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ON HOPF ALGEBRAS OF DIMENSION p^3

By

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Abstract. We discuss some general results on finite-dimensional Hopf algebras over an algebraically closed field k of characteristic zero and then apply them to Hopf algebras H of dimension p^3 over k. There are 10 cases according to the group-like elements of H and H^*. We show that in 8 of the 10 cases, it is possible to determine the structure of the Hopf algebra. We also give a partial classification of the quasitriangular Hopf algebras of dimension p^3 over k, after studying extensions of a group algebra of order p by a Taft algebra of dimension p^2. In particular, we prove that every ribbon Hopf algebra of dimension p^3 over k is either a group algebra or a Frobenius-Lusztig kernel. Finally, using some results from [1] and [4] on bounds for the dimension of the first term H_1 in the coradical filtration of H, we give the complete classification of the quasitriangular Hopf algebras of dimension 27.

1. Introduction

We work over an algebraically closed field k of characteristic zero. Let p be an odd prime number and let $\mathbf{G}_p$ be the cyclic group of p-th roots of unity. We denote by $T(q)$, the Taft algebra of parameter $q \in \mathbf{G}_p \backslash \{1\}$, see Remark 1.2 below. Hopf algebras of dimension 8 were classified by Williams [36]; Masouka [15] and Stefan [33] gave later a different proof of this fact. In general, the classification

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problem of Hopf algebras over k of dimension p^3 is still open. However, the classification is known for semisimple or pointed Hopf algebras. Semisimple Hopf algebras of dimension p^3 were classified by Masuoka [14]; there are $p+8$ isomorphism types, namely

(a) Three group algebras of abelian groups.

(b) Two group algebras of non-abelian groups, and their duals.

(c) $p+1$ self-dual Hopf algebras which are neither commutative nor cocommutative. They are extensions of $k[\mathbf{Z}/(p) \times \mathbf{Z}/(p)]$ by $k[\mathbf{Z}/(p)]$.

Pointed non-semisimple Hopf algebras of dimension p^3 were classified in [3], [5] and [34], by different methods. The explicit list is the following, where $q \in \mathbf{G}_p \backslash \{1\}$:

(d) The tensor-product Hopf algebra $T(q) \otimes k[\mathbf{Z}/(p)]$.

(e) $\widetilde{T(q)} := k\langle g, x \mid gxg^{-1} = q^{1/p}x, g^{p^2} = 1, x^p = 0\rangle$ ($q^{1/p}$ a p-th root of q), with comultiplication $\Delta(x) = x \otimes g^p + 1 \otimes x$, $\Delta(g) = g \otimes g$.

(f) $\widehat{T(q)} := k\langle g, x \mid gxg^{-1} = qx, g^{p^2} = 1, x^p = 0\rangle$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(g) = g \otimes g$.

(g) $\mathbf{r}(q) := k\langle g, x \mid gxg^{-1} = qx, g^{p^2} = 1, x^p = 1 - g^p\rangle$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(g) = g \otimes g$.

(h) The Frobenius-Lusztig kernel $\mathbf{u}_q(\mathfrak{sl}_2) := k\langle g, x, y \mid gxg^{-1} = q^2x, gyg^{-1} = q^{-2}y, g^p = 1, x^p = 0, y^p = 0, xy - yx = g - g^{-1}\rangle$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(y) = y \otimes 1 + g^{-1} \otimes y$, $\Delta(g) = g \otimes g$.

(i) The book Hopf algebra $\mathbf{h}(q,m) := k\langle g, x, y \mid gxg^{-1} = qx, gyg^{-1} = q^m y, g^p = 1, x^p = 0, y^p = 0, xy - yx = 0\rangle$, $m \in \mathbf{Z}/(p)\backslash\{0\}$, with comultiplication $\Delta(x) = x \otimes g + 1 \otimes x$, $\Delta(y) = y \otimes 1 + g^m \otimes y$, $\Delta(g) = g \otimes g$.

Furthermore, there are two examples of non-semisimple but also non-pointed Hopf algebras of dimension p^3, namely

(j) The dual of the Frobenius-Lusztig kernel, $\mathbf{u}_q(\mathfrak{sl}_2)^*$.

(k) The dual of the case (g), $\mathbf{r}(q)^*$.

There are no isomorphisms between different Hopf algebras in the list. Moreover, the Hopf algebras in cases (d), ..., (k) are not isomorphic for different values of $q \in \mathbf{G}_p \backslash \{1\}$, except for the book algebras, where $\mathbf{h}(q,m)$ is isomorphic to $\mathbf{h}(q^{-m^2}, m^{-1})$. In particular, the Hopf algebra $\widetilde{T(q)}$ does not depend, modulo isomorphisms, upon the choice of the p-th root of q. The Hopf algebra $\mathbf{r}(q)$ was first considered by Radford (see [3], [28]).

We conjecture that any Hopf algebra H of dimension p^3 is semisimple or pointed or its dual is pointed, that is, H is one of the Hopf algebras of the list (a), ..., (k).

In this paper we prove this conjecture under additional assumptions. In

Theorem 2.1 and Corollary 2.2 we show the simple modules of a crossed product of a Taft algebra of dimension p^2 and a group algebra of order p are one-dimensional, that is the dual of the crossed product is pointed.

In Section 3 we discuss some general results on finite-dimensional Hopf algebras and then apply them to Hopf algebras of dimension p^3. There are 10 cases according to the group-like elements of H and H^*. We show that in 8 of the 10 cases, it is possible to determine the structure of the Hopf algebra.

Let us say that a Hopf algebra H of dimension p^3 is *strange* if H is simple as a Hopf algebra, not semisimple and if H and H^* are not pointed. It turns out that a Hopf algebra H of dimension p^3 is isomorphic to a Hopf algebra of the list (a), ..., (k), or

(I) H is strange, $G(H) \simeq \mathbf{Z}/(p)$ and $G(H^*) = 1$, or

(II) H is strange, $G(H) \simeq \mathbf{Z}/(p)$, and $G(H^*) \simeq \mathbf{Z}/(p)$.

It is not known whether a strange Hopf algebra exists.

In the subsection 3.2, we study non-semisimple Hopf algebras of dimension p^3 with $G(H) \simeq G(H^*) \simeq \mathbf{Z}/(p)$. The order of the antipode of such a Hopf algebra is necessarily $2p$ or $4p$. If the order is $2p$, then H is a bosonization of the group algebra $k[G(H)]$. In this case we believe that H is isomorphic to a book Hopf algebra $\mathbf{h}(q,m)$, for some $q \in \mathbf{G}_p \backslash \{1\}$, and $m \in \mathbf{Z}/(p)\backslash\{0\}$. If the order is $4p$, then H satisfies (II), and all skew primitive elements of H are trivial, that is, contained in $k[G(H)]$.

Radford and Schneider [30] conjectured that the square of the antipode of any finite-dimensional Hopf algebra must satisfy a certain condition, which they called the strong vanishing trace condition. If H is a finite-dimensional Hopf algebra and B is the unique maximal semisimple Hopf subalgebra of H, then it follows from the conjecture that the order of the square of the antipode of H must divide $\dim H/\dim B$, see [30, Thm. 6]. In particular, if the dimension of H is p^3 and $|G(H)| = |G(H^*)| = p$ or $|G(H)| = p$ and $|G(H^*)| = 1$, then the order of the antipode should be $2p$.

It is well-known that the Frobenius-Lusztig kernels $\mathbf{u}_q(\mathfrak{sl}_2)$ and the group algebras admit a quasitriangular structure (see e.g. [9, IX. 7]). We prove in Section 4 that these two are the only quasitriangular Hopf algebras from the list above. We also prove in Theorem 4.9 that there is no quasitriangular Hopf algebra of dimension p^3 which satisfies condition (I). Namely, if H is a quasitriangular Hopf algebra of dimension p^3, then

(i) H is isomorphic to a group algebra or to $\mathbf{u}_q(\mathfrak{sl}_2)$, or

(ii) H satisfies (II) and the map $f_R : H^{*cop} \to H$ is an isomorphism.

Moreover, H and H^* are minimal quasitriangular, $1 = \langle \beta, x \rangle$, for all $\beta \in G(H^*)$, $x \in G(H)$ and $\operatorname{ord} \mathcal{S} = 4p$.

As a consequence, we show in Corollary 4.10 that every ribbon Hopf algebra of dimension p^3 is either a group algebra or a Frobenius-Lusztig kernel.

Finally, using some results from [1] and [4] on bounds for the dimension of the first term H_1 in the coradical filtration of H, we classify quasitriangular Hopf algebras of dimension 27.

1.1. Conventions and Preliminaries. Our references for the theory of Hopf algebras are [18], [9], [31] and [35]. The antipode of a Hopf algebra H is denoted by $\mathcal{S}$, or $\mathcal{S}_H$ if special emphasis is needed. The Sweedler notation is used for the comultiplication of H but dropping the summation symbol. The group of group-like elements of a coalgebra C is denoted by $G(C)$. The modular group-like elements $g \in H$ and $\alpha \in H^*$ are defined by

$$\Lambda x = \langle \alpha, x \rangle \Lambda, \quad \text{for all } x \in H, \qquad \text{and} \qquad \beta\lambda = \langle \beta, g \rangle \lambda, \quad \text{for all } \beta \in H^*,$$

where $\Lambda \in H$ is a non-zero left integral and $\lambda \in H^*$ is a non-zero right integral. There is a formula for $\mathcal{S}^4$ in terms of α and g:

$$\mathcal{S}^4(h) = g(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1})g^{-1}, \quad \text{for all } h \in H,$$

where H^* acts on H on the left by $\beta \rightharpoonup h = h_{(1)}\beta(h_{(2)})$ and on the right by $h \leftharpoonup \beta = \beta(h_{(1)})h_{(2)}$, for all $\beta \in H^*$ and $h \in H$. Moreover, if λ and Λ are such that $\langle \lambda, \Lambda \rangle = 1$, then there are formulas for the trace of any linear endomorphism f on H:

$$\operatorname{Tr} f = \langle \lambda, \mathcal{S}(\Lambda_{(1)}) f(\Lambda_{(2)}) \rangle = \langle \lambda, (\mathcal{S} \circ f)(\Lambda_{(1)}) \Lambda_{(2)} \rangle.$$

The formulas above are due to many authors, including Radford (see [18], [23], [31]). Let C be a coalgebra and $a, b \in G(C)$. The set of (a, b)-skew primitive elements of C is defined by

$$P_{a,b} = \{c \in C \mid \Delta(c) = a \otimes c + c \otimes b\};$$

in particular, $k(a - b) \subseteq P_{a,b}$. We say that a skew primitive element $c \in C$ is trivial if $c \in k[G(C)]$.

A Hopf algebra H is called *simple* if it does not contain any proper normal Hopf subalgebra in the sense of [18, 3.4.1]; H is called *semisimple*, respectively *cosemisimple*, if it is semisimple as an algebra, respectively if it is cosemisimple as a coalgebra. The sum of all simple subcoalgebras is called the *coradical* of H and it is denoted by H_0. If all simple subcoalgebras of H are one-dimensional, then

H is called *pointed* and $H_0 = k[G(H)]$. A finite-dimensional Hopf algebra H is pointed if and only if all the simple H^*-modules are one-dimensional. We also consider the coradical filtration $H_0 \subset H_1 \subset \cdots$, of H (see [18, Chapter 5], [35, Chapter IX]).

Let H be a finite-dimensional Hopf algebra over k, then the following are equivalent (see [11], [12], [25, Prop. 2] and [31, Cor. 3.5]):

(a) H is semisimple,
(b) H is cosemisimple,
(c) $\mathscr{S}^2 = \mathrm{id}$,
(d) $\mathrm{Tr}\, \mathscr{S}^2 \neq 0$,

where Tr denote the trace map.

Remark 1.1 [37]. If H is an odd-dimensional Hopf algebra and $\mathscr{S}^4 = \mathrm{id}$, then H is semisimple. Therefore, if H is a non-semisimple Hopf algebra of odd dimension, either $G(H)$ or $G(H^*)$ is non-trivial.

Remark 1.2. Let $N \geq 2$ be an integer and let $q \in k$ be a primitive N-th root of unity. Recall that the *Taft algebra* $T(q)$ is the k-algebra presented by generators g and x with relations $g^N = 1$, $x^N = 0$ and $gx = qxg$; $T(q)$ carries a Hopf algebra structure, determined by

$$\Delta g = g \otimes g, \quad \Delta x = x \otimes 1 + g \otimes x.$$

Then $\varepsilon(g) = 1$, $\varepsilon(x) = 0$, $\mathscr{S}(g) = g^{-1}$, and $\mathscr{S}(x) = -g^{-1}x$. It is known that $T(q)$ is a pointed Hopf algebra, with $G(T(q)) = \langle g \rangle \simeq \mathbf{Z}/(N)$. The proper Hopf subalgebras of $T(q)$ are contained in $k\langle g \rangle$, whence they are semisimple. We also have:

(i) $T(q) \simeq T(q)^*$,
(ii) $T(q) \simeq T(q')$ if and only if $q = q'$.

It is not difficult to see that $T(q)^{*\mathrm{cop}} \simeq T(q)^{\mathrm{op}} \simeq T(q^{-1})$.

The square of the antipode of $T(q)$ coincides with the inner automorphism induced by g. Therefore, $\mathscr{S}^4 \neq \mathrm{id}$ if $N > 2$.

2. Extensions of a Taft Algebra by the Group Algebra of Order p

In the list of Hopf algebras of dimension p^3 given above, the cases (d), (f) and (g) are extensions of Taft algebras by the group algebra of order p; that is, they fit into an exact sequence of finite-dimensional Hopf algebras

$$1 \to k[\mathbf{Z}/(p)] \xrightarrow{\iota} H \xrightarrow{\pi} T(q) \to 1.$$

We show that in some sense the converse is also true, that is, if H is an extension of a Taft algebra by the group algebra of order p, then H is necessarily pointed.

We recall some definitions and formulas for cleft extensions. For examples and a characterization of these extensions see [18, Chapter 7].

By [32, Thm. 2.4], every extension $1 \to A \xrightarrow{\iota} H \xrightarrow{\pi} B \to 1$, of finite-dimensional Hopf algebras is cleft; *i.e.* there exists a right B-comodule map $\gamma : B \to H$, which is convolution invertible and preserve the unit and the counit. Using this map, one can construct a weak action of B on A and a convolution invertible 2-cocycle $\sigma \in \mathrm{Hom}(B \otimes B, A)$, that give H the structure of a crossed product $A \#_\sigma B$ of A with B. As vector spaces $A \#_\sigma B = A \otimes B$. The weak action of B on A is defined by

$$b \cdot a = \gamma(b_{(1)}) a \gamma^{-1}(b_{(2)}),$$

for all $a \in A$, $b \in B$, and satisfies

$$b \cdot (aa') = (b_{(1)} \cdot a)(b_{(2)} \cdot a') \quad \text{and} \quad b \cdot 1 = \varepsilon(b) 1_A,$$

for all $b \in B$, $a, a' \in A$. The convolution invertible 2-cocycle σ is given by

$$\sigma(b, c) = \gamma(b_{(1)}) \gamma(c_{(1)}) \gamma^{-1}(b_{(2)} c_{(2)}),$$

for all $b, c \in B$, and satisfies

$$[b_{(1)} \cdot \sigma(b'_{(1)}, b''_{(1)})] \sigma(b_{(2)}, b'_{(2)} b''_{(2)}) = \sigma(b_{(1)}, b'_{(1)}) \sigma(b_{(2)} b'_{(2)}, b'')$$
$$\text{and} \quad \sigma(b, 1) = \sigma(1, b) = \varepsilon(b) 1,$$

for all $b, b', b'' \in B$. The multiplication on the crossed product $A \#_\sigma B$ is given by

$$(a \# b)(c \# d) = a(b_{(1)} \cdot c) \sigma(b_{(2)}, d_{(1)}) \# b_{(3)} d_{(2)},$$

for all $a, c \in A$, $b, d \in B$. The unit in $A \#_\sigma H$ is $1 \# 1$. Here we write $a \# h$ for $a \otimes h$ as an element in $A \#_\sigma B$.

If $B = k[\Gamma]$ is a group algebra of a group Γ, then $A *_\sigma \Gamma := A \#_\sigma k[\Gamma]$ is called the Γ-crossed product of Γ over A. Observe that $A *_\sigma \Gamma$ is a Γ-graded algebra. Moreover, it is easy to see that $A *_\sigma \Gamma$ is characterized as the Γ-graded algebra which contains an invertible element in each component and whose 1-component is A. In this case, we say that A is the neutral component of $A *_\sigma \Gamma$.

Theorem 2.1. *Let H be a finite-dimensional Hopf algebra which fits into an extension of the form*

$$1 \to A \xrightarrow{\iota} H \xrightarrow{\pi} k[\Gamma] \to 1, \tag{1}$$

where $\Gamma = \mathbf{Z}/(p)$ is the group algebra of order p, A^* is pointed and the group of group-likes of A^* has order $|G(A^*)| \leq p$. Then H^* is pointed.

Proof. To prove that H^* is pointed, we show that every simple H-module is one-dimensional. To see this it is enough to show that the algebra $H/\mathrm{Rad}\, H$ is commutative. Indeed, we have that $H/\mathrm{Rad}\, H$ is a semisimple algebra. If it is commutative, then by the Artin-Wedderburn theorem it follows that every simple $H/\mathrm{Rad}\, H$-module, and therefore every simple H-module, is one-dimensional.

Hence, we prove actually the following. Let $H = A *_\sigma \Gamma$ be a finite-dimensional Γ-crossed product with neutral component A and suppose that

(i) $A/\mathrm{Rad}\, A \simeq \mathrm{Map}(X, k)$, and

(ii) there exists an epimorphism $\pi : H \to k[\Gamma]$ of Γ-graded algebras,

where $\mathrm{Map}(X, k)$ is the set of functions on the finite set X and $|X| \leq p$. Then $H/\mathrm{Rad}\, H$ is commutative.

Let $\bar{A} = A/\mathrm{Rad}\, A$. X can be naturally identified with the set $\{\delta_0, \dots, \delta_m\}$ of primitive idempotents of $\bar{A}$.

Since for every $g \in \Gamma$, the map $r(g) : A \to A$ given by $r(g)(a) = g \cdot a$ for all $a \in A$ is an algebra map, the radical $\mathrm{Rad}\, A$ of A is stable under the weak action of Γ on A and whence $(\mathrm{Rad}\, A) * \Gamma$ is a Γ-graded ideal in H, which is nilpotent. Therefore we have a quotient algebra $\bar{H} = \bar{A} * \Gamma$, which is also a Γ-crossed product. Since $\mathrm{char}\, k = 0$, $\bar{H}$ is semisimple, whence $\bar{H} = H/\mathrm{Rad}\, H$. Moreover, since $k[\Gamma]$ is a semisimple algebra, it follows that $\pi(\mathrm{Rad}\, H) = 0$ and π factorizes through $\bar{H}$. Denote by $\bar{\pi} : \bar{H} \to k[\Gamma]$ the map induced by this factorization.

Let δ be a primitive idempotent of $\bar{A}$. Then for all $g \in \Gamma$, $g \cdot \delta$ is also a primitive idempotent of $\bar{A}$. Hence, the weak action of Γ associated to $\bar{H}$ arises from a group homomorphism, say $\alpha : \Gamma \to \mathrm{Perm}(X)$. Since for all $g \in \Gamma$ and $\delta_i \in X$,

$$\bar{\pi}(\alpha(g)(\delta_i) * 1) = \bar{\pi}(g \cdot \delta_i * 1) = \bar{\pi}((1 * g)(\delta_i * 1)(1 * g^{-1})) = \varepsilon(\delta_i)1 = \delta_{i,0},$$

it follows that Γ must fix the unique element δ_0 in X that does not vanish under $\bar{\pi}$. Since $\Gamma = \mathbf{Z}/(p)$ and p does not divide $(|X| - 1)!$, the homomorphism α must be trivial. This implies that the weak action of Γ on $\bar{A}$ is trivial and therefore that the Γ-crossed product $\bar{H}$ is commutative. □

Corollary 2.2. *Let H be a Hopf algebra over k of dimension p^3 which is an extension of a Taft algebra $T(q)$ by a group algebra $k[\mathbf{Z}/(p)]$, that is, H fits into the following exact sequence of finite-dimensional Hopf algebras:*

$$(2) \qquad 1 \to k[\mathbf{Z}/(p)] \xrightarrow{\iota} H \xrightarrow{\pi} T(q) \to 1.$$

Then H is pointed.

PROOF. If we dualize the sequence (2), we see that H^* is an extension of a group algebra of order p by a Taft algebra of order p^2,

$$1 \to T(q) \xrightarrow{\iota^*} H^* \xrightarrow{\pi^*} k[\mathbf{Z}/(p)] \to 1.$$

Since the Taft algebra satisfies the hypothesis on A in Theorem 2.1, by Theorem 2.1 the simple H^*-modules are one-dimensional, which implies that H is a pointed Hopf algebra. □

The following corollary states that if there exist a Hopf algebra H of dimension p^3 which is not isomorphic to a Hopf algebra of the list (a), ..., (k), then H is necessarily strange.

COROLLARY 2.3. *Let H be a non-semisimple Hopf algebra of dimension p^3 such that H and H^* are both non-pointed. Then H is simple as a Hopf algebra.*

PROOF. Suppose that H is not simple, then it contains a proper Hopf subalgebra K which is normal and non-trivial. Hence, we have an extension of Hopf algebras

$$(3) \qquad 1 \to K \xrightarrow{\iota} H \xrightarrow{\pi} D \to 1,$$

where $D = H/K^+H$. Then it follows that $p^3 = \dim K \dim D$, by Nichols-Zoeller, and the dimension of K is p or p^2.

If $\dim K = p^2$, then $\dim D = p$. Moreover, by [37, Thm. 2], $D \simeq k[\mathbf{Z}/(p)]$ and by [21, Thm. 5.5], $K \simeq T(q)$, a Taft algebra, since H is non-semisimple by assumption. Hence H is an extension of a group algebra by a Taft algebra and by the Corollary 2.2, H^* must be pointed, which is a contradiction.

If $\dim K = p$, applying the same argument to the dual extension of (3) we get that H is pointed, which is also a contradiction. □

3. On Hopf Algebras of Dimension p^3

In this section we first discuss some general results on Hopf algebras of finite dimension and then we apply them to Hopf algebras of dimension p^3.

3.1. General results. The order of the antipode plays a very important rôle in the theory of finite-dimensional Hopf algebras. The following linear algebra lemma will help us to determine this order in some particular cases. Note that part (c) generalizes (b).

LEMMA 3.1. (a) [2, Lemma 2.6, (i)] *Let T be a linear transformation of a finite-dimensional vector space $V \neq 0$ such that* $\mathrm{Tr}\, T = 0$ *and* $T^p = \mathrm{id}$. *Let $q \in \mathbf{G}_p \backslash \{1\}$ and let $V(i)$ be the eigenspace of T of eigenvalue q^i, then* $\dim V(i)$ *is constant; in particular p divides* $\dim V$.

(b) [2, Lemma 2.6, (ii)] *If V is a vector space of dimension p and $T \in$* End V *satisfies* $T^{2p} = \mathrm{id}$, *and* $\mathrm{Tr}\, T = 0$, *then* $T^p = \pm\mathrm{id}$.

(c) *Let $n \in \mathbf{N}$ and let ω be a p^n-th primitive root of unity in k. If V is a vector space of dimension p and $T \in$* End V *satisfies* $T^{2p^n} = \mathrm{id}$, *and* $\mathrm{Tr}\, T = 0$, *then there exists m, $0 \leq m \leq p^{n-1} - 1$ such that* $T^p = \pm\omega^{mp}\, \mathrm{id}$.

PROOF. To prove (c) we follow the approach of [2]. The crucial point here is that the minimal polynomial over $\mathbf{Q}$ of a p^n-th root of unity in k is known and that V is p-dimensional.

Since $T^{2p^n} = \mathrm{id}$, the eigenvalues of T are of the form $(-1)^a\omega^i$, where $a \in \{0, 1\}$ and $0 \leq i \leq p^n - 1$. Let $V_{a,i} := \{v \in V : T(v) = (-1)^a\omega^i v\}$ be the eigenspace of T of eigenvalue $(-1)^a\omega^i$. Since $\mathrm{Tr}\, T = 0$, we have

$$0 = \mathrm{Tr}\, T = \sum_{i=0}^{p^n-1} (\dim V_{0,i} - \dim V_{1,i})\omega^i. \tag{1}$$

Define now $P \in \mathbf{Z}[X]$ by $P(X) = \sum_{i=0}^{p^n-1} (\dim V_{0,i} - \dim V_{1,i})X^i$. Then $\deg P \leq p^n - 1$, $P(\omega) = 0$ and the number of coefficients of P different from zero is less or equal than p, since $V = \bigoplus_{a,i} V_{a,i}$ and $\dim V = p$. Moreover, the minimal polynomial Φ_ω of ω over $\mathbf{Q}$ must divide P, since ω is a root of P. Hence there exists $Q \in \mathbf{Z}[X]$ such that $P = Q\Phi_\omega$, where Q is the zero polynomial or $\deg Q \leq p^{n-1} - 1$, since $\deg \Phi_\omega = \varphi(p^n) = p^{n-1}(p - 1)$ (where φ is the Euler's function).

Define $V_+ = \bigoplus_{i=0}^{p^n-1} V_{0,i}$ and $V_- = \bigoplus_{i=0}^{p^n-1} V_{1,i}$, then it is clear that $V = V_+ \oplus V_-$. If Q is the zero polynomial, then P is also the zero polynomial and it follows that $\dim V_{0,i} = \dim V_{1,i}$, for all $0 \leq i \leq p^n - 1$. But this implies that $\dim V_+ = \dim V_-$, from which follows that the dimension of V is even, a contradiction.

Therefore we can assume that Q is not the zero polynomial. Suppose that $Q(X) = \sum_{j=0}^{p^{n-1}-1} d_j X^j$, where $d_m \neq 0$ for some $0 \leq m \leq p^{n-1} - 1$, and recall that $\Phi_\omega(X) = X^{p^{n-1}(p-1)} + X^{p^{n-1}(p-2)} + \cdots + X^{p^{n-1}} + 1$. Then we have that

$$P(X) = \left(\sum_{i=0}^{p^{n-1}-1} d_i X^i\right)\left(\sum_{j=0}^{p-1} (X^{p^{n-1}})^j\right) = \sum_{i=0}^{p^{n-1}-1} d_i \left(\sum_{j=0}^{p-1} X^{i+jp^{n-1}}\right).$$

Since the number of non-zero coefficients of P is not zero and less or equal than p, there exists a unique l, $0 \leq l \leq p^{n-1} - 1$ such that $d_l \neq 0$ and $d_i = 0$ for all $i \neq l$, $0 \leq i \leq p^{n-1} - 1$. Hence, $l = m$ and then $P(X) = d_m X^m \Phi_\omega$, implying for all $0 \leq j \leq p-1$, $0 \leq i \leq p^n - 1$ such that $i \neq m + jp^{n-1}$ that

$$\dim V_{0,m+jp^{n-1}} - \dim V_{1,m+jp^{n-1}} = d_m, \quad \text{and} \quad \dim V_{0,i} - \dim V_{1,i} = 0.$$

From these equalities it follows that $\dim V_+ - \dim V_- = d_m p$; as $\dim V_+ + \dim V_- = p$, one concludes that $V_+ = \bigoplus_{j=0}^{p-1} V_{0,m+jp^{n-1}} = V$ or $V_- = \bigoplus_{j=0}^{p-1} V_{1,m+jp^{n-1}} = V$, and this implies that $T^p(v) = \pm(\omega^m)^p v = \pm \omega^{mp} v$, for all $v \in V$. □

PROPOSITION 3.2 [2, Prop. 5.1]. *Let H be a finite-dimensional Hopf algebra whose antipode $\mathcal{S}$ has order $2p$. Assume also that H contains a cosemisimple Hopf subalgebra B such that* $\dim H = p \dim B$. *Then B is the coradical of H.* □

We use Lemma 3.1 (c) to improve a result due to Radford and Schneider [29]. As shown in the remark below, this result gives an alternative proof of the classification of Hopf algebras of dimension p^2.

PROPOSITION 3.3. *Let H be a finite-dimensional non-semisimple Hopf algebra which contains a commutative Hopf subalgebra B such that* $\dim H = p \dim B$. *If* $\mathcal{S}^{4p^n} = \mathrm{id}$, *for some $n \in \mathbf{N}$, then* $\mathcal{S}^{2p} = \mathrm{id}$, *and consequently B is the coradical of H.*

PROOF. We follow the proof of [29]. By assumption on k, we know that B is semisimple, since it is commutative. Let $\{e_j\}_{1 \leq j \leq s}$, be the central primitive idempotents of B such that $1_B = e_1 + \cdots + e_s$. Let $I_j = He_j$; then we can write $H = \bigoplus_{j=1}^s I_j$.

CLAIM 1. $\mathrm{Tr}(\mathcal{S}^2|_{I_j}) = 0$, $\forall 1 \leq j \leq s$.
It is clear that $\mathcal{S}^2(I_j) \subseteq I_j$, since

$$\mathcal{S}^2(I_j) = \mathcal{S}^2(He_j) = \mathcal{S}^2(H)\mathcal{S}^2(e_j) = \mathcal{S}^2(H)(e_j) \subseteq He_j = I_j.$$

Let $P_j : H \to I_j$ be the linear projection given by $P_j(a) = ae_j$, for all $a \in H$, and $r(h)$ the linear map given by $r(h)(x) = xh$, $\forall x, h \in H$. Then $\mathcal{S}^2|_{I_j} = \mathcal{S}^2 \circ P_j = \mathcal{S}^2 \circ r(e_j)$, since these maps coincide on I_j. Let $\lambda \in H^*$ be a right integral and $\Lambda \in H$ a left integral such that $\langle \lambda, \Lambda \rangle = 1$; then $(\Lambda_{(1)}, \mathcal{S}(\Lambda_{(2)}))$ are dual bases for the Frobenius homomorphism λ, (see [31]). By Radford's trace formula, we get that

$$\begin{aligned}\operatorname{Tr}(\mathscr{S}^2|_{I_j}) &= \operatorname{Tr}((\mathscr{S}^2)\circ r(e_j)) = \langle \lambda, \mathscr{S}(\Lambda_{(2)})[\mathscr{S}^2\circ r(e_j)](\Lambda_{(1)})\rangle \\ &= \langle \lambda, \mathscr{S}(\Lambda_{(2)})\mathscr{S}^2(\Lambda_{(1)}e_j)\rangle \\ &= \langle \lambda, \mathscr{S}(\Lambda_{(2)})\mathscr{S}^2(\Lambda_{(1)})\mathscr{S}_B^2(e_j)\rangle \\ &= \langle \lambda, \mathscr{S}(\mathscr{S}(\Lambda_{(1)})\Lambda_{(2)})e_j\rangle = \langle \lambda, e_j\rangle\langle \varepsilon, \Lambda\rangle = 0,\end{aligned}$$

where the last equality follows from the fact that H is non-semisimple.

CLAIM 2. $\dim I_j = p$, $1 \le j \le s$.

Since for all idempotents $e \in B$, $e \neq 0$, $He \cong H \otimes_B Be$, as vector spaces over k, we have that $\dim He = \dim_B H \dim Be$. But this dimension is p, since $\dim Be = 1$, because B is commutative.

Let $T_j = \mathscr{S}^2|_{I_j}$, $1 \le j \le s$. By the claims above, it follows that $\operatorname{Tr}(T_j) = 0$, $T_j^{2p^n} = \mathrm{id}_{I_j}$ and $\dim I_j = p$, which implies by Lemma 3.1 (c) that $T_j^p = \pm\omega_j^{m_jp}\,\mathrm{id}_{I_j}$, where ω_j a p^n-th root of unity and $0 \le m_j \le p^{n-1}-1$. Since $T_j(e_j) = \mathscr{S}^2|_{I_j}(e_j) = e_j$, it follows that $m_j = 0$ and $T_j^p = \mathrm{id}_{I_j}$, $\forall 1 \le j \le s$, that is, $\mathscr{S}^{2p}|_{I_j} = \mathrm{id}_{I_j}$, for all $1 \le j \le s$. Finally, the proposition follows in view of Proposition 3.2. □

REMARK 3.4 [29]. Radford and Schneider proved Proposition 3.3 in the case where $n = 1$ using Lemma 3.1 (b). This result provides an alternative proof of the classification of Hopf algebras of dimension p^2, which was recently finished by Ng [21]. Indeed, if H is semisimple, then by a result of Masuoka [16], it is isomorphic to a group algebra of order p^2. Now suppose that H is non-semisimple. By Remark 1.1, $G(H)$ or $G(H)^*$ is not trivial and the order of each group is less than p^2. Assume that $G(H)$ is not trivial. Then $\dim H = p|G(H)|$ and it follows from Radford's formula for $\mathscr{S}^4$ that $\mathscr{S}^{4p} = \mathrm{id}$. Hence, by the proposition above, H must be pointed and by [2, Thm. A (ii)] H is isomorphic to a Taft algebra.

Let H be a finite-dimensional Hopf algebra and suppose that $G(H)$ is abelian and its order is p^n, $n \ge 0$. Following the classification of finite abelian groups, we say that $G(H)$ is of type $(p^{i_1}, p^{i_2}, \dots, p^{i_s})$ if $G(H) \simeq \mathbf{Z}/(p^{i_1}) \times \cdots \times \mathbf{Z}/(p^{i_s})$. Suppose in addition that $G(H^*)$ is also abelian and its order is a power of p. We say that the Hopf algebra H is of *type* $(p^{i_1}, \dots, p^{i_s}; p^{j_1}, \dots, p^{j_t})$ if $G(H)$ and $G(H^*)$ are of type $(p^{i_1}, \dots, p^{i_s})$ and $(p^{j_1}, \dots, p^{j_t})$, respectively.

If H is a non-semisimple Hopf algebra of dimension p^3, then $G(H)$ and $G(H^*)$ are abelian and their orders are powers of p, by Nichols-Zoeller. Up to duality, we have only 10 possible types. Next we prove that it is possible to classify all Hopf algebras of 8 of the 10 possible types.

Theorem 3.5. (a) *There is no Hopf algebra H of dimension p^3 such that H or H^* is of one of the following types*:

$$(1;1),\ (p,p;1),\ (p,p;p),\ (p,p;p^2),\ (p^2;1).$$

(b) *Let H be a non-semisimple Hopf algebra of dimension p^3.*

(1) *If H is of type $(p,p;p,p)$, then $H \simeq T(q) \otimes k[\mathbf{Z}/(p)]$.*

(2) *If H is of type $(p^2;p)$, then $H \simeq \mathbf{r}(q)$.*

(3) *If H is of type $(p^2;p^2)$, then either $H \simeq \widehat{T(q)}$ or $H \simeq \widetilde{T(q)}$.*

Proof. It follows from Masuoka's classification that no semisimple Hopf algebra satisfies any of the conditions in (a). Then, if there exists a Hopf algebra H which satisfies one of the conditions above, H should be non-semisimple. Type $(1;1)$ is not possible by Remark 1.1. For the other types, we have that $|G(H)| = p^2$ and the order of the antipode divides $4p^2$, by Radford's formula. Hence, by Proposition 3.3, H should be pointed. By inspection, the cases in (a) are impossible and in case (1), H should be isomorphic to $T(q) \otimes k[\mathbf{Z}/(p)]$, in case (2), H should be isomorphic to $\mathbf{r}(q)$ and in case (3), H should be isomorphic either to $\widehat{T(q)}$ or $\widetilde{T(q)}$. □

Remark 3.6. It follows from Theorem 3.5 and Corollary 2.3 that a Hopf algebra H of dimension p^3 is isomorphic to a Hopf algebra of the list (a), …, (k) or H satisfies condition (I) or (II), *i.e.* H is strange and of type $(p;1)$ or H is strange and of type $(p;p)$.

From Masuoka's classification it follows that there is no semisimple Hopf algebra of dimension p^3 of type $(p;1)$ or $(p;p)$. The Frobenuis-Lusztig kernels $\mathbf{u}_q(\mathfrak{sl}_2)$, $q \in \mathbf{G}_p \backslash \{1\}$ are of type $(p;1)$ and the book algebras $\mathbf{h}(q,m)$, $q \in \mathbf{G}_p \backslash \{1\}$, $m \in \mathbf{Z}/(p) \backslash \{0\}$ are of type $(p;p)$. These are the only non-semisimple pointed Hopf algebras of type $(p;1)$ or $(p;p)$ (see [2, Section 6]).

3.2. Hopf algebras of type $(p;p)$. In order to complete the classification of Hopf algebras of dimension p^3, we give in this section some results on Hopf algebras of type $(p;p)$.

The following theorem will be also useful in the quasitriangular case.

Theorem 3.7. *Let H be a finite-dimensional non-semisimple Hopf algebra such that $G(H)$ is abelian, $\mathcal{S}^{4p} = \mathrm{id}$ and $\langle \alpha, x \rangle = 1$ for all $x \in G(H)$, where α is the modular element of H^*. Assume further that there exists a surjective Hopf* *algebra map* $\pi : H \to L$, *such that* $\pi(x) = 1$ *for all* $x \in G(H)$, *where* L *is a semisimple Hopf algebra such that* $\dim L = \dim H/p|G(H)|$. *Then* $\operatorname{ord} \mathcal{S} = 4p$.

PROOF. Since H is non-semisimple, the order of the antipode is bigger than 2 and divides $4p$, by Radford's formula. Clearly it cannot be p since the order is even. Assume that $\mathcal{S}^{2p} = \mathrm{id}$.

Let $e_0, \dots, e_s$ be the primitive central idempotents of $k[G(H)]$. As in the proof of Proposition 3.3, we can write $1 = e_0 + \dots + e_s$ and hence $H = \bigoplus_{j=0}^{s} I_j$, where $I_j = He_j$. Since $G(H)$ is abelian, it follows that $\dim k[G(H)]e_j = 1$ for all $0 \le j \le s$, and this implies that $\dim I_j = \dim H/|G(H)|$ for all $0 \le j \le s$, since

$$\dim I_j = \dim H \otimes_{k[G(H)]} k[G(H)]e_j = \dim_{k[G(H)]} H \dim k[G(H)]e_j.$$

It was also shown in the proof of Proposition 3.3 that these spaces are invariant under the action of $\mathcal{S}^2$ and $\mathrm{Tr}(S^2|_{I_j}) = 0$. Let $q \in \mathbf{G}_p \backslash \{1\}$. Since $\mathcal{S}^{2p} = \mathrm{id}$, by Lemma 3.1 (a) follows that $I_j = \bigoplus_{m=0}^{p-1} I_{j,m}$ for all $0 \le j \le s$, where $I_{j,m} = \{h \in I_j : \mathcal{S}^2(h) = q^m h\}$, and $p \dim I_{j,m} = \dim I_j$ for all $0 \le m \le p-1$. In particular,

$$\dim I_{0,0} = \dim I_0/p = \dim H/p|G(H)| = \dim L,$$

and we can decompose H as $H = \bigoplus_{j,m=0}^{p-1} I_{j,m}$.

Since $\pi(x) = 1$ for all $x \in G(H)$, it follows that $\pi(e_j) = 0$ for all $j \ne 0$, that is $I_j \subseteq \operatorname{Ker} \pi$ for all $j \ne 0$. Moreover, $I_{0,m} \subseteq \operatorname{Ker} \pi$ for all $m \ne 0$, since for all $h \in I_{0,m}$, with $m \ne 0$, we have that $q^m \pi(h) = \pi(\mathcal{S}^2(h)) = \mathcal{S}^2(\pi(h)) = \pi(h)$, because L is semisimple. Therefore $\bigoplus_{(j,m) \ne (0,0)} I_{j,m} \subseteq \operatorname{Ker} \pi$, which implies that

$$\operatorname{Ker} \pi = \bigoplus_{(j,m) \ne (0,0)} I_{j,m}, \tag{2}$$

since both have the same dimension.

Let $\Lambda \in H$ be a non-zero left integral, then $\Lambda \in I_{0,0}$. Indeed, since $H = \bigoplus_{j=0}^{s} I_j$, there exist $h_0, \dots, h_s$ in H such that $\Lambda = h_0 e_0 + \dots + h_s e_s$. Hence $\Lambda = \langle \alpha, e_0 \rangle \Lambda = \Lambda e_0 = h_0 e_0$, since $\langle \alpha, x \rangle = 1$ for all $x \in G(H)$ and $e_0 = \frac{1}{|G(H)|} \sum_{x \in G(H)} x$. Moreover, by [25, Prop. 3, (d)], we know that $\mathcal{S}^2(\Lambda) = \langle \alpha, g^{-1} \rangle \Lambda$, where $g \in G(H)$ is the modular element of H. This implies that $\Lambda \in I_{0,0}$, since by assumption $\langle \alpha, g^{-1} \rangle = 1$ and $\Lambda = h_0 e_0 \in I_0$.

On the other hand, since H is non-semisimple, we have that $\langle \varepsilon, \Lambda \rangle = \langle \varepsilon, \pi(\Lambda) \rangle = 0$ and this implies that $\pi(\Lambda) = 0$, since $\pi(\Lambda)$ is a left integral in L and L is semisimple. Therefore, $\Lambda \in \operatorname{Ker} \pi \cap I_{0,0}$, implying by (2) that $\Lambda = 0$, which is a contradiction to our choice of Λ. $\square$

Remark 3.8. Let H, I_0, Λ and $\alpha \in G(H^*)$ be as in Theorem 3.7 and let $e_0, \dots, e_s$, be the primitive idempotents of $k[G(H)]$. Then $\Lambda \in I_0$ if and only if $\langle \alpha, x \rangle = 1$, for all $x \in G(H)$.

Proof. Suppose that $\Lambda \in I_0$. Then there exists $h \in H$ such that $\Lambda = he_0$. In particular, for all $x \in G(H)$ we have that $\Lambda x = \langle \alpha, x \rangle \Lambda = he_0 x = he_0 = \Lambda$, and this implies that $\langle \alpha, x \rangle = 1$, for all $x \in G(H)$.

Conversely, suppose that $\langle \alpha, x \rangle = 1$, for all $x \in G(H)$. Since $H = He_0 + \cdots + He_s$, there exist $h_0, \dots, h_s \in H$ such that $\Lambda = h_0 e_0 + \cdots + h_s e_s$. Since $\langle \alpha, e_0 \rangle = 1$, it follows that $\Lambda = \langle \alpha, e_0 \rangle \Lambda = \Lambda e_0 = h_0 e_0$, which implies that $\Lambda \in I_0$. $\square$

Remark 3.9. The proof of the theorem above was inspired in some results of Ng; the spaces $I_{j,m}$, $0 \leq j, m \leq p-1$ are the spaces $H^w_{0,m,j}$, $w = q \in \mathbf{G}_p \backslash \{1\}$, defined in [21, Section 3] in the special case where $\mathscr{S}^{2p} = \mathrm{id}$.

Corollary 3.10. *Let H be a non-semisimple Hopf algebra of dimension p^3 and type $(p; p)$.*

(1) *Then the order of the antipode is $2p$ or $4p$.*

(2) *If $\langle \beta, x \rangle = 1$, for all $\beta \in G(H^*)$, $x \in G(H)$, then the order of the antipode is $4p$.*

Proof. (1) Since H is non-semisimple, the order of the antipode is bigger than 2 and divides $4p$, by Radford's formula. Since ord $\mathscr{S}$ is even and p is odd, it is necessarily $2p$ or $4p$.

(2) Consider the surjective Hopf algebra map $H \xrightarrow{\pi} k^{G(H^*)}$, defined by $\langle \pi(h), \beta \rangle = \langle \beta, h \rangle$, for all $h \in H$, $\beta \in G(H^*)$. Then $\pi(x) = 1$ for all $x \in G(H)$, since by assumption $\langle \beta, x \rangle = 1$, for all $\beta \in G(H^*)$, $x \in G(H)$. The claim follows directly from Theorem 3.7, since $k^{G(H^*)}$ is semisimple, $\dim k^{G(H^*)} = p = \dim H/p|G(H)|$ and $\mathscr{S}^{4p} = \mathrm{id}$. $\square$

Let H be a Hopf algebra provided with a projection $H \xrightarrow{\pi} B$, which admits a section of Hopf algebras $B \xrightarrow{\gamma} H$. Then $A = H^{co\,\pi}$ is a Hopf algebra in the category of Yetter-Drinfel'd modules over B and H is isomorphic to the smash product $A \# B$. In this case, following the terminology of Majid, we say that H is a *bosonization* of B. For references on the correspondence between Hopf algebras with a projection and Hopf algebras in the category of Yetter-Drinfel'd modules see [13], [27].

PROPOSITION 3.11. *Let H be a finite-dimensional non-semisimple Hopf algebra. Assume that $G(H)$ is non-trivial and abelian and $G(H) \simeq G(H^*)$.*

(1) *If $|G(H)| = p$, then H is a bosonization of $k[G(H)]$ if and only if there exist $\beta \in G(H^*)$ and $x \in G(H)$ such that $\langle \beta, x\rangle \neq 1$.*
(2) *If $\dim H = p|G(H)|$, then H is a bosonization of $k[G(H)]$.*

PROOF. (1) Suppose that H is a bosonization of $k[G(H)]$. Then there exists a Hopf algebra projection $H \xrightarrow{\pi} k[G(H)]$ which admits a Hopf algebra section $k[G(H)] \xrightarrow{\gamma} H$, that is $\pi \circ \gamma = \mathrm{id}$. Let $G = G(H)$ and denote by $\hat{G}$ the group of characters of G. We identify $\hat{G}$ with $\mathrm{Alg}(k[G], k) = G(k[G]^*)$. Since $G \neq 1$, there exist $x \in G$, $\beta \in \hat{G}$ such that $\langle \beta, x\rangle \neq 1$. Consider the group homomorphism $\hat{G} \xrightarrow{\varphi} G(H^*)$ given by $\varphi(\chi) = \chi \circ \pi$, for all $\chi \in \hat{G}$. Then $\beta \circ \pi \in G(H^*)$, $\gamma(x) \in G$ and $\langle \beta \circ \pi, \gamma(x)\rangle = \langle \beta, \pi \circ \gamma(x)\rangle = \langle \beta, x\rangle \neq 1$.

Suppose now that H is not a bosonization of $k[G]$. We show that for all $\beta \in G(H^*)$, $x \in G$, $\langle \beta, x\rangle = 1$. Denote by $k[G] \xrightarrow{\imath} H$ the inclusion of the group algebra in H. Since $G(H^*)$ is also non-trivial, we have a surjection of Hopf algebras $H \xrightarrow{\pi} k^{G(H^*)}$, given by $\langle \pi(h), \beta\rangle = \langle \beta, h\rangle$, for all $h \in H$, $\beta \in G(H^*)$.

CLAIM 1. $\pi(x) = 1_{k^{G(H^*)}}$, for all $x \in G$. Since $G(H^*)$ is abelian, we have that $k^{G(H^*)} \simeq k[\widehat{G(H^*)}] \simeq k[G]$ as Hopf algebras. Moreover, the composition of these isomorphisms with π induce a Hopf algebra surjection $H \xrightarrow{\tau'} k[G]$. If there exists $x \in G$ such that $\pi(x) \neq 1_{k^{G(H^*)}}$, then $\tau'(x) \neq 1$ and the restriction $\tau' \circ \imath$ of τ' to $k[G]$ defines an automorphism of $k[G]$, since $|G| = p$, which is a prime number. Define now $\tau : H \to k[G]$ via $\tau = (\tau' \circ \imath)^{-1} \circ \tau'$. Then $\tau \circ \imath = \mathrm{id}_{k[G]}$ and H is a bosonization of $k[G]$, which is a contradiction to our assumption.

Therefore for all $\beta \in G(H^*)$, $x \in G$ we have $\langle \beta, x\rangle = \langle \pi(x), \beta\rangle = \langle 1_{k^{G(H^*)}}, \beta\rangle = 1$, which proves (1).

(2) Consider the Hopf algebra surjection $H \xrightarrow{\tau'} k[G]$ defined in the proof of Claim 1.

CLAIM 2. $\tau'|_G$ defines a group automorphism of G. Suppose on the contrary, that $\tau'|_G$ does not define an automorphism of G. Then there exists $h \in G$ such that $h \neq 1$ and $\tau'(h) = 1$; in particular, $h \in H^{co\,\tau'}$.

On the other hand, $\dim H^{co\,\tau'} = p$, since $\dim k^{G(H^*)} = |\widehat{G(H^*)}| = |G|$, and $\dim H = p|G| = \dim H^{co\,\tau'} \dim k^{G(H^*)}$, by Nichols-Zoeller. Since p is a prime number, it follows that $\mathrm{ord}\, h = p$ and $k\langle h\rangle = H^{co\,\tau'}$. In particular, one has the exact sequence of finite-dimensional Hopf algebras $1 \to k\langle h\rangle \xrightarrow{\imath} H \xrightarrow{\tau'} k[G] \to 1$, implying as it is well-known that H is semisimple, which is a contradiction to our assumption.

Therefore the automorphism $\tau'|_G$ defines a Hopf algebra automorphism in $k[G]$, and H is a bosonization of $k[G]$. $\square$

REMARK 3.12. If we examine the Hopf algebras in the list (a), ..., (k) in the introduction, we see that the cases (d), (e), (f) and (i) are also bosonizations. In the case (d) it is clear that the product Hopf algebra is the bosonization of $k[\mathbf{Z}/(p)]$ and by the proposition above it is also a bosonization of $k[\mathbf{Z}/(p) \times \mathbf{Z}/(p)]$. The cases (e) and (f) are bosonizations of $k[\mathbf{Z}/(p^2)]$ and the book algebras $\mathbf{h}(q, m)$ in case (i) are bosonizations of $k[\mathbf{Z}/(p)]$. In the case of the book algebras, Andruskiewitsch and Schneider proved in [2] that these algebras are also bosonizations of a Taft algebra, *i.e.* they are isomorphic to a smash product $R \# T(q)$. Moreover, they also proved that the algebras R exhaust the list of non-semisimple Hopf algebras of order p in the Yetter-Drinfel'd category over $T(q)$.

We now prove that a Hopf algebra H of dimension p^3 is a bosonization of $k[\mathbf{Z}/(p)]$ in the following two cases. Although some properties of H are known, we cannot determine its structure, since the classification of Hopf algebras of dimension p^3 which are bosonizations of $k[\mathbf{Z}/(p)]$, that is, of braided Hopf algebras of dimension p^2 in the category of Yetter-Drinfel'd modules over $k[\mathbf{Z}/(p)]$, is not known.

COROLLARY 3.13. *Let H be a non-semisimple Hopf algebra of dimension p^3 and type $(p; p)$. If $\mathscr{S}^{2p} = \mathrm{id}$ then H is a bosonization of $k[\mathbf{Z}/(p)]$.*

PROOF. Follows from Corollary 3.10 (2) and Proposition 3.11 (1). $\square$

COROLLARY 3.14. *Let H be a Hopf algebra of dimension p^3 and type $(p; p)$ such that H contains a non-trivial skew primitive element. Then H is a bosonization of $k[\mathbf{Z}/(p)]$.*

PROOF. Suppose on the contrary that H is not a bosonization of $k[\mathbf{Z}/(p)]$. By [1, Prop. 1.8], H contains a Hopf subalgebra B which is isomorphic to a Taft algebra of dimension p^2. Let $x, h \in H$, be the generators of B, such that $1 \neq h \in G(H)$ and $x \in P_{1,h}$.

Consider now the Hopf algebra surjection $\pi : H \to k^{G(H^*)}$ given by $\langle \pi(t), \beta \rangle = \langle \beta, t \rangle$, for all $t \in H$, $\beta \in G(H^*)$. If π is not trivial on $G(H)$, then H is a bosonization of $k[\mathbf{Z}/(p)]$, by Proposition 3.11 (1). If π is trivial on $G(H)$, then $t \in H^{co\,\pi}$, for all $t \in G(H)$. Since π is a Hopf algebra map and $\Delta(x) = x \otimes 1 + h \otimes x$, we see that $\pi(x)$ is a primitive element in $k^{G(H^*)}$. Since $G(H^*)$ has finite order, $\pi(x) = 0$ and therefore $(\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1$, that is $x \in H^{co\,\pi}$. Since B is generated by x and $h \in G(H)$, we have that $B \subseteq H^{co\,\pi}$ and hence $B = H^{co\,\pi}$, because both have dimension p^2. Therefore H fits into the extension of Hopf algebras

$$1 \to B \xrightarrow{\iota} H \xrightarrow{\pi} k^{G(H^*)} \to 1.$$

As $G(H^*) \simeq \mathbf{Z}/(p)$, H^* is pointed by Theorem 2.1. Since the only non-semisimple pointed Hopf algebras of dimension p^3 of type $(p;p)$ are the book algebras, it follows that $H^* \simeq \mathbf{h}(q,-m)$, where $q \in \mathbf{G}_p \backslash \{1\}$, and $m \in \mathbf{Z}/(p) \backslash \{0\}$. Hence $H \simeq \mathbf{h}(q,m)$ and therefore pointed, since for all $q \in \mathbf{G}_p \backslash \{1\}$, $m \in \mathbf{Z}/(p) \backslash \{0\}$ we have that $\mathbf{h}(q,-m)^* \simeq \mathbf{h}(q,m)$ (see [2, Section 6]). Hence H is a bosonization of $k[\mathbf{Z}/(p)]$, by Remark 3.12. □

Finally we note a very special result on Hopf algebras of dimension p^3 following [20, Prop. 1.3] and Theorem 2.1.

Proposition 3.15. *Let H be a non-semisimple Hopf algebra of dimension p^3 and assume that H contains a simple subcoalgebra C of dimension* 4 *such that $\mathcal{S}(C) = C$. Then H^* is pointed. In particular, H cannot be of type $(p;p)$.*

Proof. Let B be the algebra generated by C; clearly B is a Hopf subalgebra of H and it follows that $\dim B | p^3$, by Nichols-Zoeller. Since $C \subseteq B \subseteq H$, B is non-semisimple and non-pointed. Then necessarily $B = H$, since by [37, Thm. 2] and [21, Thm. 5.5], the only non-semisimple Hopf algebras whose dimension is a power of p with exponent less than 3 are the Taft algebras, which are pointed. Hence H is generated as an algebra by a simple coalgebra of dimension 4 which is stable by the antipode. By [20, Prop. 1.3], H fits into an extension $1 \to k^G \to H \to A \to 1$, where G is a finite group and A^* is a pointed non-semisimple Hopf algebra.

Since H is not semisimple, it follows that $|G(H)| = 1$ and H^* is pointed, or $|G(H)| = p$ and H is pointed by Theorem 2.1, which is impossible by assumption. Moreover, if H^* is pointed and of type $(p;p)$, then H^* is isomorphic to book algebra $\mathbf{h}(q,m)$, for some $q \in \mathbf{G}_p \backslash \{1\}$, $m \in \mathbf{Z}/(p) \backslash \{0\}$. Hence H is also pointed and it cannot contain a simple subcoalgebra of dimension 4. □

4. Quasitriangular Hopf Algebras of Dimension p^3

Let H be a finite-dimensional Hopf algebra and let $R \in H \otimes H$. As usual, we use for R the symbolic notation $R = R^{(1)} \otimes R^{(2)}$. Define a linear map $f_R : H^* \to H$ by $f_R(\beta) = \langle \beta, R^{(1)} \rangle R^{(2)}$, for $\beta \in H^*$. The pair (H, R) is said to be a quasitriangular Hopf algebra [6] if the following axioms hold:

(QT.1) $\Delta^{\mathrm{cop}}(h)R = R\Delta(h)$, $\forall h \in H$,

(QT.2) $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$,

(QT.3) $(\varepsilon \otimes \mathrm{id})(R) = 1$,

(QT.4) $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$,

(QT.5) $(\mathrm{id} \otimes \varepsilon)(R) = 1$;

or equivalently if $f_R : H^{*cop} \to H$ is a bialgebra map and (QT.1) is satisfied.

We have used the notation R_{12} to indicate the element $R \otimes 1 \in H^{\otimes 3}$, similarly for R_{13} and R_{23}. Note that $(H^{\mathrm{cop}}, R_{21})$ and $(H^{\mathrm{op}}, R_{21})$ are also quasitriangular, where $R_{21} := R^{(2)} \otimes R^{(1)}$.

We refer to a pair (H, R) which satisfies the five axioms above as a quasitriangular Hopf algebra or simply saying that H admits a quasitriangular structure. See for example [24] for a fuller discussion and references.

We define a morphism $f : (H, R) \to (H', R')$ of quasitriangular Hopf algebras over k to be a Hopf algebra map $f : H \to H'$ such that $R' = (f \otimes f)(R)$.

Let $\tilde{R} = R_{21}$. There is another Hopf algebra map $f_{\tilde{R}} : H^* \to H^{op}$, given by $f_{\tilde{R}}(\beta) = \langle \beta, R^{(2)} \rangle R^{(1)}$, for all $\beta \in H^*$. With the usual identification of vector spaces of H and H^{**}, the maps $f_{\tilde{R}}$ and f_R are related by the equation $f_{\tilde{R}} = f_R^*$.

Remark 4.1. Let L and K denote, respectively, the images of f_R and $f_{\tilde{R}}$. Then L and K are Hopf subalgebras of H of dimension $n > 1$, unless H is cocommutative and $R = 1 \otimes 1$; this dimension is called the rank of R. By [24, Prop. 2], we have $L \simeq K^{*\mathrm{cop}}$.

Let H_R be the Hopf subalgebra of H generated by L and K. If B is a Hopf subalgebra of H such that $R \in B \otimes B$, then $H_R \subseteq B$. Hence, we say that (H, R) is a *minimal* quasitriangular Hopf algebra if $H = H_R$. It is shown in [24, Thm. 1] that $H_R = LK = KL$. If L is semisimple, then K is semisimple and therefore H_R is semisimple. Minimal quasitriangular Hopf algebras were first introduced and studied in [24].

We recall some fundamental properties of finite-dimensional quasitriangular Hopf algebras (see for example [6], [18]). First R is invertible with inverse $R^{-1} = (\mathcal{S} \otimes I)(R) = (I \otimes \mathcal{S}^{-1})(R)$, and $R = (\mathcal{S} \otimes \mathcal{S})(R)$. Set $u = \mathcal{S}(R^{(2)})R^{(1)}$, then u is also invertible where

$$u^{-1} = R^{(2)}\mathcal{S}^2(R^{(1)}), \quad \Delta(u) = (u \otimes u)(\tilde{R}R)^{-1} = (\tilde{R}R)^{-1}(u \otimes u),$$

$$\varepsilon(u) = 1 \quad \text{and} \quad \mathcal{S}^2(h) = uhu^{-1} = (\mathcal{S}(u))^{-1}h\mathcal{S}(u), \quad \text{for all } h \in H.$$

Consequently, $u\mathcal{S}(u)$ is a central element of H. Since $\mathcal{S}^2(h) = h$, for all $h \in G(H)$, it follows that u commutes with the group-like elements of H. The element $u \in H$ is called the *Drinfel'd element* of H.

We say that $v \in H$ is a *ribbon element* of (H, R) if the following conditions are satisfied:

(R.1) $v^2 = u\mathcal{S}(u)$,
(R.2) $\mathcal{S}(v) = v$,
(R.3) $\varepsilon(v) = 1$,
(R.4) $\Delta(v) = (\tilde{R}R)^{-1}(v \otimes v)$ and
(R.5) $vh = hv$, for all $h \in H$.

If H contains a ribbon element, then the triple (H, R, v) or simply H is called a *ribbon Hopf algebra* (see [9, XIV.6], [10], [26, Section 2.2]).

The following theorem is due to Natale and it will be crucial to prove some results in the case where the dimension of the Hopf algebra is p^3.

THEOREM 4.2 [19, Thm. 2.3]. *Let H be a Hopf algebra of dimension pq over k, where p and q are odd primes which are not necessarily distinct. Assume that H admits a quasitriangular structure. Then H is semisimple and isomorphic to a group algebra $k[F]$, where F is a group of order pq.*

REMARK 4.3. The theorem above implies the known fact that the Taft algebra $T(q)$ does not admit any quasitriangular structure if $\dim T(q) = p^2$, with p odd prime.

The following result is due to Gelaki.

THEOREM 4.4. *Let (H, R) be a finite-dimensional quasitriangular Hopf algebra with antipode $\mathcal{S}$ over a field k of characteristic* 0.

(a) [7, Thm. 3.3] *If the Drinfel'd element u of H acts as a scalar in any irreducible representation of H (e.g. when H^* is pointed), then $u = \mathcal{S}(u)$ and in particular $\mathcal{S}^4 = \mathrm{id}$.*
(b) [8, Thm. 1.3.5] *If H_R is semisimple, then $u = \mathcal{S}(u)$ and $\mathcal{S}^4 = \mathrm{id}$.*
(c) *If H^* is pointed then either H is semisimple or* $\dim H$ *is even.*

PROOF. (c) follows from (a) and Remark 1.1. □

We can now prove our first assertion.

Proposition 4.5. *Among the Hopf algebras in the list* (a), ..., (k) *in the introduction, only the group algebras and the Frobenius-Lusztig kernels* $\mathbf{u}_q(\mathfrak{sl}_2)$, *where* $q \in \mathbf{G}_p \backslash \{1\}$, *admit a quasitriangular structure.*

Proof. It is well-known that the group algebras and the Frobenius-Lusztig kernels are quasitriangular (see [9, IX.7]). We show next that the other Hopf algebras in the list cannot admit a quasitriangular structure.

Let $q \in \mathbf{G}_p \backslash \{1\}$. The Hopf algebras in cases (d), (f) and (g) admit no quasitriangular structure, since they have a Hopf algebra surjection to the Taft algebra $T(q)$ and by Remark 4.3, $T(q)$ is not quasitriangular. (see [3, Section 1]).

Let H be one of the Hopf algebras $\widetilde{T(q)}$, $\mathbf{h}(q,m)$, $m \in \mathbf{Z}/(p) \backslash \{1\}$, $\mathbf{u}_q(\mathfrak{sl}_2)^*$, $\mathbf{r}(q)^*$, *i.e.* H is one of the cases (e), (i), (j), (k) of the list. Then H^* is pointed and H is not semisimple of odd dimension. Hence by Theorem 4.4 (a) and Remark 1.1, H cannot admit a quasitriangular structure.

Let G be a finite group. If $H = k^G$ admits a quasitriangular structure, then G must be abelian by (QT1), and H is isomorphic to a group algebra.

Finally, the semisimple Hopf algebras of dimension p^3 in (c) are not quasitriangular by [17, Thm. 1]. □

Remark 4.6. The case of $\widetilde{T(q)}$ of Proposition 4.5 also follows from [26, Section 5]. There, Radford defines Hopf algebras which depends on certain parameters. He proved that these algebras admit a quasitriangular structure if and only if these parameters satisfy specific relations. One can see that the algebras are of this type and the relations needed to have a quasitriangular structure do not hold.

In the following, we give a partial description of the quasitriangular Hopf algebras of dimension p^3.

Let $D(H)$ be the Drinfel'd double of H (see [18] or [24] for its definition and properties). We identify $D(H) = H^{*cop} \otimes H$ as vector spaces and for $\beta \in H^*$ and $h \in H$, the element $\beta \otimes h \in D(H)$ is denoted by $\beta \# h$. We may also identify $D(H)^* = H^{op} \otimes H^*$, and for $h \in H^{op}$ and $\beta \in H^*$, the element $h \otimes \beta \in D(H)^*$ is denoted by $h \# \beta$ if no confusion arrives.

For every finite-dimensional quasitriangular Hopf algebra (H,R), there is a Hopf algebra surjection

$$D(H) \xrightarrow{F} H, \quad F(\beta \# h) = \langle \beta, R^{(1)} \rangle R^{(2)} h,$$

which induces by duality an inclusion of Hopf algebras $H^* \xrightarrow{F^*} D(H)^*$. Moreover, by [24, Prop. 10] all the group-like elements of $D(H)^*$ have the form $x\#\beta$, for some $x \in G(H)$, $\beta \in G(H^*)$, and there is a central extension of Hopf algebras

$$(1) \qquad 1 \to k[G(D(H)^*)] \xrightarrow{\iota} D(H) \xrightarrow{\pi} A \to 1,$$

where $\iota(x\#\beta) = \beta\#x$, for all $x\#\beta \in G(D(H)^*)$ and A is the Hopf algebra given by the quotient $D(H)/D(H)k[G(D(H)^*)]^+$.

We need the following lemma due to Natale.

Lemma 4.7 [19, Lemma 3.2]. *Let $1 \to A \xrightarrow{\iota} H \xrightarrow{\pi} B \to 1$, be an extension of finite-dimensional Hopf algebras. Let also $L \subseteq H$ be a Hopf subalgebra. If L is simple then either $L \subseteq \iota(A)$ or $L \cap \iota(A) = k1$. In the last case, the restriction $\pi|_L : L \to B$ is injective.*

Lemma 4.8. *Let H be a quasitriangular Hopf algebra of dimension p^3 such that the Hopf algebra map f_R defined above is an isomorphism. Assume further that H is simple as a Hopf algebra. Then $G(H)$ has order p and $\langle \alpha, g \rangle = 1$, where α and g are the modular elements of H^* and H, respectively.*

Proof. H cannot be semisimple, since otherwise H would admit a non-trivial central group-like element by [16, Thm. 1] and this would contradict our assumption on the simplicity of H. Moreover, H cannot be unimodular, since otherwise H^* would be also unimodular and by Radford's formula, $\mathcal{S}^4 = \mathrm{id}$, which implies by Remark 1.1 that H is semisimple. Since f_R is an isomorphism, H^* is also non-semisimple, simple as a Hopf algebra and $G(H) \simeq G(H^*)$. Hence, by Nichols-Zoeller we have to deal only with the cases where $|G(H)| = |G(H^*)| = p$ or $|G(H)| = |G(H^*)| = p^2$. But the order of $G(H)$ cannot be p^2, since otherwise H would be pointed by Proposition 3.3 and hence isomorphic to a Frobenius-Lusztig kernel $\mathbf{u}_q(\mathfrak{sl}_2)$, for some $q \in \mathbf{G}_p \backslash \{1\}$, by Proposition 4.5, which is a contradiction, since $|G(\mathbf{u}_q(\mathfrak{sl}_2))| = p$. Therefore, the only possibility is $|G(H)| = |G(H^*)| = p$.

By [8, Cor. 2.10, 1)], we know that $f_{\tilde{R}}(\alpha) = g^{-1}$ and by [26, Prop. 3], $f_{\tilde{R}R} = f_{\tilde{R}} * f_R$ and $f_{\tilde{R}R}(\alpha) = 1$. This implies that necessarily $f_R(\alpha) = g$ and hence the order of g and α must be equal.

Consider now the Hopf algebra surjection $F : D(H) \to H$ and the Hopf algebra inclusion $F^* : H^* \to D(H)^*$, defined above. Since $G(H^*) \neq 1$, we have that $G = G(D(H)^*) \neq 1$, and dualizing extension (1), we get another extension of Hopf algebras given by

$$(2) \qquad 1 \to A^* \xrightarrow{\pi^*} D(H)^* \xrightarrow{\iota^*} k^G \to 1.$$

Define $L = F^*(H^*) \subseteq D(H^*)$. Since H^* is simple as a Hopf algebra, by Lemma 4.7 we have that $L \subseteq \pi^*(A^*)$ or $L \cap \pi^*(A^*) = k1$. But the last case cannot occur, since otherwise the restriction $\imath^*|_L : L \to k^G$ would be injective, implying that H^* is semisimple. Hence $L \subseteq \pi^*(A^*)$.

Then there are $\beta \in G(H^*)$, $\beta \neq \varepsilon$ and $x \in G(H)\backslash\{1\}$ such that $x \# \beta \in G(\pi^*(A^*)) \subseteq G$. Moreover, since the image of G in $D(H)$ is central and F is a Hopf algebra surjection, it follows that $F(\beta \# x) \in G(H) \cap Z(H)$ and by simplicity of H we have that necessarily $F(\beta \# x) = 1$.

Since both modular elements are not trivial, we have that $G(H^*) = \langle \alpha \rangle$ and $G(H) = \langle g \rangle$ and hence $|G(D(H))| = p^2$. Moreover, $|G(H^*)| = |G(A^*)| = p$, since otherwise $|G(A^*)| = |G(D(H)^*)| = |G(D(H))| = p^2$ and this would imply that H has a central non-trivial group-like element, which contradicts our assumption on the simplicity of H. Hence the group-like element $\beta \# x$ generates $G(\pi^*(A^*))$, and $\beta = \alpha^j$, $x = g^i$ for some $1 \leq i, j \leq p-1$. In particular,

$$1 = F(\alpha^j \# g^i) = \langle \alpha^j, R^{(1)} \rangle R^{(2)} g^i = f_R(\alpha^j) g^i = f_R(\alpha)^j g^i = g^j g^i.$$

Then we have that $j \equiv -i \bmod p$ and $\pi^*(G(A^*)) = \langle g \# \alpha^{-1} \rangle$. Moreover, by [20, Cor. 2.3.2], it follows that $\langle \alpha^{-1}, g \rangle^2 = 1$, and this implies that $1 = \langle \alpha^{-1}, g \rangle = \langle \alpha, g \rangle^{-1}$, since $|G(H)| = |G(H^*)| = p$ and p is odd. □

We prove next one of our main results.

Theorem 4.9. *Let H be a quasitriangular Hopf algebra of dimension p^3. Then*

(i) *H is a group algebra, or*
(ii) *H is isomorphic to $\mathbf{u}_q(\mathfrak{sl}_2)$, for some $q \in \mathbf{G}_p\backslash\{1\}$, or*
(iii) *H is a strange Hopf algebra of type $(p; p)$ and the map f_R is an isomorphism. Moreover, H and H^* are minimal quasitriangular, $1 = \langle \beta, x \rangle$, for all $\beta \in G(H^*)$, $x \in G(H)$, and* $\operatorname{ord} \mathcal{S} = 4p$.

Proof. If H is semisimple, the claim follows by [17, Thm. 1], since H must be isomorphic to a group algebra.

Assume now that H is non-semisimple and let H_R be the minimal quasitriangular Hopf subalgebra of H. Recall that $H_R = KL = LK$, where $K = \operatorname{Im} f_R$ and $L = \operatorname{Im} f_{\tilde{R}}$. By Theorem 4.4 (b) and Remark 1.1, H_R is necessarily non-semisimple. Since the only non-semisimple Hopf algebra whose dimension is a power of p with exponent less than 3 are the Taft algebras, by [37, Thm. 2] and [21, Thm. 5.5], and the Taft algebras are not quasitriangular by Remark 4.3, we conclude that $\dim H_R = p^3$.

Therefore the only possible case is when $H_R = H$ and (H, R) is a minimal quasitriangular Hopf algebra. Then by [24, Cor. 3], it follows that $\dim H \mid (\dim K)^2$; hence the dimension of K can only be p^2 or p^3.

Suppose that the dimension of K is p^2. Since H is not semisimple, K is not semisimple by Remark 4.1. Moreover, by [21, Thm. 5.5] again, K must be isomorphic to a Taft algebra $T(q)$, where $q \in \mathbf{G}_p \backslash \{1\}$. Since $L \simeq K^{*\mathrm{cop}}$, L must be also isomorphic to a Taft algebra and by Remark 1.2, $L \simeq T(q^{-1})$.

It is clear that $G(K) \subseteq G(H)$ and $G(L) \subseteq G(H)$, where the order of $G(H)$ is p or p^2. Since H is a product of two Taft algebras, we have that $\mathcal{S}^{4p} = \mathrm{id}$. If the order of $G(H)$ is p^2, then by Proposition 3.3, H must be pointed. Then, by [3, Thm. 0.1] and Proposition 4.5, H is isomorphic to a Frobenius-Lusztig kernel $\mathbf{u}_q(\mathfrak{sl}_2)$; but this cannot occur, since $|G(\mathbf{u}_q(\mathfrak{sl}_2))| = p$.

Therefore $|G(H)| = p$, and consequently $G(H) = G(K) = G(L)$. Denote by $g \in G(H)$, and $x \in K$ the generators of K, and $g' \in G(H)$ and $y \in L$ the generators of L; they are subject to similar relations as in Remark 1.2, since both are isomorphic to Taft algebras, but for different roots of unity. Moreover, g' must be a power of g, since $|G(H)| = p$.

Hence H is generated as an algebra by g, x and y, which implies that H is pointed by [18, Lemma 5.5.1]. Therefore, by [3, Thm. 0.1] and Proposition 4.5, H is isomorphic to $\mathbf{u}_q(\mathfrak{sl}_2)$, for some $q \in \mathbf{G}_p \backslash \{1\}$.

Assume now that $\dim K = p^3$, then the map $f_R : H^{*cop} \to H$ is an isomorphism of Hopf algebras. Moreover, H must be non-pointed and hence simple as a Hopf algebra by Corollary 2.3, since otherwise H would be isomorphic to a Frobenius-Lusztig kernel $\mathbf{u}_q(\mathfrak{sl}_2)$, which is a contradiction, since $\mathbf{u}_q(\mathfrak{sl}_2)^*$ is not pointed nor quasitriangular. Then by Lemma 4.8, we get that $|G(H)| = p$ and $\langle \alpha, g \rangle = 1$, where α and g are the modular elements of H^* and H respectively. It was shown in the proof of Lemma 4.8 that $f_R(\alpha) = g$; then it follows by Remark 1.1, that H and H^* are both not unimodular. Hence, $\langle \beta, x \rangle = 1$, for all $\beta \in G(H^*)$, $x \in G(H)$, which implies by Corollary 3.10 (2) that $\operatorname{ord} \mathcal{S} = 4p$. Therefore, H is a strange Hopf algebra of type $(p; p)$ which satisfies all conditions in (iii). □

It is well-known that the group algebras and the Frobenius-Lusztig kernels are ribbon Hopf algebras (see [9]). Next we prove that there are no other ribbon Hopf algebras of dimension p^3.

COROLLARY 4.10. *Let H be a ribbon Hopf algebra of dimension p^3. Then H is a group algebra or H is isomorphic to $\mathbf{u}_q(\mathfrak{sl}_2)$, for some $q \in \mathbf{G}_p \backslash \{1\}$.*

Proof. Suppose on the contrary that H is a ribbon Hopf algebra of dimension p^3 which is not a group algebra and H is not isomorphic to a Frobenius-Lusztig kernel. Then by the preceding theorem, H is of type $(p;p)$ and $\operatorname{ord}\mathscr{S} = 4p$. But this cannot occur, since by [10, Thm. 2], the square of the antipode must have odd order. □

As another application of Theorem 4.9 we classify quasitriangular Hopf algebras of dimension 27 using some results from [1] and [4]. As in [4], we denote by $M^c(n,k)$ the simple matrix coalgebras contained in the coradical.

Let H be a finite-dimensional Hopf algebra and let H_0 be the coradical of H. Then we have that $H_0 \simeq \bigoplus_{\tau\in\hat{H}} H_\tau$, where H_τ is a simple subcoalgebra of dimension d_τ^2, $d_\tau \in \mathbf{Z}$, and $\hat{H}$ is the set of isomorphism types of simple left H-comodules. Define

$$H_{0,d} = \bigoplus_{\tau\in\hat{H}:d_\tau=d} H_\tau.$$

For instance $H_{0,1} = k[G(H)]$ and $H_{0,2}$ is the sum of all 4-dimensional simple subcoalgebras of H. By [1, Lemma 2.1 (i)], the order of $G(H)$ divides the dimension of $H_{0,d}$ for all $d \geq 1$.

Theorem 4.11. *Let H be a quasitriangular Hopf algebra of dimension* 27. *Then H is a group algebra or H is isomorphic to $\mathbf{u}_q(\mathfrak{sl}_2)$, for some $q \in \mathbf{G}_3\backslash\{1\}$. Precisely, H is isomorphic to one and only one Hopf algebra of the following list, where $q \in \mathbf{G}_3\backslash\{1\}$,*

(a) $k[\mathbf{Z}/(27)]$
(b) $k[\mathbf{Z}/(9) \times \mathbf{Z}/(3)]$
(c) $k[\mathbf{Z}/(3) \times \mathbf{Z}/(3) \times \mathbf{Z}/(3)]$
(d) $\mathbf{u}_q(\mathfrak{sl}_2) := k\langle g,x,y \mid gxg^{-1} = q^2x, gyg^{-1} = q^{-2}y,\ g^3 = 1,\ x^3 = 0,\ y^3 = 0,\ xy - yx = g - g^2\rangle$.

Proof. By Theorem 4.9, we have to show that there is no quasitriangular Hopf algebra H of dimension 27 which satisfies (iii).

Suppose that such a Hopf algebra exists. Since H is not semisimple, H is also not cosemisimple. Moreover, H has no non-trivial skew primitive elements, by Corollary 3.14 and Proposition 3.11 (1).

Suppose now that

$$H_0 = k[G(H)] \oplus M^c(n_1,k) \oplus \cdots \oplus M^c(n_t,k),$$

where $2 \leq n_1 \leq \cdots \leq n_t \leq 3$, since $3 \mid \dim H_{0,d}$, for all $d \geq 1$. Moreover, since H is not cosemisimple, we have the following possibilities for H_0:

(1) $H_0 = k[G(H)] \oplus M^c(3,k)$, with dim $H_0 = 12$,
(2) $H_0 = k[G(H)] \oplus M^c(2,k)^3$, with dim $H_0 = 15$,
(3) $H_0 = k[G(H)] \oplus M^c(3,k)^2$, with dim $H_0 = 21$,
(4) $H_0 = k[G(H)] \oplus M^c(2,k)^3 \oplus M^c(3,k)$, with dim $H_0 = 24$.

Since all skew primitive elements of H are trivial, by [4, Cor. 4.3] we have that

$$27 = \dim H > \dim H_1 \geq (1 + 2n_1)3 + \sum_{i=1}^{t} n_i^2. \tag{3}$$

Replacing the corresponding dimensions in equation (3), it follows that no one of the cases (1), ..., (4) is possible, obtaining in this way a contradiction to our assumption. □

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