y) = (t_{n+1}(x) - \bar{s}_n(y), l_n(y))$, $n \geq 0$ (with $\bar{s}_n : P'_n \to C_{n+1}$ such that $u_n \circ l_n - t_n \circ v_n = \bar{s}_{n-1} \circ f'_n + g_{n+1} \circ \bar{s}_n$, $\forall n \geq 1$, $u_0 \circ l_0 - t_0 \circ v_0 = g_1 \circ \bar{s}_0$) and $\bar{\psi} : C'_0 \to C_0$, $\bar{\psi} = t_0$.

We prove that $\psi \circ \omega$ is homotopic to $Id_{M(u)}$.

Since $t \circ k : \mathbf{C.} \to \mathbf{C.}$ is a chain map induced by Id_M, we have $t \circ k \sim Id_{\mathbf{C.}}$. So there exist maps $\beta_i \in Hom(C_i, C_{i+1})$, $i \geq 0$ such that $t_0 \circ k_0 - Id = g_1 \circ \beta_0$ and $t_i \circ k_i - Id = \beta_{i-1} \circ g_i + g_{i+1} \circ \beta_i$, $\forall i \geq 1$.

Let $\chi_0 : C_0 \to C_1 \oplus P_0$, $\chi_0(x) = (\beta_0(x), 0)$, $\forall x \in C_0$. Then $\overline{\delta_1} \circ \chi_0(x) = \overline{\delta_1}(\beta_0(x), 0) = g_1(\beta_0(x)) + u_0(0) = (t_0 \circ k_0 - Id)(x) = (\bar{\psi} \circ \bar{\omega} - Id)(x)$, $\forall x \in C_0$.

We have $\overline{\delta_1} \circ (\psi_0 \circ \omega_0 - \chi_0 \circ \overline{\delta_1} - Id) = \overline{\delta_1} \circ \psi_0 \circ \omega_0 - (\overline{\delta_1} \circ \chi_0) \circ \overline{\delta_1} - \overline{\delta_1} = t_0 \circ k_0 \circ \overline{\delta_1} - (t_0 \circ k_0 - Id) \circ \overline{\delta_1} - \overline{\delta_1} = 0$.

Let $r_0 : P_0 \to C_1 \oplus P_0$, $r_0 = (\psi_0 \circ \omega_0 - Id - \chi_0 \circ \overline{\delta_1}) \circ e_0$ with $e_0 : P_0 \to C_1 \oplus P_0$, $e_0(y) = (0, y)$. We have $\overline{\delta_1} \circ r_0 = \overline{\delta_1} \circ (\psi_0 \circ \omega_0 - Id - \chi_0 \circ \overline{\delta_1}) \circ e_0 = 0$. Since $r_0 \in Ker\, Hom(P_0, \overline{\delta_1}) = Im\, Hom(P_0, \delta_2)$ (by (2)) it follows that $r_0 = \delta_2 \circ \gamma_1$ for some $\gamma_1 \in Hom(P_0, C_2 \oplus P_1)$. Hence $(\psi_0 \circ \omega_0 - Id - \chi_0 \circ \overline{\delta_1})(0, y) = \delta_2(\gamma_1(y))$.

Also we have $(\psi_0 \circ \omega_0 - Id - \chi_0 \circ \overline{\delta_1})(x, 0) = \psi_0(\omega_0(x, 0)) - (x, 0) - \chi_0(\overline{\delta_1}(x, 0)) = \psi_0(k_1(x), 0) - (x, 0) - \chi_0(g_1(x)) = ((t_1 \circ k_1 - Id - \beta_0 \circ g_1)(x), 0) = ((g_2 \circ \beta_1)(x), 0) = \delta_2(\beta_1(x), 0)$.

So $(\psi_0 \circ \omega_0 - Id - \chi_0 \circ \overline{\delta_1})(x, y) = \delta_2 \circ \chi_1(x, y)$ where $\chi_1 : C_1 \oplus P_0 \to C_2 \oplus P_1$, $\chi_1(x, y) = (\beta_1(x), 0) + \gamma_1(y)$. Hence $\psi_0 \circ \omega_0 - Id = \chi_0 \circ \overline{\delta_1} + \delta_2 \circ \chi_1$.

Similarly, there exists $\chi_i \in Hom(C_i \oplus P_{i-1}, C_{i+1} \oplus P_i)$ such that $\psi_i \circ \omega_i - Id = \chi_i \circ \delta_{i+1} + \delta_{i+2} \circ \chi_{i+1}$, $\forall i \geq 1$.

Thus $\psi \circ \omega \sim Id_{M(u)}$. Similarly, $\omega \circ \psi \sim Id_{M(v)}$. Then $H^n(Hom(M(v), N)) \simeq H^n(Hom(M(u), N))$ for any ${}_RN$, for any $n \geq 0$.

Remark 1. *The proof above does not depend on $\mathscr{P}$, $\mathscr{C}$ containing all the projective R-modules. It works for any two classes $\mathscr{P}$, $\mathscr{C}$ of left R-modules such that $\mathscr{P} \subset \mathscr{C}$. And even without assuming that $\mathscr{P}$, $\mathscr{C}$ contain the projectives we still get an Avramov-Martsinkovsky type sequence. Let $\mathscr{P}$, $\mathscr{C}$ be two classes of left R-modules such that $\mathscr{P} \subset \mathscr{C}$. If the R-module M has a $\mathscr{P}$-resolution* **P** *and a $\mathscr{C}$-resolution* **C** *then Id_M induces a chain map $u : \mathbf{P.} \to \mathbf{C.}$ and we have an exact sequence of complexes $0 \to \mathbf{C.} \to M(u) \to \mathbf{P.}[1] \to 0$ which is split exact in each degree, so $0 \to Hom(\mathbf{P.}[1], N) \to Hom(M(u), N) \to Hom(\mathbf{C.}, N) \to 0$ is still exact for any R-module N. Its associated long exact sequence is: $0 \to H^0(Hom(M(u), N)) \to Ext^0_{\mathscr{C}}(M, N) \to Ext^0_{\mathscr{P}}(M, N) \to \widehat{Ext}^0_{\mathscr{C},\mathscr{P}}(M, N) \to Ext^1_{\mathscr{C}}(M, N) \to \cdots$ (with $\widehat{Ext}^n_{\mathscr{C},\mathscr{P}}(M, N) = H^{n+1}(Hom(M(u), N))$ for any $n \geq 0$).*

Example 1. *Let $R = \boldsymbol{Z}$, $\mathscr{P}$ = the class of projective $\boldsymbol{Z}$-modules, $\mathscr{T}$ = the class of torsion free modules (so $\mathscr{P} \subset \mathscr{T}$), $M = \boldsymbol{Z}/_{2\boldsymbol{Z}}$, $N = \boldsymbol{Z}/_{2\boldsymbol{Z}}$. A $\mathscr{P}$-resolution of M is $0 \to \boldsymbol{Z} \xrightarrow{2} \boldsymbol{Z} \xrightarrow{\pi} \boldsymbol{Z}/_{2\boldsymbol{Z}} \to 0$. A $\mathscr{T}$-resolution of M is $0 \to 2\hat{\boldsymbol{Z}}_2 \to \hat{\boldsymbol{Z}}_2 \xrightarrow{\varphi} \boldsymbol{Z}/_{2\boldsymbol{Z}} \to 0$, with $\varphi\left(\sum_{i=0}^{\infty} \alpha_i \cdot 2^i\right) = a_0$. There is a map of complexes $u : P. \to T.$ ($P.$, $T.$ are the deleted $\mathscr{P}$, $\mathscr{T}$-resolutions) and the mapping cone $M(u) : 0 \to \boldsymbol{Z} \to 2\hat{\boldsymbol{Z}}_2 \oplus \boldsymbol{Z} \to \hat{\boldsymbol{Z}}_2 \to 0$ is exact. Since the class $\mathscr{T}$ of torsion free $\boldsymbol{Z}$-modules coincides with the class of flat $\boldsymbol{Z}$-modules and $\mathscr{P} \subset \mathscr{T}$, $M(u)$ is an exact sequence of flat Z-modules. We have $Hom(\boldsymbol{Z}/_{2\boldsymbol{Z}}, \boldsymbol{Q}/_{\boldsymbol{Z}}) \simeq \boldsymbol{Z}/_{2\boldsymbol{Z}}$. So $\boldsymbol{Z}/_{2\boldsymbol{Z}}$ is pure injective and therefore cotorsion. It follows that $Hom(M(u), \boldsymbol{Z}/_{2\boldsymbol{Z}})$ is an exact complex and therefore $\widehat{Ext}^n_{\mathscr{C},\mathscr{P}}(\boldsymbol{Z}/_{2\boldsymbol{Z}}, \boldsymbol{Z}/_{2\boldsymbol{Z}}) = 0$ for all n. So, in this case, the exact sequence $0 \to Ext^1_{\mathscr{T}}(\boldsymbol{Z}/_{2\boldsymbol{Z}}, \boldsymbol{Z}/_{2\boldsymbol{Z}}) \to Ext^1_{\boldsymbol{Z}}(\boldsymbol{Z}/_{2\boldsymbol{Z}}, \boldsymbol{Z}/_{2\boldsymbol{Z}}) \to \widehat{Ext}^1_{\mathscr{T},\mathscr{P}}(\boldsymbol{Z}/_{2\boldsymbol{Z}}, \boldsymbol{Z}/_{2\boldsymbol{Z}}) \to Ext^2_{\mathscr{T}}(\boldsymbol{Z}/_{2\boldsymbol{Z}}, \boldsymbol{Z}/_{2\boldsymbol{Z}}) \to \cdots$ is $0 \to Ext^1_{\mathscr{T}}(\boldsymbol{Z}/_{2\boldsymbol{Z}}, \boldsymbol{Z}/_{2\boldsymbol{Z}}) \to Ext^1_{\boldsymbol{Z}}(\boldsymbol{Z}/_{2\boldsymbol{Z}}, \boldsymbol{Z}/_{2\boldsymbol{Z}}) \to 0$ with $Ext^1_{\boldsymbol{Z}}(\boldsymbol{Z}/_{2\boldsymbol{Z}}, \boldsymbol{Z}/_{2\boldsymbol{Z}}) \simeq \boldsymbol{Z}/_{2\boldsymbol{Z}}$.*

3. Avramov-Martsinkovsky's Exact Sequence

For the rest of the article R denotes a left noetherian ring (unless otherwise specified) and R-module means left R-module. For unexplained terminology and notation please see [1] and [3].

Proposition 1 below shows that when $\mathscr{P}$ is the class of projective R-modules, $\mathscr{G}$ is the class of Gorenstein projective R-modules and M is an R-module of finite Gorenstein projective dimension, the modules $\widehat{Ext}^n_{\mathscr{G},\mathscr{P}}(M,N)$ are the usual Tate cohomology modules for any $n \geq 1$.

We recall first the following:

DEFINITION 4 ([1]). *A complete resolution of an R-module M is a diagram* $\mathbf{T} \xrightarrow{u} \mathbf{P} \xrightarrow{\pi} M$ *where* $\mathbf{P} \xrightarrow{\pi} M$ *is a projective resolution of* M, $\mathbf{T}$ *is a totally acyclic complex,* u *is a morphism of complexes and* u_n *is bijective for all* $n \gg 0$. *If* $\mathbf{T} \xrightarrow{u} \mathbf{P} \xrightarrow{\pi} M$ *is such a complete resolution of M then for each left R-module N and for each* $n \in \mathbf{Z}$ *the usual Tate cohomology module* $\widehat{Ext}^n_R(M,N)$ *is defined by the equality* $\widehat{Ext}^n_R(M,N) = H^n(Hom(\mathbf{T},N))$.

PROPOSITION 1. *If M is an R-module with Gor proj dim* $M < \infty$ *then for each R-module N we have* $\widehat{Ext}^n_{\mathscr{G},\mathscr{P}}(M,N) \simeq \widehat{Ext}^n_R(M,N)$ *for any* $n \geq 1$.

PROOF. Let $g = Gor\,proj\,dim\,M$.

We start by constructing a complete resolution of M.

If $0 \longrightarrow C \xrightarrow{i} P_{g-1} \xrightarrow{f_{g-1}} P_{g-2} \xrightarrow{f_{g-2}} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$ is a partial projective resolution of M then C is a Gorenstein projective module ([5], Theorem 2.20). Hence there exists an exact sequence $\mathbf{T} : \cdots \longrightarrow P^{-2} \xrightarrow{d_{-2}} P^{-1} \xrightarrow{d_{-1}} P^0 \xrightarrow{d_0} P^1 \longrightarrow \cdots$ of projective modules such that $C = Ker\,d_0$ and $Hom(\mathbf{T},P)$ is an exact complex for any projective R-module P. In particular $Hom(\mathbf{T},R)$ is exact. Since each P^n is a projective module and $H_n(\mathbf{T}) = 0 = H_n(\mathbf{T}^*)$ for any integer n, the complex $\mathbf{T}$ is totally acyclic.

Since $C = Im\,d_{-1} = Ker\,f_{g-1}$ and $\cdots \longrightarrow P^{-2} \xrightarrow{d_{-2}} P^{-1} \xrightarrow{d_{-1}} C \longrightarrow 0$ is exact, the complex $\mathbf{P} : \cdots \longrightarrow P^{-2} \xrightarrow{d_{-2}} P^{-1} \xrightarrow{i \circ d_{-1}} P_{g-1} \xrightarrow{f_{g-1}} P_{g-2} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$ is a projective resolution of M.

$$\begin{array}{ccccccccccccccc}
\mathbf{T}: \cdots \longrightarrow & P^{-1} & \xrightarrow{d_{-1}} & P^0 & \xrightarrow{d_0} & P^1 & \xrightarrow{d_1} & \cdots \longrightarrow & P^{g-2} & \xrightarrow{d_{g-2}} & P^{g-1} & \xrightarrow{d_{g-1}} & P^g & \longrightarrow \cdots \\
& \| & & \downarrow{\scriptstyle u_{g-1}} & & \downarrow{\scriptstyle u_{g-2}} & & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \downarrow & \\
\mathbf{P}: \cdots \longrightarrow & P^{-1} & \xrightarrow{i \circ d_{-1}} & P_{g-1} & \xrightarrow{f_{g-1}} & P_{g-2} & \xrightarrow{f_{g-2}} & \cdots \longrightarrow & P_1 & \xrightarrow{f_1} & P_0 & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

Since P_{g-1} is projective, the complex $Hom(\mathbf{T},P_{g-1})$ is exact. We have $i \circ d_{-1} \in Ker\,Hom(d_{-2},P_{g-1}) = Im\,Hom(d_{-1},P_{g-1})$. So there exists $u_{g-1} \in Hom(P^0,P_{g-1})$ such that $i \circ d_{-1} = u_{g-1} \circ d_{-1}$. Similarly there exist $u_{g-2},\ldots,u_0$ that make the diagram commutative. Since $u : \mathbf{T} \to \mathbf{P}$ (with u_0, $u_1,\ldots,u_{g-1}$ as above and $u_n =$

$Id_{P^{g-1-n}}$ for $n \geq g$) is a morphism of complexes, u_n is bijective for $n \geq g$, $\mathbf{T}$ is a totally acyclic complex and $\mathbf{P} \to M$ is a projective resolution of M, it follows that $\mathbf{T} \xrightarrow{u} \mathbf{P} \xrightarrow{\pi} M$ is a complete resolution of M.

We use now the projective resolution $\mathbf{P}$ and the complete resolution $\mathbf{T}$ to construct a Gorenstein projective resolution of M.

Let $D = \operatorname{Im} d_{g-1}$. Then D is a Gorenstein projective module ([5], Obs. 2.2) and there is a commutative diagram:

$$\begin{array}{ccccccccccccccccccc}
0 & \longrightarrow & C & \longrightarrow & P^0 & \xrightarrow{d_0} & P^1 & \xrightarrow{d_1} & P^2 & \xrightarrow{d_2} & \cdots \longrightarrow & P^{g-2} & \xrightarrow{d_{g-2}} & P^{g-1} & \xrightarrow{d_{g-1}} & D & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle u_{g-1}} & & \downarrow{\scriptstyle u_{g-2}} & & \downarrow{\scriptstyle u_{g-3}} & & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & C & \longrightarrow & P_{g-1} & \xrightarrow{f_{g-1}} & P_{g-2} & \xrightarrow{f_{g-2}} & P_{g-3} & \xrightarrow{f_{g-3}} & \cdots \longrightarrow & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{\pi} & M & \longrightarrow & 0
\end{array}$$

with u defined by: $u(d_{g-1}(x)) = \pi(u_0(x))$.

Since both rows are exact complexes, the associated mapping cone $\overline{\mathscr{C}} : 0 \longrightarrow C \xrightarrow{\Delta} C \oplus P^0 \xrightarrow{\delta_0} P_{g-1} \oplus P^1 \xrightarrow{\delta_1} P_{g-2} \oplus P^2 \longrightarrow \cdots \longrightarrow P_1 \oplus P^{g-1} \xrightarrow{\delta_{g-1}} P_0 \oplus D \xrightarrow{\beta} M \longrightarrow 0$ is also an exact complex.

$\overline{\mathscr{C}}$ has the exact subcomplex: $0 \to C \xrightarrow{\sim} C \to \underline{0}$. Forming the quotient complex, we get an exact complex: $0 \longrightarrow 0 \longrightarrow P^0 \xrightarrow{\overline{\delta_0}} P_{g-1} \oplus P^1 \xrightarrow{\delta_1} P_{g-2} \oplus P^2 \longrightarrow \cdots \longrightarrow P_1 \oplus P^{g-1} \xrightarrow{\delta_{g-1}} P_0 \oplus D \xrightarrow{\beta} M \longrightarrow 0$.

Let L be a Gorenstein projective module. Since $proj\,dim\,Ker\,\beta < \infty$, we have $Ext^1_R(L, Ker\,\beta) = 0$ ([5], Proposition 2.3). The sequence $0 \to Ker\,\beta \to P_0 \oplus D \to M \to 0$ is exact, so we have the associated exact sequence: $0 \to Hom(L, Ker\,\beta) \to Hom(L, P_0 \oplus D) \to Hom(L, M) \to Ext^1_R(L, Ker\,\beta) = 0$. Thus $P_0 \oplus D \to M$ is a Gorenstein projective precover. Similarly $P_1 \oplus P^{g-1} \to Ker\,\beta$ is a Gorenstein projective precover, $\ldots, P^0 \to Ker\,\delta_1$ is a Gorenstein projective precover, so $\mathbf{G}$: $0 \to P^0 \to P_{g-1} \oplus P^1 \to P_{g-2} \oplus P^2 \to \cdots \to P_0 \oplus D \to M \to 0$ is a Gorenstein projective resolution of M.

There is a map of complexes $e : \mathbf{P} \to \mathbf{G}$

$$\begin{array}{ccccccccccccccc}
\cdots \longrightarrow & P^{-2} & \xrightarrow{d_{-2}} & P^{-1} & \xrightarrow{d_{-1}} & P_{g-1} & \xrightarrow{f_{g-1}} \cdots \longrightarrow & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & \downarrow & & \downarrow{\scriptstyle d_{-1}} & & \downarrow{\scriptstyle e_{g-1}} & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_0} & & \| & & \\
\cdots \longrightarrow & 0 & \longrightarrow & P^0 & \xrightarrow{\overline{\delta_0}} & P_{g-1} \oplus P^1 & \xrightarrow{\delta_1} \cdots \longrightarrow & P_1 \oplus P^{g-1} & \xrightarrow{\delta_{g-1}} & P_0 \oplus D & \xrightarrow{\beta} & M & \longrightarrow & 0
\end{array}$$

with

$$e_0 : P_0 \to P_0 \oplus D, \quad e_0(x) = (x, 0)$$

$$e_j : P_j \to P_j \oplus P^{g-j}, \quad e_j(x) = (x, 0) \quad 1 \leq j \leq g-1$$

P is a projective resolution of M, **G** is a Gorenstein projective resolution of M and $e : \mathbf{P} \to \mathbf{G}$ is a chain map induced by Id_M, so $\widehat{Ext}^n_{\mathscr{G},\mathscr{P}}(M,N) = H^{n+1}(Hom(M(e),N))$, $\forall n \geq 0$, where $M(e)$ is the mapping cone of $e : \mathbf{P.} \to \mathbf{G.}$.

Let

$$\overline{\mathbf{T}} : \cdots \xrightarrow{d_{-2}} P^{-1} \xrightarrow{d_{-1}} P^0 \longrightarrow \cdots \longrightarrow P^{g-2} \xrightarrow{d_{g-2}} P^{g-1} \xrightarrow{d_{g-1}} D \longrightarrow 0.$$

We prove that $M(e)$ and $\overline{\mathbf{T}}[1]$ are homotopically equivalent.

There is a map of complexes $\alpha : \overline{\mathbf{T}}[1] \to M(e)$ with

$\alpha_0 : P^0 \to P^0 \oplus P_{g-1}$, $\alpha_0(x) = (x, -u_{g-1}(x))$ $\forall x \in P^0$;

$\alpha_j : P^j \to P_{g-j} \oplus P^j \oplus P_{g-j-1}$, $\alpha_j(x) = (0, x, -u_{g-j-1}(x))$, $\forall x \in P^j$, $1 \leq j \leq g-1$

$\alpha' : D \to P_0 \oplus D$, $\alpha'(x) = (0,x)$ $\forall x \in D$; $\alpha_j = -Id_{P^j}$ if $j \leq -1$ is odd; $\alpha_j = Id_{P^j}$ if $j \leq -1$ is even.

There is also a map of complexes $l : M(e) \to \mathbf{T}[1]$:

$l_0 : P^0 \oplus P_{g-1} \to P^0$ $l_0(x,y) = x$ $\forall (x,y) \in P^0 \oplus P_{g-1}$

$l_j : P_{g-j} \oplus P^j \oplus P_{g-j-1} \to P^j$ $l_j(x,y,z) = y$ $\forall (x,y,z) \in P_{g-j} \oplus P^j \oplus P_{g-j-1}$ $1 \leq j \leq g-1$

$l' : P^0 \oplus D \to D$ $l'(x,y) = y$ $\forall (x,y) \in P^0 \oplus D$

$l_j = -Id_{P^j}$ if $j \leq -1$ is odd; $l_j = Id_{P^j}$ if $j \leq -1$ is even.

We have

$$l \circ \alpha = Id_{\overline{\mathbf{T}}[1]} \quad \text{and} \quad \alpha \circ l \sim Id_{M(e)} \tag{3}$$

(a chain homotopy between $\alpha \circ l$ and Id_M is given by the maps:

$\chi_0 : P_0 \oplus D \to P_1 \oplus P^{g-1} \oplus P_0$, $\chi_0(x,y) = (0,0,-x)$

$\chi_j : P_j \oplus P^{g-j} \oplus P_{j-1} \to P_{j+1} \oplus P^{g-j-1} \oplus P_j$, $\chi_j(x,y,z) = (0,0,-x)$, $1 \leq j \leq g-2$

$\chi_{g-1} : P_{g-1} \oplus P^1 \oplus P_{g-2} \to P^0 \oplus P_{g-1}$, $\chi_{g-1}(x,y,z) = (0,-x)$)

By (3) we have $H^{n+1}(Hom(M(e),N)) \simeq H^{n+1}(Hom(\overline{\mathscr{T}}[1],N))$ that is $\widehat{Ext}^n_{\mathscr{G},\mathscr{P}}(M,N) = \widehat{Ext}^n_R(M,N)$, for any ${}_RN$, for all $n \geq 1$. □

Corollary 1 (Avramov-Martsinkovsky). *Let M be an R-module with $Gor\,proj\,dim\, M = g < \infty$. For each R-module N there is an exact sequence*: $0 \to Ext^1_{\mathscr{G}}(M,N) \to Ext^1_R(M,N) \to \widehat{Ext}^1_R(M,N) \to \cdots \to Ext^n_{\mathscr{G}}(M,N) \to Ext^n_R(M,N) \to \widehat{Ext}^n_R(M,N) \to \cdots \to Ext^g_R(M,N) \to \widehat{Ext}^g_R(M,N) \to 0$.

Proof. By (1) there is an exact sequence: $0 \to Ext^1_{\mathscr{G}}(M,N) \to Ext^1_R(M,N) \to \widehat{Ext}^1_{\mathscr{G},\mathscr{P}}(M,N) \to \cdots$.

By Proposition 1 we have $\widehat{Ext}^i_{\mathscr{G},\mathscr{P}}(M,N) \simeq \widehat{Ext}^i_R(M,N)$, $\forall i \geq 1$.

Since $Ext_{\mathscr{G}}^{g+i}(M,N)=0$, $\forall i \geq 1$ the exact sequence above gives us: $0 \to Ext_{\mathscr{G}}^{1}(M,N) \to Ext_{R}^{1}(M,N) \to \widehat{Ext}_{R}^{1}(M,N) \to \cdots \to Ext_{\mathscr{G}}^{n}(M,N) \to Ext_{R}^{n}(M,N) \to \widehat{Ext}_{R}^{n}(M,N) \to \cdots \to Ext_{R}^{g}(M,N) \to \widehat{Ext}_{R}^{g}(M,N) \to 0.$ □

4. Computing the Tate Cohomology Using Complete Injective Resolutions

The classical groups $Ext_{R}^{n}(M,N)$ can be computed using either a projective resolution of M or an injective resolution of N. In this section we want to prove an analogous result for the groups $\widehat{Ext}_{R}^{n}(M,N)$. We note that we cannot use a straightforward modification of the proof in classical case. This is basically because the associated double complex in our case is not a first (or third) quadrant one and so we cannot use the usual machinery of spectral sequences.

We start by defining a complete injective resolution.

Let N be an R-module with $Gor\,inj\,dim\,N = d < \infty$.

If $0 \longrightarrow N \longrightarrow E^0 \xrightarrow{f_0} E^1 \xrightarrow{f_1} \cdots \longrightarrow E^{d-1} \xrightarrow{f_{d-1}} H \longrightarrow 0$ is a partial injective resolution of N, then H is a Gorenstein injective module ([5], Theorem 2.22). Hence there exists a $Hom(Inj,-)$ exact sequence

$$\mathscr{E}: \cdots \longrightarrow E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \xrightarrow{d_{-2}} \cdots$$

of injective modules such that $\mathscr{E}$ is exact and $H = Ker\,d_0$ ([3], 10.1.1).

We say that $\mathscr{E}$ is a *complete injective resolution* of N.

For each module ${}_RM$ and each $i \in \boldsymbol{Z}$ let $\overline{Ext}_{R}^{i}(M,N) \overset{\text{def}}{=} H^i(Hom(M,\mathscr{E}))$.

We prove that any two complete injective resolutions of N are homotopically equivalent.

Let $\mathscr{E}: \cdots \longrightarrow I^{-1} \xrightarrow{g_{-1}} I^0 \xrightarrow{g_0} I^1 \xrightarrow{g_1} I^2 \longrightarrow \cdots$ and $\bar{\mathscr{E}}: \cdots \longrightarrow \bar{I}^{-1} \xrightarrow{g'_{-1}} \bar{I}^0 \xrightarrow{g'_0} \bar{I}^1 \xrightarrow{g'_1} \cdots$ be two complete injective resolutions of N corresponding to two injective resolutions, $\mathscr{N}$ and $\bar{\mathscr{N}}$, of N ($H = Ker\,g_0 = Im\,g_{-1}$ is the dth cosyzygy of $\mathscr{N}$ and $\bar{H} = Ker\,g'_0 = Im\,g'_{-1}$ is the dth cosyzygy of $\bar{\mathscr{N}}$).

If $\mathscr{H}$ is the injective resolution of H obtained from $\mathscr{N}$ and $\bar{\mathscr{H}}$ is the injective resolution of $\bar{H}$ obtained from $\bar{\mathscr{N}}$ then $\mathscr{H}$ and $\bar{\mathscr{H}}$ are homotopically equivalent (since the two injective resolutions of N, $\mathscr{N}$ and $\bar{\mathscr{N}}$, are homotopically equivalent).

Since $\mathscr{E}': 0 \to H \to I^0 \to I^1 \to \cdots$ is an injective resolution of H it follows that $\mathscr{E}'$ and $\mathscr{H}$ are homotopically equivalent. Similarly $\bar{\mathscr{E}}': 0 \to \bar{H} \to \bar{I}^0 \to \bar{I}^1 \to \cdots$ is homotopically equivalent to $\bar{\mathscr{H}}$. Then, by the above, $\mathscr{E}'$ and $\bar{\mathscr{E}}'$ are homotopically equivalent. So there exist chain maps $u: \mathscr{E}' \to \bar{\mathscr{E}}'$ and $v: \bar{\mathscr{E}}' \to \mathscr{E}'$ (u defined by $\bar{u} \in Hom(H,\bar{H})$, $u_j \in Hom(I^j,\bar{I}^j)$, $j \geq 0$ and v defined by $\bar{v} \in$

$Hom(\overline{H}, H)$ and $v_j \in Hom(\overline{I^j}, I^j)$), there exist $\beta \in Hom(I^0, H)$, $\beta_j \in Hom(I^j, I^{j-1})$, $j \geq 1$ such that $\bar{v} \circ \bar{u} - Id = \beta \circ i$ (where $i : H \to I^0$ is the inclusion map), and

$$v_0 \circ u_0 - Id = \beta_1 \circ g_0 + i \circ \beta,$$

$$v_j \circ u_j - Id = g_{j-1} \circ \beta_j + \beta_{j+1} \circ g_j, \quad \forall j \geq 1.$$

Since $\mathscr{E}'' : \cdots \to I^{-2} \to I^{-1} \to H \to 0$ is an injective resolvent of H ([2], 1.3) and $\overline{\mathscr{E}''} : \cdots \to \bar{I}^{-2} \to \bar{I}^{-1} \to \overline{H} \to 0$ is an injective resolvent of $\overline{H}$, $\bar{u} \in Hom(H, \overline{H})$ induces a map of complexes $u : \mathscr{E}'' \to \overline{\mathscr{E}''}$, $u = (u_j)_{j \leq -1}$. Similarly, there is a map of complexes $v : \overline{\mathscr{E}''} \to \mathscr{E}''$, $v = (v_j)_{j \leq -1}$, induced by $\bar{v} \in Hom(\overline{H}, H)$.

Since I^0 is injective and $g_{-1} : I^{-1} \to H$ is an injective precover, there exists $\beta_0 \in Hom(I^0, I^{-1})$ such that $\beta = g_{-1} \circ \beta_0$. So $v_0 \circ u_0 - Id = \beta_1 \circ g_0 + i \circ \beta = \beta_1 \circ g_0 + g_{-1} \circ \beta_0$.

We have $g_{-1} \circ (v_{-1} \circ u_{-1} - Id - \beta_0 \circ g_{-1}) = 0 \Leftrightarrow Im(v_{-1} \circ u_{-1} - Id - \beta_0 \circ g_{-1}) \subset Ker\, g_{-1}$. Since I^{-1} is injective and $I^{-2} \xrightarrow{g_{-2}} Ker\, g_{-1}$ is an injective precover, there is $\beta_{-1} \in Hom(I^{-1}, I^{-2})$ such that $v_{-1} \circ u_{-1} - Id - \beta_0 \circ g_{-1} = g_{-2} \circ \beta_{-1}$.

Similarly, there exist $\beta_j \in Hom(I^j, I^{j-1})$, $\forall j \leq -1$ such that $v_j \circ u_j - Id = \beta_{j+1} \circ g_j + g_{j-1} \circ \beta_j$, $\forall j \leq -1$. Thus $v \circ u \sim Id_{\mathscr{E}}$. Similarly $u \circ v \sim Id_{\bar{\mathscr{E}}}$.

Hence $H^i(Hom(M, \mathscr{E})) \simeq H^i(Hom(M, \bar{\mathscr{E}}))$ for any ${}_RM$, for all $i \in \mathbf{Z}$.

So $\overline{Ext}_R^n(-, N)$ is well-defined.

If $\mathscr{N}.$ is a deleted injective resolution of N, $\mathscr{G}.$ is a deleted Gorenstein injective resolution of N and $v : \mathscr{G}. \to \mathscr{N}.$ is a chain map induced by Id_N then a dual argument of the proof of Theorem 1 shows that the cohomology of $Hom(M, M(v))$ gives us the functor $\overline{Ext}_R(M, N)$ and that there is an exact sequence

$$0 \to Ext^1_{\mathscr{G}\mathscr{I}}(M,N) \to Ext^1_R(M,N) \to \overline{Ext}^1_R(M,N) \to Ext^2_{\mathscr{G}\mathscr{I}}(M,N)$$

$$\to \cdots \to Ext^d_{\mathscr{G}\mathscr{I}}(M,N) \to Ext^d_R(M,N) \to \overline{Ext}^d_R(M,N) \to 0$$

where $Ext^i_{\mathscr{G}\mathscr{I}}(M,N) = H^i(Hom(M, \mathscr{G}.))$ for any $i \geq 0$.

If *Gorproj dim* $M < \infty$ then $Ext^i_{\mathscr{G}}(M,N) \simeq Ext^i_{\mathscr{G}\mathscr{I}}(M,N)$ for any $i \geq 0$ ([4], Theorem 3.6).

Thus we have:

THEOREM 1. *Let N be an R-module with Gor inj dim $N = d < \infty$. For each R-module M with Gorproj dim $M < \infty$ there is an exact sequence*:

$$0 \to Ext^1_{\mathscr{G}}(M,N) \to Ext^1_R(M,N) \to \overline{Ext}^1_R(M,N) \to \cdots$$

Theorem 2 shows that over Gorenstein rings $\overline{Ext}_R^n(M,N) \simeq \widehat{Ext}_R^n(M,N)$ for any left R-modules M and N, for any $n \in \mathbf{Z}$.

Theorem 2. *If R is a Gorenstein ring then $\overline{Ext}_R^n(M,N) \simeq \widehat{Ext}_R^n(M,N)$ for any R-modules M, N for any $n \in \mathbf{Z}$.*

Proof. Let $g = Gor\,proj\,dim\,M$ and $d = Gor\,inj\,dim\,N$. R is a Gorenstein ring, so $g < \infty$ ([3], Corollary 11.5.8) and $d < \infty$ (this follows from [3], Theorem 11.2.1).

We are using the notations of Proposition 1 and Theorem 1.

• We prove first that if M is Gorenstein projective then $\widehat{Ext}_R^n(M,N) \simeq \overline{Ext}_R^n(M,N)$ for any $n \in \mathbf{Z}$.

Since M is Gorenstein projective we have a complete resolution $\mathbf{T} \xrightarrow{u} \mathbf{P} \xrightarrow{\pi} M$ with $T^n = P^n$, $\forall n \geq 0$ and $u_n = id_{P^n}$, $\forall n \geq 0$.

So

$$\widehat{Ext}_R^n(M,N) \simeq Ext_R^n(M,N) \quad \forall n \geq 1 \tag{4}$$

We have the exact sequence (by Theorem 1):

$$0 \to Ext_{\mathscr{G}}^1(M,N) \to Ext_R^1(M,N) \to \overline{Ext}_R^1(M,N) \to Ext_{\mathscr{G}}^2(M,N) \to \cdots$$

Since $Ext_{\mathscr{G}}^i(M,N) = 0$, $\forall i \geq 1$ it follows that

$$\overline{Ext}_R^i(M,N) \simeq Ext_R^i(M,N), \quad \forall i \geq 1 \tag{5}$$

By (4) and (5) we have $\overline{Ext}_R^i(M,N) \simeq \widehat{Ext}_R^i(M,N) \simeq Ext_R^i(M,N)$, for all $i \geq 1$.

• Case $n \leq 0$

Let $n = -k$, $k \geq 0$.

Let $\mathscr{E}$ be a complete injective resolution of N.

Since $\mathbf{T}: \cdots \longrightarrow P^{-2} \xrightarrow{d_{-2}} P^{-1} \xrightarrow{d_{-1}} P^0 \xrightarrow{d_0} P^1 \xrightarrow{d_1} P^2 \longrightarrow \cdots$ is exact with each P^i projective and such that $Hom(\mathbf{T}, Q)$ is exact for any projective module Q, it follows that $M^i = Im\,d_i$ is a Gorenstein projective module for any $i \in \mathbf{Z}$ ([5], Obs. 2.2).

Let $M^1 = Im\,d_1$. Since $0 \to M \to P^1 \to M^1 \to 0$ is exact and all the terms of $\mathscr{E}$ are injective modules, we have an exact sequence of complexes $0 \to Hom(M^1, \mathscr{E}) \to Hom(P^1, \mathscr{E}) \to Hom(M, \mathscr{E}) \to 0$ and therefore an associated long exact sequence:

(6) $$\cdots \to H^i(Hom(P^1,\mathscr{E})) \to H^i(Hom(M,\mathscr{E})) \to H^{i+1}(Hom(M^1,\mathscr{E})) \to H^{i+1}(Hom(P^1,\mathscr{E})) \to \cdots$$

Since a complete injective resolution $\mathscr{E}$ of N is exact and P^1 is projective, the complex $Hom(P^1,\mathscr{E})$ is exact. Then, by (6), we have $H^i(Hom(M,\mathscr{E})) \simeq H^{i+1}(Hom(M^1,\mathscr{E})) \Leftrightarrow \overline{Ext}^i_R(M,N) \simeq \overline{Ext}^{i+1}_R(M^1,N)$ for any ${}_RN$, for any $i \in \boldsymbol{Z}$.

Similarly,

(7) $$\overline{Ext}^i_R(M,N) \simeq \overline{Ext}^{i+k+1}_R(M^{k+1},N)$$

for any ${}_RN$ for all $i \in \boldsymbol{Z}$ where $M^{k+1} = Im\, d_{k+1} \in Gor\, Proj$.

Since R is a Gorenstein ring there is an exact sequence $0 \to G' \to L' \to N \to 0$ with $proj\, dim\, L' < \infty$ and G' a Gorenstein injective module ([3], Exercise 6, pp. 277).

Since each term of a complete resolution $\mathbf{T}$ is a projective module, we have an exact sequence of complexes $0 \to Hom(\mathbf{T},G') \to Hom(\mathbf{T},L') \to Hom(\mathbf{T},N) \to 0$ and therefore an associated long exact sequence:

(8) $$\cdots \to H^i(Hom(\mathbf{T},G')) \to H^i(Hom(\mathbf{T},L')) \to H^i(Hom(\mathbf{T},N)) \to H^{i+1}(Hom(\mathbf{T},G')) \to H^{i+1}(Hom(\mathbf{T},L')) \to \cdots$$

Since $proj\, dim\, L' < \infty$ it follows that $Hom(\mathbf{T},L')$ is an exact complex ([5], Proposition 2.3). Then, by (8), we have $H^i(Hom(\mathbf{T},N)) \simeq H^{i+1}(Hom(\mathbf{T},G'))$ that is

(9) $$\widehat{Ext}^i_R(M,N) \simeq \widehat{Ext}^{i+1}_R(M,G')$$

for any $i \in \boldsymbol{Z}$ and for any ${}_RM$.

Let $\bar{\mathscr{E}} : \cdots \longrightarrow \bar{E}_{-2} \xrightarrow{g_{-2}} \bar{E}_{-1} \xrightarrow{g_{-1}} \bar{E}_0 \xrightarrow{g_0} \bar{E}_1 \xrightarrow{g_1} \bar{E}_2 \longrightarrow \cdots$ be a complete injective resolution of the Gorenstein injective module G' ($G' = Ker\, g_0 = Im\, g_{-1}$) and let $G_i = Ker\, g_i$.

We have (same argument as above)

(10) $$\widehat{Ext}^i_R(M,N) \simeq \widehat{Ext}^{i+k+1}_R(M,G_{-k}), \quad \forall i \in \boldsymbol{Z}$$

for any ${}_RM$, where $G_{-k} = Ker\, g_{-k}$.

By (7), $\overline{Ext}^{-k}_R(M,N) \simeq \overline{Ext}^1_R(M^{k+1},N) \simeq Ext^1_R(M^{k+1},N) \simeq \widehat{Ext}^1_R(M^{k+1},N)$, (since M^{k+1} is Gorenstein projective). Then, by (10), $\widehat{Ext}^1_R(M^{k+1},N) \simeq \widehat{Ext}^{k+2}_R(M^{k+1},G_{-k}) \simeq Ext^{k+2}_R(M^{k+1},G_{-k})$.

So $\overline{Ext}^{-k}_R(M,N) \simeq Ext^{k+2}_R(M^{k+1},G_{-k})$.

By (10), $\widehat{Ext}^{-k}_R(M,N) \simeq \widehat{Ext}^1_R(M,G_{-k}) \simeq Ext^1_R(M,G_{-k}) \simeq \overline{Ext}^1_R(M,G_{-k})$, (since M is Gorenstein projective). Then, by (7), $\overline{Ext}_R^1(M, G_{-k}) \simeq \overline{Ext}_R^{k+2}(M^{k+1}, G_{-k}) \simeq Ext_R^{k+2}(M^{k+1}, G_{-k})$.

So $\widehat{Ext}_R^{-k}(M,N) \simeq Ext_R^{k+2}(M^{k+1}, G_{-k}) \simeq \overline{Ext}_R^{-k}(M,N)$ for any $k \in \mathbf{Z}$, $k \geq 0$.

Hence $\overline{Ext}_R^n(M,N) \simeq \widehat{Ext}_R^n(M,N)$ for any $n \in \mathbf{Z}$, if M is Gorenstein projective.

Similarly, $\overline{Ext}_R^n(M,N) \simeq \widehat{Ext}_R^n(M,N)$ for any $n \in \mathbf{Z}$, if N is Gorenstein injective.

- Case $g = Gor\,proj\,dim\, M \geq 1$

R is a Gorenstein ring, so there is an exact sequence $0 \to M \to L \to C' \to 0$ with $proj\,dim\, L < \infty$ and C' a Gorenstein projective module (the same argument used in [6], Corollary 3.3.7, gives this result for R-modules).

Since $proj\,dim\, L < \infty$ it follows that

$$Hom(L, \mathscr{E}) \text{ is an exact complex.} \tag{11}$$

Since $0 \to M \to L \to C' \to 0$ is exact and each term of $\mathscr{E}$ is an injective module we have an exact sequence of complexes $0 \to Hom(C', \mathscr{E}) \to Hom(L, \mathscr{E}) \to Hom(M, \mathscr{E}) \to 0$ and therefore an associated long exact sequence: $\cdots \to H^n(Hom(C', \mathscr{E})) \to H^n(Hom(L, \mathscr{E})) \to H^n(Hom(M, \mathscr{E})) \to H^{n+1}(Hom(C', \mathscr{E})) \to H^{n+1}(Hom(L, \mathscr{E})) \to \cdots$

By (11) we have $H^n(Hom(L, \mathscr{E})) = 0$ $\forall n \in \mathbf{Z}$. So

$$H^n(Hom(M, \mathscr{E})) \simeq H^{n+1}(Hom(C', \mathscr{E})) \tag{12}$$

$$\Leftrightarrow \overline{Ext}_R^n(M,N) \simeq \overline{Ext}_R^{n+1}(C',N)$$

for any ${}_RN$, for any $n \in \mathbf{Z}$.

So $\overline{Ext}_R^n(M,N) \simeq \overline{Ext}_R^{n+1}(C',N) \simeq \widehat{Ext}_R^{n+1}(C',N)$ (since $C' \in Gor\,Proj$) for any ${}_RN$, for all $n \in \mathbf{Z}$.

By (9) $\widehat{Ext}_R^{n+1}(C',N) \simeq \widehat{Ext}_R^{n+2}(C',G')$ $\forall n \in \mathbf{Z}$. (where $0 \to G' \to L' \to N \to 0$ is exact, $G' \in Gor\,Inj$, $L \in \mathscr{L}$)

Hence $\overline{Ext}_R^n(M,N) \simeq \widehat{Ext}_R^{n+2}(C',G')$ $\forall n \in \mathbf{Z}$.

By (9) $\widehat{Ext}_R^n(M,N) \simeq \widehat{Ext}_R^{n+1}(M,G') \simeq \overline{Ext}_R^{n+1}(M,G')$ (since G' is Gorenstein injective), for all $n \in \mathbf{Z}$. Then, by (12) $\overline{Ext}_R^{n+1}(M,G') \simeq \overline{Ext}_R^{n+2}(C',G') \simeq \widehat{Ext}_R^{n+2}(C',G')$ (since C' is Gorenstein projective) for all $n \in \mathbf{Z}$.

Hence $\overline{Ext}_R^n(M,N) \simeq \widehat{Ext}_R^{n+2}(C',G') \simeq \widehat{Ext}_R^n(M,N)$ $\forall n \in \mathbf{Z}$. □

Remark 2. *Theorem 2 shows that over Gorenstein rings there is a new way of computing the Tate cohomology, i.e. by using a complete injective resolution of N.*

In a subsequent publication we hope to show how we can exploit this procedure to gain new information about Tate cohomology modules.

Theorem 1 together with Theorem 2 give us the following result:

Let R be a Gorenstein ring, let N be an R-module with $Gor\,inj\,dim\,N = d < \infty$. For each R-module M there is an exact sequence:

$$0 \to Ext^1_{\mathscr{R}}(M,N) \to Ext^1_R(M,N) \to \widehat{Ext}^1_R(M,N) \to \cdots$$

$$\to Ext^d_R(M,N) \to \widehat{Ext}^d_R(M,N) \to 0.$$

Theorem 2 allows us to give an easy proof of the existence of a long exact sequence of Tate cohomology associated with any short exact sequence $0 \to M' \to M \to M'' \to 0$.

Theorem 3. *Let R be a Gorenstein ring. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. For any R-module N there exists a long exact sequence of Tate cohomology modules $\cdots \to \widehat{Ext}^n_R(M'',N) \to \widehat{Ext}^n_R(M,N) \to \widehat{Ext}^n_R(M',N) \to \widehat{Ext}^{n+1}_R(M'',N) \to \cdots$*

Proof. Let $\mathscr{E}$ be a complete injective resolution of N. Then, by Theorem 2, $\widehat{Ext}^n_R(M,N) \simeq H^n(Hom(M,\mathscr{E}))$ for any ${}_RM$ and any $n \in \boldsymbol{Z}$.

Since $0 \to M' \to M \to M'' \to 0$ is exact and each term of $\mathscr{E}$ is an injective module, we have an exact sequence of complexes: $0 \to Hom(M'',\mathscr{E}) \to Hom(M,\mathscr{E}) \to Hom(M',\mathscr{E}) \to 0$.

Its associated cohomology exact sequence is the desired long exact sequence. □

Remark 3. *J. Asadollahi and Sh. Salarian also have a proof of the claim of Theorem 2 in a recent preprint (Gorenstein Local Cohomology Modules) of theirs.*

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