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EXTENDING POINTWISE BOUNDED EQUICONTINUOUS COLLECTIONS OF FUNCTIONS

By

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Abstract. We prove that for a subspace A of a space X, the following statements are equivalent: (1) for any Fréchet space Y, every pointwise bounded equicontinuous subset of $C(A, Y)$ can be extended to a pointwise bounded equicontinuous subset of $C(X, Y)$; (2) every pointwise bounded equicontinuous subset of $C(A)$ can be extended to a pointwise bounded equicontinuous subset of $C(X)$; (3) for any Fréchet space Y, every function $f \in C(A, Y)$ can be extended to a function $g \in C(X, Y)$. This theorem and other results obtained in this paper generalize several known theorems due to Flood, Frantz and Heath-Lutzer-Zenor, etc.

1. Introduction and Preliminaries

All spaces are assumed to be T_1-spaces. For short, we call a topological vector space a *TV-space*, and a locally convex TV-space an *LCTV-space* (see [2], [12]). In a TV-space, 0 stands for its origin. For topological spaces X and Y, $C(X, Y)$ denotes the set of all continuous functions from X into Y. In particular, the set of all continuous real-valued (resp. continuous bounded real-valued) functions is denoted by $C(X)$ (resp. $C^*(X)$). Let X be a space, Y a TV-space and $\mathscr{F} = \{f_\alpha : \alpha \in \Omega\} \subset C(X, Y)$. For a point $x \in X$, $\mathscr{F}$ is said to be *equicontinuous at* x if for every neighborhood V of 0 in Y, there exists a neighborhood O of x in X such that $f_\alpha(y) - f_\alpha(x) \in V$ for every $y \in O$ and every $\alpha \in \Omega$. The collection $\mathscr{F}$ is said to be *equicontinuous* if $\mathscr{F}$ is equicontinuous at every point $x \in X$. The collection $\mathscr{F}$ is said to be *pointwise bounded* if for every $x \in X$ and every neighborhood V of 0 in Y, there exists $e_x > 0$ such that $r \cdot f_\alpha(x) \in V$ for every r with $|r| < e_x$ and every $\alpha \in \Omega$. The collection $\mathscr{F}$ is said to be *pointwise totally bounded* if for every $x \in X$ and every neighborhood V of 0 in Y, there exists a

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finite subset F of Y such that $\{f_\alpha(x) : \alpha \in \Omega\} \subset F + V$. As is known, every pointwise totally bounded subset $\mathscr{F}$ of $C(X, Y)$ is pointwise bounded, and the converse holds when $Y = \mathbf{R}$. Let X be a space, A a subspace of X and Y a space. For $\mathscr{F}(= \{f_\alpha : \alpha \in \Omega\}) \subset C(A, Y)$ and $\mathscr{G} \subset C(X, Y)$, we say that $\mathscr{F}$ is *extended* to $\mathscr{G}$ (or $\mathscr{G}$ is an *extension* of $\mathscr{F}$) if $\mathscr{G}$ is expressed as $\{g_\alpha : \alpha \in \Omega\}$ and $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$.

The problem "When can a pointwise bounded equicontinuous subset of $C(A, Y)$ be extended to the one of $C(X, Y)$?" was studied by M. Frantz [8], which was motivated by the Dugundji extension theorem [5].

THEOREM 1.1 (Frantz [8]). *For a metrizable space X, a closed subspace A of X and a metrizable LCTV-space Y, every pointwise bounded equicontinuous subset of $C(A, Y)$ can be extended to an equicontinuous subset of $C(X, Y)$.*

It was shown in [8] that the equicontinuous subset $\{f_n : n \in \mathbf{N}\}$ of $C(\{0, 1\})$, defined by $f_n(0) = 0$, and $f_n(1) = n$ for every $n \in \mathbf{N}$, admits no equicontinuous extension over $C([0, 1])$. Thus, the pointwise boundedness can not be dropped in the above theorem.

In this paper, we study the above problem from the following points of view.

In Section 2, we show that Theorem 1.1 remains true if 'metrizable space' is weakened to 'decreasing (G) space' in the sense of Collins-Roscoe [4] (Theorem 2.1). Stares [18] proved that the Dugundji extension theorem also holds for decreasing (G) spaces. Our result is along this direction.

In Section 3, we prove the equivalence stated in the abstract (Theorem 3.1). This generalizes some known results due to Flood [7] and Heath-Lutzer-Zenor [10], and establishes some incomplete results due to Aló [1] and Gutev [9] (see Section 3 for details). In particular, Lemma 3.5, which is a key lemma to prove Theorem 3.1, shows that for a P-embedded subspace A of a space X and a Fréchet space Y, every poinwise bounded equicontinuous subset $\mathscr{F}$ of $C(A, Y)$ has an extender which well behaves like Dugundji's one in [5]. Some applications characterizing collectionwise normality are also given (Corollaries 3.7 and 3.9).

In the final part of this section, we show that every (not necessarily pointwise bounded) equicontinuous subset of $C(X, Y)$ can be extended to an equicontinuous subset of $C(\gamma X, Y)$, where X is a Tychonoff space and γX is its Dieudonné completion (Theorem 3.11). The result slightly improves the one of Sanchis [15].

Let us recall some definitions. A *Fréchet space* is a completely metrizable LCTV-space. Note that every Banach space is a Fréchet space.

Let X be a topological space and Y a TV-space. Let $\mathscr{B}$ be a collection of subsets of X which is closed under finite unions. For $B \in \mathscr{B}$, a neighborhood V of 0 in Y and $f \in C(X, Y)$, define $N(f, B, V) = \{g \in C(X, Y) : f(x) - g(x) \in V$ for every $x \in B\}$. The collection $\{N(f, B, V) : B \in \mathscr{B}, V$ is a neighborhood of 0 in $Y\}$ can be taken as a neighborhood base of f and the topology is called as *the topology of uniform convergence* if $\mathscr{B} = \{X\}$, *the compact-open topology* if $\mathscr{B} = \{K \subset X$: K is compact$\}$, and *the topology of pointwise convergence* if $\mathscr{B} = \{F \subset X$: F is finite$\}$. For a metric LCTV-space (Y, ρ), denote the open ε-ball and the closed ε-ball by $B(0; \varepsilon)$ and $\bar{B}(0; \varepsilon)$, respectively; that is, $B(0; \varepsilon) = \{y \in Y : \rho(0, y) < \varepsilon\}$ and $\bar{B}(0; \varepsilon) = \{y \in Y : \rho(0, y) \leq \varepsilon\}$. For a metric LCTV-space (Y, ρ), $\varepsilon > 0$, $B \in \mathscr{B}$ and $f \in C(X, Y)$, $N(f, B, \varepsilon)$ denotes $N(f, B, B(0; \varepsilon))$.

The symbols $C_k(X, Y)$, $C_k(X)$ or $C_k^*(X)$ stand for $C(X, Y)$, $C(X)$ or $C^*(X)$ with the compact-open topology. Similarly, the symbols $C_p(X, Y)$, $C_p(X)$ or $C_p^*(X)$ stand for $C(X, Y)$, $C(X)$ or $C^*(X)$ with the topology of pointwise convergence.

A space X is said to be a *k-space* if for every $S \subset X$, the set S is closed in X provided that the intersection of S with any compact subspace Z of X is closed in Z.

Let X be a space and A a subspace of X. For a collection $\mathscr{W}$ of subsets of X, $\mathscr{W} \wedge A$ stands for $\{W \cap A : W \in \mathscr{W}\}$. A subspace A is said to be C (resp. C^*)-*embedded* in X if every real-valued (resp. bounded real-valued) continuous function on A can be continuously extended over X. A subspace A is said to be P^{γ}-*embedded* in X if for every normal open cover $\mathscr{U}$ of A with $|\mathscr{U}| \leq \gamma$, there exists a normal open cover $\mathscr{V}$ of X such that $\mathscr{V} \wedge A$ refines $\mathscr{U}$. A subspace A is said to be *P-embedded* in X if A is P^{γ}-embedded in X for every γ. It is known that A is P-embedded in X if and only if every continuous function from A into any Banach space Y can be extended to a continuous one over X [2]. Moreover, it is known that 'Banach space' in the above can be replaced by 'Fréchet space' (see [2]). A subspace A is said to be *well-embedded* in X if every zero-set of X disjoint from A can be completely separated from A in X (see [2], [11]). We use the following facts without references; (*i*) A is $P^{\aleph_0}$-embedded in X if and only if A is C-embedded in X; (*ii*) A is C-embedded in X if and only if A is C^*-embedded and well-embedded in X; (*iii*) X is collectionwise normal if and only if every closed subspace A of X is P-embedded in X. For these results, see [2], [11] and [13].

Other terminology and basic facts are referred to [2], [6], [11], [12] and [13].

2. A Generalization of Theorem 1.1 to Decreasing (G) Spaces X

Let X be a metrizable space, A a closed subspace of X and Y an LCTV-space. Let $\Psi : C(A, Y) \to C(X, Y)$ be Dugundji's extender constructed in the proof of [5, Theorem 4.1]. Theorem 1.1 actually shows that for a pointwise bounded equicontinuous subset $\{f_\alpha : \alpha \in \Omega\}$ of $C(A, Y)$, the extended collection $\{\Psi(f_\alpha) : \alpha \in \Omega\}$ is also equicontinuous.

A space X is said to be *decreasing* (G) if there exists a collection $\{\mathscr{W}(x) : x \in X\}$, where $\mathscr{W}(x)$ is a collection of sets of the form $\mathscr{W}(x) = \{W(n, x) : n \in \mathbf{N}\}$ such that (i) $W(n+1, x) \subset W(n, x) \subset X$ for all x and n and (ii) for every $x \in X$ and every open neighborhood U of x, there exists an open neighborhood $V(x, U)$ of x such that for every $y \in V(x, U)$ there is n with $x \in W(n, y) \subset U$ ([4]). Note that every stratifiable space ([3]) is decreasing (G), and every decreasing (G) space is hereditarily paracompact.

Extending Theorem 1.1, we have the following:

THEOREM 2.1. *For a decreasing (G) space X, a closed subspace A of X and an LCTV-space Y, every pointwise bounded (resp. pointwise totally bounded) equicontinuous subset of $C(A, Y)$ can be extended to a pointwise bounded (resp. pointwise totally bounded) equicontinuous subset of $C(X, Y)$.*

PROOF. The proof is based on Stares [18]. Let X be a decreasing (G) space, A a non-empty closed subspace of X and Y an LCTV-space. We will actually show in the following that for every pointwise bounded (resp. pointwise totally bounded) equicontinuous subset $\{f_\alpha : \alpha \in \Omega\}$ of $C(A, Y)$, the collection $\{\Phi(f_\alpha) : \alpha \in \Omega\}$ is pointwise bounded (resp. pointwise totally bounded) equicontinuous, where $\Phi : C(A, Y) \to C(X, Y)$ is Dugundji's extender constructed by Stares in [18].

Let $\{\mathscr{W}(x) : x \in X\}$, where $\mathscr{W}(x) = \{W(n, x) : n \in \mathbf{N}\}$, be a collection satisfying (i) and (ii) in the definition of a decreasing (G) space. Let

$$B = \{x \in X - A : x \in V(a, U) \text{ for some } a \in A \text{ and some open subset } U \text{ of } X \text{ with } a \in U\}.$$

Moreover, for every $x \in B$, let

$$\mathscr{B}_x = \{V(a, U) : x \in V(a, U), a \in A, \text{ and } U \text{ is open in } X\}, \quad \text{and}$$

$$m(x) = \max\{n \in \mathbf{N} : a \in W(n, x) \subset U \text{ for some } V(a, U) \in \mathscr{B}_x\},$$

the well-definedness of $m(x)$ is due to [18]. Since $X - A$ is paracompact, there exists a locally finite open cover $\mathscr{U}$ of $X - A$ such that $\mathscr{U}$ refines $\{V(x, X - A) : x \in X - A\}$. Let $\{p_U : U \in \mathscr{U}\}$ be a locally finite partition of unity on $X - A$ subordinated to $\mathscr{U}$. For every $U \in \mathscr{U}$, fix $x_U \in X - A$ so as to satisfy $p_U^{-1}((0, 1]) \subset V(x_U, X - A)$. Fix $a_0 \in A$ arbitrarily. For every $U \in \mathscr{U}$, take $a_U \in A$ and a neighborhood O_U of a_U in X as follows:

If $x_U \notin B$, set $a_U = a_0$ and $O_U = X$.

If $x_U \in B$, select $a_U \in A$ and a neighborhood O_U of a_U in X such that $a_U \in W(m(x_U), x_U) \subset O_U$ and $x_U \in V(a_U, O_U)$.

Let $\{f_\alpha : \alpha \in \Omega\}$ be a pointwise bounded equicontinuous subset of $C(A, Y)$. Define functions $g_\alpha : X \to Y$, $\alpha \in \Omega$, by

$$g_\alpha(x) = \begin{cases} f_\alpha(x) & \text{if } x \in A \\ \sum_{U \in \mathscr{U}} p_U(x) \cdot f_\alpha(a_U) & \text{otherwise.} \end{cases}$$

Then, $\{g_\alpha : \alpha \in \Omega\}$ is the required extension of $\{f_\alpha : \alpha \in \Omega\}$.

To prove $\{g_\alpha : \alpha \in \Omega\}$ is equicontinuous, let $x \in X$ and W be a neighborhood of 0 in Y. We may assume W is convex.

Case 1. $x \in A$. Let O be a neighborhood of x in X satisfying that $f_\alpha(y) \in f_\alpha(x) + W$ for every $y \in O \cap A$ and every $\alpha \in \Omega$. Then, we shall show that $g_\alpha(y) \in g_\alpha(x) + W$ for every $y \in V(x, V(x, O))$ and every $\alpha \in \Omega$. Fix $y \in V(x, V(x, O))$ and $\alpha \in \Omega$. Since $V(x, V(x, O)) \subset O$, we may assume $y \in V(x, V(x, O)) - A$. Then, by the similar way to [18], $a_U \in O$ holds for every $U \in \mathscr{U}$ with $y \in p_U^{-1}((0, 1])$. Hence, it follows that $g_\alpha(y) - g_\alpha(x) = \sum_{U \in \mathscr{U}} p_U(y) \cdot (f_\alpha(a_U) - f_\alpha(x)) \in W$, the last inclusion is due to the convexity of W. Hence, $g_\alpha(y) \in g_\alpha(x) + W$ holds for every $y \in V(x, V(x, 0))$ and every $\alpha \in \Omega$.

Case 2. $x \in X - A$. There exist a neighborhood O_1 of x in $X - A$ and finitely many elements $U_1, \dots, U_n \in \mathscr{U}$ such that $O_1 \cap U = \varnothing$ for every $U \in \mathscr{U} - \{U_1, \dots, U_n\}$. Since $\{f_\alpha : \alpha \in \Omega\}$ is pointwise bounded, there exists $e_x > 0$ such that $r \cdot f_\alpha(a_{U_i}) \in W$ for every r with $|r| < e_x$, every $\alpha \in \Omega$ and every $i = 1, \dots, n$. Then, there exists a neighborhood O_2 of x in $X - A$ such that $|p_{U_i}(y) - p_{U_i}(x)| < e_x/n$ for every $y \in O_2$ and every $i = 1, \dots, n$. Let $y \in O_1 \cap O_2$ and $\alpha \in \Omega$. Then, $g_\alpha(y) - g_\alpha(x) = (1/n) \sum_{i=1}^n n \cdot (p_{U_i}(y) - p_{U_i}(x)) \cdot f_\alpha(a_{U_i})$. Then, we have $n \cdot (p_{U_i}(y) - p_{U_i}(x)) \cdot f_\alpha(a_{U_i}) \in W$ for every $i = 1, \dots, n$. Since W is convex, it follows that $g_\alpha(y) - g_\alpha(x) \in W$.

Hence, these complete the proof that $\{g_\alpha : \alpha \in \Omega\}$ is equicontinuous.

To see $\{g_\alpha : \alpha \in \Omega\}$ is pointwise bounded, let $x \in X$. We may assume $x \in X - A$. Let W be a neighborhood of 0 in Y. We may assume W is convex. Since $\mathscr{U}$ is point-finite, there exist finite elements $U_1, \dots, U_n \in \mathscr{U}$ such that $x \notin U$ for every $U \in \mathscr{U} - \{U_1, \dots, U_n\}$. Since $\{f_\alpha : \alpha \in \Omega\}$ is pointwise bounded, there exists $e_x > 0$ such that $r \cdot f_\alpha(a_{U_i}) \in W$ for every r with $|r| < e_x$, every $\alpha \in \Omega$ and every $i = 1, \dots, n$. For every r with $|r| < e_x$ and every $\alpha \in \Omega$, we have $r \cdot g_\alpha(x) = \sum_{i=1,\dots,n} p_{U_i}(x) \cdot (r \cdot f_\alpha(a_{U_i})) \in W$, the last inclusion is due to the convexity of W. Hence $\{g_\alpha : \alpha \in \Omega\}$ is pointwise bounded.

The case of pointwise total boundedness is left to the reader. This completes the proof. □

COROLLARY 2.2. *For a stratifiable space X, a closed subspace A of X and an LCTV-space Y, every pointwise bounded (resp. pointwise totally bounded) equicontinuous subset of $C(A, Y)$ can be extended to a pointwise bounded (resp. pointwise totally bounded) equicontinuous subset of $C(X, Y)$.*

Note that, on Theorem 2.1, the assumption of being decreasing (G) can not be weakened to being hereditarily paracompact. For example, let X be the Michael line [6, 5.1.32] and A the set of all rationals. For a (complete) LCTV-space $Y = C_k(\mathbf{P})$, where $\mathbf{P}$ is the set of all irrationals and a continuous function $f : A \to Y$ defined by $f(x)(y) = 1/(x - y)$, $x \in A$ and $y \in \mathbf{P}$, f can not be extended over X (see Sennott [16]).

3. Extending Pointwise Bounded Equicontinuous Collections of Functions with Values in Fréchet Spaces

In this section, we describe subspaces which admit extending pointwise bounded equicontinuous collections of functions with values in Fréchet spaces. The main result is the following Theorem 3.1. The equivalence (2) ⇔ (3) was announced without proofs by Aló [1], but later this was withdrawn under a review of Sennott [17]. Assuming that X is a Tychonoff space, (2) ⇔ (3) was proved by Flood [7, Theorem 5.9.2] by categorical methods, and it seems to be essential to assume being Tychonoff spaces in his proof. The equivalence (3) ⇔ (4) was first stated by Gutev [9], with incomplete proof, for Banach spaces Y.

THEOREM 3.1. *Let X be a space and A a subspace of X. Then, the following statements are equivalent:*

(1) *for any Fréchet space Y, every pointwise bounded equicontinuous subset of $C(A, Y)$ can be extended to a pointwise bounded equicontinuous subset of $C(X, Y)$;*

(2) *every pointwise bounded equicontinuous subset of $C(A)$ can be extended to a pointwise bounded equicontinuous subset of $C(X)$;*

(3) *A is P-embedded in X (that is, for any Fréchet space Y, every function* $f \in C(A, Y)$ *can be extended to a function* $g \in C(X, Y)$*);*

(4) *for any Fréchet space Y, every pointwise totally bounded equicontinuous subset of* $C(A, Y)$ *can be extended to a pointwise totally bounded equicontinuous subset of* $C(X, Y)$.

For the proof, we prepare some lemmas. For a subspace S of a space X, $\mathrm{Int}_X S$ stands for the interior of S in X.

LEMMA 3.2. *Let X be a space, Y a TV-space and* $\{f_\alpha : \alpha \in \Omega\}$ *a subset of* $C(X, Y)$. *For every neighborhood V of* 0 *in Y and every* $x \in X$, *define*

$$O_x(V) = \mathrm{Int}_X(\bigcap\{f_\alpha^{-1}(f_\alpha(x) + V) : \alpha \in \Omega\}),$$

and put $\mathcal{O}_V = \{O_x(V) : x \in X\}$. *Then, the following hold.*

(1) $\{f_\alpha : \alpha \in \Omega\}$ *is equicontinuous if and only if* $\mathcal{O}_V$ *is an open cover of X for every neighborhood V of* 0 *in Y.*

(2) *If V and W are neighborhoods of* 0 *in Y satisfying that* $W + W \subset V$ *and W is symmetric, then* $\mathrm{St}(x, \mathcal{O}_W)(= \bigcup\{O \in \mathcal{O}_W : x \in O\}) \subset O_x(V)$ *for every* $x \in X$.

The proof of Lemma 3.2 is straightforward. By this lemma, we immediately have the following:

LEMMA 3.3. *Let X be a space and Y a TV-space. Let* $\{f_\alpha : \alpha \in \Omega\}$ *be an equicontinuous subset of* $C(X, Y)$, *and V a neighborhood of* 0 *in Y. Then, the collection* $\mathcal{O}_V$ *defined as in Lemma* 3.2 *is a normal open cover of X.*

The proof of the following lemma is easy and omitted.

LEMMA 3.4. *Let X be a space and A a well-embedded subspace of X. Assume that F is the intersection of a zero-set and a cozero-set of X, and F is disjoint from A. Then, there exists a cozero-set U of X such that* $F \subset U \subset X - A$.

The following lemma is essential for the proof, and seems to be interesting in itself. For a space X, a subspace A of X and $\mathcal{F} \subset C(A, Y)$, a map $\Phi : \mathcal{F} \to C(X, Y)$ is said to be an *extender* if $\Phi(f) \mid A = f$ for every $f \in \mathcal{F}$.

LEMMA 3.5. *Let X be a space, A a P-embedded subspace of X and Y a Fréchet space. Let $\mathscr{F} = \{f_\alpha : \alpha \in \Omega\}$ be a pointwise bounded equicontinuous subset of $C(A, Y)$. Then, there exists an extender $\Phi : \mathscr{F} \to C(X, Y)$ satisfying that*

(i) *$\Phi(f_\alpha)(X)$ is contained in the closed convex hull of $f_\alpha(A)$ for every $\alpha \in \Omega$;*

(ii) *$\Phi(\mathscr{F})$ is pointwise bounded equicontinuous;*

(iii) *Φ is continuous when $C(A, Y)$ and $C(X, Y)$ carry either one of the compact-open topology, the topology of pointwise convergence and the topology of uniform convergence, where $\mathscr{F}$ has the subspace topology of $C(A, Y)$.*

If, in addition, $\mathscr{F}$ is pointwise totally bounded, then $\Phi(\mathscr{F})$ is also pointwise totally bounded.

PROOF OF LEMMA 3.5. Let X be a space, A a P-embedded subspace of X and (Y, ρ) a Fréchet space, where ρ is an invariant metric on Y (see [12]). Let $\mathscr{F} = \{f_\alpha : \alpha \in \Omega\}$ be a pointwise bounded equicontinuous subset of $C(A, Y)$. For every $n \in \mathbf{N}$, there exists a convex symmetric neighborhood S_n of 0 in Y such that $S_n + S_n \subset B(0; 1/2^n)$. For every $n \in \mathbf{N}$ and every $a \in A$, define $O_a(S_n) = \mathrm{Int}_A(\bigcap\{f_\alpha^{-1}(f_\alpha(a) + S_n) : \alpha \in \Omega\})$ like in Lemma 3.2.

First, we shall define a sequence $\{\mathscr{V}_n : n \in \mathbf{N}\}$ of locally finite cozero-set covers of X, for some index set B, such that

(i) $\mathscr{V}_n$ is expressed as $\mathscr{V}_n = \{V_{(\beta_1,\dots,\beta_n)} : (\beta_1, \dots, \beta_n) \in B^n\}$;

(ii) $\mathscr{V}_n \wedge A$ refines $\{O_a(S_n) : a \in A\}$;

(iii) $\bigcup\{V_{(\beta_1,\dots,\beta_n,\beta_{n+1})} : \beta_{n+1} \in B\} = V_{(\beta_1,\dots,\beta_n)}$ for all $(\beta_1, \dots, \beta_n) \in B^n$;

(iv) If $V_{(\beta_1,\dots,\beta_n)} \neq \varnothing$, then $V_{(\beta_1,\dots,\beta_n)} \cap A \neq \varnothing$.

Let B be any infinite set with $|B| \geq |2^X|$ and fix it. From Lemma 3.3, $\{O_a(S_i) : a \in A\}$ is a normal open cover of A for every $i \in \mathbf{N}$. Since A is P-embedded in X, for every $i \in \mathbf{N}$, there exists a locally finite cozero-set cover $\{W_\beta : \beta \in B_i\}$ of X such that $\{W_\beta : \beta \in B_i\} \wedge A$ refines $\{O_a(S_i) : a \in A\}$. We may assume $B = B_i$ for every $i \in \mathbf{N}$, because we can regard B_i as a subset of B and set $W_\beta = \varnothing$ for $\beta \in B - B_i$. Since the first step constructing $\mathscr{V}_1$ can be similarly proved if we put $V_{(\ )} = X$ in the following proof, we only show the general step. Assume that $\mathscr{V}_1, \dots, \mathscr{V}_n$ have been constructed so as to satisfy the conditions from (i) to (iv) above. Let $(\beta_1, \dots, \beta_n) \in B^n$ be fixed. Put $B_{(\beta_1,\dots,\beta_n)} = \{\beta \in B : V_{(\beta_1,\dots,\beta_n)} \cap W_\beta \cap A \neq \varnothing\}$. By Lemma 3.4, there exists a cozero-set $D_{(\beta_1,\dots,\beta_n)}$ of X such that

$$V_{(\beta_1,\dots,\beta_n)} - \bigcup\{W_\beta : \beta \in B_{(\beta_1,\dots,\beta_n)}\} \subset D_{(\beta_1,\dots,\beta_n)} \subset X - A.$$

In case $B_{(\beta_1,\dots,\beta_n)} \neq \varnothing$, pick up and fix a $\beta_{(\beta_1,\dots,\beta_n)} \in B_{(\beta_1,\dots,\beta_n)}$, and define for every $\beta \in B$,

$$V_{(\beta_1,\dots,\beta_n,\beta)} = \begin{cases} V_{(\beta_1,\dots,\beta_n)} \cap (W_\beta \cup D_{(\beta_1,\dots,\beta_n)}) & \text{if } \beta = \beta_{(\beta_1,\dots,\beta_n)} \\ V_{(\beta_1,\dots,\beta_n)} \cap W_\beta & \text{if } \beta \in B_{(\beta_1,\dots,\beta_n)} - \{\beta_{(\beta_1,\dots,\beta_n)}\} \\ \varnothing & \text{if } \beta \notin B_{(\beta_1,\dots,\beta_n)}. \end{cases}$$

In case $B_{(\beta_1,\dots,\beta_n)} = \varnothing$, define $V_{(\beta_1,\dots,\beta_n,\beta)} = \varnothing$ for all $\beta \in B$.

Then, $\{V_{(\beta_1,\dots,\beta_n,\beta_{n+1})} : (\beta_1,\dots,\beta_n,\beta_{n+1}) \in B^{n+1}\}$ is the required $\mathscr{V}_{n+1}$.

Second, we show that there exists a locally finite partition of unity $\mathscr{P}_n = \{p_{(\beta_1,\dots,\beta_n)} : (\beta_1,\dots,\beta_n) \in B^n\}$ on X for each $n \in \mathbf{N}$ such that $p^{-1}_{(\beta_1,\dots,\beta_n)}((0,1]) = V_{(\beta_1,\dots,\beta_n)}$ and $\sum_{\beta\in B} p_{(\beta_1,\dots,\beta_n,\beta)}(x) = p_{(\beta_1,\dots,\beta_n)}(x)$ for $x \in X$.

Indeed, assume that $\mathscr{P}_n$ is constructed. Let $(\beta_1,\dots,\beta_n) \in B^n$ be fixed. Since $\{V_{(\beta_1,\dots,\beta_n,\beta)} : \beta \in B\}$ is a locally finite cozero-set cover of $V_{(\beta_1,\dots,\beta_n)}$, there exists a locally finite partition of unity $\{q_{(\beta_1,\dots,\beta_n,\beta)} : \beta \in B\}$ on $V_{(\beta_1,\dots,\beta_n)}$ such that $q^{-1}_{(\beta_1,\dots,\beta_n,\beta)}((0,1]) = V_{(\beta_1,\dots,\beta_n,\beta)}$ for every $\beta \in B$. For every $(\beta_1,\dots,\beta_n) \in B^n$ and $\beta \in B$, define

$$p_{(\beta_1,\dots,\beta_n,\beta)}(x) = \begin{cases} p_{(\beta_1,\dots,\beta_n)}(x) \cdot q_{(\beta_1,\dots,\beta_n,\beta)}(x) & \text{if } x \in V_{(\beta_1,\dots,\beta_n)}, \\ 0 & \text{otherwise} \end{cases}$$

for every $x \in X$. Then, one can show that the function $p_{(\beta_1,\dots,\beta_n,\beta)} : X \to [0,1]$ is continuous and $\mathscr{P}_{n+1} = \{p_{(\beta_1,\dots,\beta_n,\beta_{n+1})} : (\beta_1,\dots,\beta_n,\beta_{n+1}) \in B^{n+1}\}$ is now the required partition of unity.

Third, we shall construct an extension of f_α over X for every $\alpha \in \Omega$. If $V_{(\beta_1,\dots,\beta_n)} \neq \varnothing$, pick up an element $a_{(\beta_1,\dots,\beta_n)} \in V_{(\beta_1,\dots,\beta_n)} \cap A$ and fix it. For every $n \in \mathbf{N}$ and every $\alpha \in \Omega$, define a continuous function $g^n_\alpha : X \to Y$ by $g^n_\alpha(x) = \sum_{(\beta_1,\dots,\beta_n)\in B^n} p_{(\beta_1,\dots,\beta_n)}(x) \cdot f_\alpha(a_{(\beta_1,\dots,\beta_n)})$ for every $x \in X$. Then, for every $n \in \mathbf{N}$, every $\alpha \in \Omega$ and every $x \in A$, note that:

$$\text{If } p_{(\beta_1,\dots,\beta_n)}(x) > 0, \text{ then } f_\alpha(x) - f_\alpha(a_{(\beta_1,\dots,\beta_n)}) \in S_n + S_n. \tag{1}$$

Indeed, if $p_{(\beta_1,\dots,\beta_n)}(x) > 0$, then $x, a_{(\beta_1,\dots,\beta_n)} \in O_a(S_n)$ for some $a \in A$. So, (1) holds. Hence, by (1) and the convexity of $S_n + S_n$, we have

$$f_\alpha(x) - g^n_\alpha(x) \in S_n + S_n \subset B(0; 1/2^n). \tag{2}$$

Let $(\beta_1,\dots,\beta_n) \in B^n$ and $\beta \in B$. Since $a_{(\beta_1,\dots,\beta_n)}$ and $a_{(\beta_1,\dots,\beta_n,\beta)}$ is contained in $V_{(\beta_1,\dots,\beta_n)} \cap A$, it follows that $a_{(\beta_1,\dots,\beta_n)}, a_{(\beta_1,\dots,\beta_n,\beta)} \in O_a(S_n)$ for some $a \in A$. So, we have

$$f_\alpha(a_{(\beta_1,\dots,\beta_n)}) - f_\alpha(a_{(\beta_1,\dots,\beta_n,\beta)}) \in S_n + S_n. \tag{3}$$

Hence, for every $n \in \mathbf{N}$, every $\alpha \in \Omega$ and every $x \in X$, by (3) and the convexity of $S_n + S_n$, we have

$$g^n_\alpha(x) - g^{n+1}_\alpha(x) \in S_n + S_n \subset B(0; 1/2^n). \tag{4}$$

By (2), (4) and the fact that ρ is invariant, we have

$$\rho(f_\alpha(x), g_\alpha^n(x)) < \frac{1}{2^n} \text{ for every } n \in \mathbf{N}, \text{ every } \alpha \in \Omega \text{ and every } x \in A, \tag{5}$$

$$\rho(g_\alpha^n(x), g_\alpha^{n+1}(x)) < \frac{1}{2^n} \text{ for every } n \in \mathbf{N}, \text{ every } \alpha \in \Omega \text{ and every } x \in X. \tag{6}$$

Hence, by (5), (6) and the completeness of Y, the function $g_\alpha : X \to Y$ defined by $g_\alpha(x) = \lim_{n\to\infty} g_\alpha^n(x)$, $\alpha \in \Omega$ and $x \in X$, is continuous and an extension of f_α. Define an extender $\Phi : \mathscr{F} \to C(X, Y)$ by $\Phi(f_\alpha) = g_\alpha$ for every $\alpha \in \Omega$. Clearly, $g_\alpha(X) = \Phi(f_\alpha)(X)$ is contained in the closed convex hull of $f_\alpha(A)$ for every $\alpha \in \Omega$.

In particular, it follows from (6) and the invariantness of ρ that

$$g_\alpha(x) \in g_\alpha^n(x) + \bar{B}(0; 1/2^{n-1}) \tag{7}$$

for every $n \in \mathbf{N}$, every $x \in X$ and every $\alpha \in \Omega$.

Fourth, we shall prove $\{g_\alpha : \alpha \in \Omega\}$ is pointwise bounded equicontinuous. To do this, fix $x \in X$. Let W be a neighborhood of 0 in Y. Let V be a convex and circled neighborhood of 0 in Y with $V + V + V \subset W$, and $m \in \mathbf{N}$ with $\bar{B}(0; 1/2^{m-1}) \subset V$. Since $\{V_{(\beta_1,\dots,\beta_m)} : (\beta_1, \dots, \beta_m) \in B^m\}$ is locally finite, there exist a neighborhood O of x in X and a non-empty finite subset Λ of B^m such that $O \cap V_{(\beta_1,\dots,\beta_m)} = \emptyset$ for every $(\beta_1, \dots, \beta_m) \in B^m - \Lambda$. Since $\{f_\alpha : \alpha \in \Omega\}$ is pointwise bounded, there exists $e_x > 0$ such that

$$r \cdot f_\alpha(a_{(\beta_1,\dots,\beta_m)}) \in V \tag{8}$$

for every r with $|r| < e_x$, every $\alpha \in \Omega$ and every $(\beta_1, \dots, \beta_m) \in \Lambda$. Hence, $r \cdot g_\alpha^m(x) = \sum_{(\beta_1,\dots,\beta_m)\in\Lambda} p_{(\beta_1,\dots,\beta_m)}(x) \cdot (r \cdot f_\alpha(a_{(\beta_1,\dots,\beta_m)}))$. From (8) and the convexity of V, we have

$$r \cdot g_\alpha^m(x) \in V. \tag{9}$$

Let r with $|r| < e_x \wedge 1$ and $\alpha \in \Omega$. Then, by (7), (9) and being circled of V, we have that $r \cdot g_\alpha(x) \in r \cdot (g_\alpha^m(x) + \bar{B}(0; 1/2^{m-1})) \subset V + V \subset W$. This completes the proof that $\{g_\alpha : \alpha \in \Omega\}$ is pointwise bounded.

On the other hand, since $p_{(\beta_1,\dots,\beta_m)}$ is continuous, there exists a neighborhood O' of x in X such that

$$|p_{(\beta_1,\dots,\beta_m)}(y) - p_{(\beta_1,\dots,\beta_m)}(x)| < e_x/|\Lambda| \tag{10}$$

for every $y \in O'$ and every $(\beta_1, \dots, \beta_m) \in \Lambda$, where $|\Lambda|$ denotes the cardinality of Λ. Fix $y \in O \cap O'$ and $\alpha \in \Omega$. Then, it follows that

$$g_\alpha^m(y) - g_\alpha^m(x)$$
$$= \frac{1}{|\Lambda|}\left(\sum_{(\beta_1,\dots,\beta_m)\in\Lambda} |\Lambda|\cdot(p_{(\beta_1,\dots,\beta_m)}(y) - p_{(\beta_1,\dots,\beta_m)}(x))\cdot f_\alpha(a_{(\beta_1,\dots,\beta_m)})\right).$$

Hence, by (8), (10) and the convexity of V, we have

$$g_\alpha^m(y) - g_\alpha^m(x) \in V. \tag{11}$$

Moreover, by (7) and (11), we have that

$$g_\alpha(y) - g_\alpha(x) \in \bar{B}(0; 1/2^{m-1}) + V + \bar{B}(0; 1/2^{m-1}) \subset V + V + V \subset W.$$

This completes the proof that $\{g_\alpha : \alpha \in \Omega\}$ is equicontinuous at x.

Fifth, to prove Φ is continuous with respect to the compact-open topology and the topology of pointwise convergence, it suffices to show the case of the topology of pointwise convergence. For, the topology of pointwise convergence coincides with the compact-open topology on $\mathscr{F}$ and $\Phi(\mathscr{F})$. Since the proof is not difficult, we left it to the reader.

Finally, assume that $\{f_\alpha : \alpha \in \Omega\}$ is pointwise totally bounded. Let $x \in X$ be fixed and W a neighborhood of 0 in Y. Moreover, let V, m and Λ be as in the first part of the proof of the pointwise boundedness of $\{g_\alpha : \alpha \in \Omega\}$. For every $(\beta_1, \dots, \beta_m) \in \Lambda$, let $N_{(\beta_1,\dots,\beta_m)}$ be a finite subset of Y such that $\{f_\alpha(a_{(\beta_1,\dots,\beta_m)}) : \alpha \in \Omega\} \subset N_{(\beta_1,\dots,\beta_m)} + V$. Let

$$N' = \left\{\sum_{(\beta_1,\dots,\beta_m)\in\Lambda} p_{(\beta_1,\dots,\beta_m)}(x)\cdot y_{(\beta_1,\dots,\beta_m)} : y_{(\beta_1,\dots,\beta_m)} \in N_{(\beta_1,\dots,\beta_m)}, (\beta_1, \dots, \beta_m) \in \Lambda\right\}$$

Then, N' is finite. For $\alpha \in \Omega$, we can express $f_\alpha(a_{(\beta_1,\dots,\beta_m)}) = y^\alpha_{(\beta_1,\dots,\beta_m)} + u^\alpha_{(\beta_1,\dots,\beta_m)}$, where $y^\alpha_{(\beta_1,\dots,\beta_m)} \in N_{(\beta_1,\dots,\beta_m)}$ and $u^\alpha_{(\beta_1,\dots,\beta_m)} \in V$. Then, we have

$$g_\alpha^m(x) \in N' + V. \tag{12}$$

Hence, by (7) and (12),

$$g_\alpha(x) \in g_\alpha^m(x) + \bar{B}(0; 1/2^{m-1}) \subset N' + V + V \subset N' + W$$

for every $\alpha \in \Omega$. Hence, we have $\{g_\alpha(x) : \alpha \in \Omega\} \subset N' + W$. It shows that $\{g_\alpha : \alpha \in \Omega\}$ is pointwise totally bounded. This completes the proof. $\square$

We now sketch the outline of an alternative proof of Lemma 3.5 using a Dugundji extender instead of normal covers, which was suggested by the referee of the first version of this paper.

OUTLINE OF THE PROOF OF LEMMA 3.5 (ALTERNATIVE). Let X be a space, A a P-embedded subspace of X and (Y, ρ) a Fréchet space, where ρ is an invariant metric on Y (see [12]). Let $\mathscr{F} = \{f_\alpha : \alpha \in \Omega\}$ be a pointwise bounded equicontinuous subset of $C(A, Y)$. First consider a pseudo-metric $d_{\mathscr{F}}$ on A defined by $d_{\mathscr{F}}(x, x') = \sup_{\alpha \in \Omega}(\rho(f_\alpha(x), f_\alpha(x')) \wedge 1)$, $x, x' \in A$.

CLAIM 1. $d_{\mathscr{F}}$ is continuous.

This follows from the equicontinuity of $\mathscr{F}$.

By the assumption, $d_{\mathscr{F}}$ can be extended to a continuous pseudo-metric d on X. For $x, x' \in X$, define an equivalence relation xRx' by $d(x, x') = 0$. Let X/d be the set of all the equivalence classes defined by R. For classes $[x], [x'] \in X/d$, define $d^*([x], [x']) = d(x, x')$. Then, d^* defines a metric on X/d. Define $q : X \to X/d$ by $q(x) = [x]$. Then, $q : X \to (X/d, d^*)$ is a continuous map onto the metric space $(X/d, d^*)$.

Let $A_d = q(A) \subset X/d$. For every $[x] \in A_d$, choose $a_x \in A$ satisfying that $[x] = [a_x]$. For every $\alpha \in \Omega$, define a function $f_\alpha^* : A_d \to Y$ by $f_\alpha^*([x]) = f_\alpha(a_x)$. Consider the map $\Psi_1 : \mathscr{F} \to C(A_d, Y)$ defined by $\Psi_1(f_\alpha) = f_\alpha^*$, $\alpha \in \Omega$. Then we easily have the following:

CLAIM 2. For every $\varepsilon > 0$ with $\varepsilon < 1$, every $a, a' \in A$ and every $\alpha \in \Omega$,

$$d^*([a], [a']) < \varepsilon \Rightarrow \rho(f_\alpha^*([a]), f_\alpha^*([a'])) < \varepsilon.$$

For later use, we now consider the following conditions $(\mathrm{i})_j$, $(\mathrm{ii})_j$, $(\mathrm{iii})_j$ and $(\mathrm{iv})_j$ on a map $\Psi_j : \mathscr{F}_j \to C(Z_j, Y)$, $\mathscr{F}_j \subset C(X_j, Y)$ and spaces X_j and Z_j, where $j = 1, 2, 3, 4$.

$(\mathrm{i})_j$ $\Psi_j(f)(Z_j)$ is contained in the closed convex hull of $f(X_j)$ for every $f \in \mathscr{F}_j$;

$(\mathrm{ii})_j$ $\Psi_j(\mathscr{F}_j)$ is pointwise bounded equicontinuous;

$(\mathrm{iii})_j$ Ψ_j is continuous when $C(X_j, Y)$ and $C(Z_j, Y)$ carry either one of the compact-open topology, the topology of pointwise convergence and the topology of uniform convergence, where $\mathscr{F}_j$ has the subspace topology of $C(X_j, Y)$;

$(\mathrm{iv})_j$ If, in addition, $\mathscr{F}_j$ is pointwise totally bounded, then $\Psi_j(\mathscr{F}_j)$ is also pointwise totally bounded.

By Claim 2 and the formula $\rho(f_\alpha(a), f_\alpha(a')) = \rho(f_\alpha^*([a]), f_\alpha^*([a']))$ $(a, a' \in A)$, we can show that the map $\Psi_1 : \mathscr{F} \to C(A_d, Y)$, putting $\mathscr{F}_1 = \mathscr{F}$, $X_1 = A$ and $Z_1 = A_d$, satisfies the conditions $(\mathrm{i})_1$, $(\mathrm{ii})_1$, $(\mathrm{iii})_1$ and $(\mathrm{iv})_1$.

Claim 2 shows that $f_\alpha^* : A_d \to Y$ is uniformly continuous for every $\alpha \in \Omega$. Since Y is complete, it follows from [6, Theorem 4.3.17] that f_α^* can be extended to a uniformly continuous function $\widehat{f_\alpha^*} : \overline{A_d}^{X/d} \to Y$ for every $\alpha \in \Omega$.

Now we have the following Claim 3 by Claim 2.

Claim 3. For every $\varepsilon > 0$ with $\varepsilon < 1$, every $[x], [x'] \in \overline{A_d}^{X/d}$ and every $\alpha \in \Omega$,

$$d^*([x], [x']) < \varepsilon \Rightarrow \rho(\widehat{f_\alpha^*}([x]), \widehat{f_\alpha^*}([x'])) < \varepsilon.$$

Consider a map $\Psi_2 : \Psi_1(\mathscr{F}) \to C(\overline{A_d}^{X/d}, Y)$ defined by $\Psi_2(f_\alpha^*) = \widehat{f_\alpha^*}$, $\alpha \in \Omega$. Then, we can show that the map Ψ_2, putting $\mathscr{F}_2 = \Psi_1(\mathscr{F})$, $X_2 = A_d$ and $Z_2 = \overline{A_d}^{X/d}$, satisfies the conditions (i)$_2$, (ii)$_2$, (iii)$_2$ and (iv)$_2$. Indeed, (i)$_2$ holds because of its construction. The statements (ii)$_2$ and (iii)$_2$ seem to be well-known and it is not difficult to prove, and (iv)$_2$ is also easy.

Let $\Psi_3 : C(\overline{A_d}^{X/d}, Y) \to C(X/d, Y)$ be Dugundji's extender. Then, we have that the map Ψ_3, putting $\mathscr{F}_3 = C(\overline{A_d}^{X/d}, Y)$, $X_3 = \overline{A_d}^{X/d}$ and $Z_3 = X/d$, satisfies the conditions (i)$_3$, (ii)$_3$, (iii)$_3$ and (iv)$_3$. Indeed, by the facts in Section 2, we have (ii)$_3$ and (iv)$_3$. Other conditions are obtained from the construction of Dugundji's extender [5].

Finally consider a map $\Psi_4 : C(X/d, Y) \to C(X, Y)$ defined by $\Psi_4(f) = f \circ q$. Then, the map Ψ_4, putting $\mathscr{F}_4 = C(X/d, Y)$, $X_4 = X/d$ and $Z_4 = X$, satisfies the conditions (i)$_4$, (ii)$_4$, (iii)$_4$ and (iv)$_4$.

Now, define a map $\Phi : \mathscr{F} \to C(X, Y)$ by $\Phi = \Psi_4 \circ \Psi_3 \circ \Psi_2 \circ \Psi_1$. Observe that Φ is an extender. By using (i)$_j$, (ii)$_j$, (iii)$_j$ and (iv)$_j$ $(j = 1, \dots, 4)$, Φ satisfies the required conditions (i), (ii), (iii) and the additional condition in Lemma 3.5. This completes the proof. □

For collections $\mathscr{U}$ and $\mathscr{V}$ of subsets of a space X, $\mathscr{V}$ is said to be a *partial refinement* of $\mathscr{U}$ if every element V of $\mathscr{V}$ is contained in some element U of $\mathscr{U}$.

Lemma 3.6. *Let $\{f_\alpha : \alpha \in \Omega\}$ be a pointwise bounded equicontinuous subset of $C(X)$. Then, there exists a σ-discrete cozero-set collection $\mathscr{V}$ of X such that $\mathscr{V}$ is a partial refinement of $\{f_\alpha^{-1}((0, +\infty)) : \alpha \in \Omega\}$ and $\bigcup \mathscr{V} = \bigcup\{f_\alpha^{-1}((0, +\infty)) : \alpha \in \Omega\}$.*

Proof. Denote $\{f_\alpha : \alpha \in \Omega\}$ by $\{f_\alpha : \alpha < \gamma\}$ with some ordinal γ. For every $n \in \mathbf{N}$ and every $\alpha < \gamma$, put

$$U_\alpha^n = f_\alpha^{-1}((3/n, +\infty)) - \left(\sup_{\beta < \alpha} f_\beta\right)^{-1}([1/n, +\infty)).$$

Since $\sup_{\beta<\alpha} f_\beta$ is continuous, U_α^n is a cozero-set of X for every $n \in \mathbf{N}$ and every $\alpha < \gamma$. Fix $n \in \mathbf{N}$. To prove $\{U_\alpha^n : \alpha < \gamma\}$ is discrete in X, let $x \in X$. Since $\{f_\alpha : \alpha < \gamma\}$ is equicontinuous, there exists a neighborhood O of x in X such that $|f_\alpha(x) - f_\alpha(y)| < 1/n$ for every $y \in O$ and every $\alpha < \gamma$. Assume that $O \cap U_\alpha^n \neq \emptyset$, $O \cap U_\beta^n \neq \emptyset$ and $\beta < \alpha$. Let $a \in O \cap U_\alpha^n$ and $b \in O \cap U_\beta^n$. Then, it follows from $b \in U_\beta^n$ that $3/n < f_\beta(b)$. Moreover, it follows from $a \in U_\alpha^n$ that $f_\beta(a) < 1/n$. Hence, we have $2/n < |f_\beta(a) - f_\beta(b)| \leq |f_\beta(a) - f_\beta(x)| + |f_\beta(x) - f_\beta(b)| < 1/n + 1/n = 2/n$, a contradiction. Hence, $\{U_\alpha^n : \alpha < \gamma\}$ is discrete.

It is clear that $U_\alpha^n \subset f_\alpha^{-1}((0,\infty))$ for every $\alpha < \gamma$ and every $n \in \mathbf{N}$. We also have that $\bigcup\{f_\alpha^{-1}((0,+\infty)) : \alpha < \gamma\} = \bigcup\{U_\alpha^n : \alpha < \gamma, n \in \mathbf{N}\}$. This completes the proof. □

PROOF OF THEOREM 3.1. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): Let $\mathscr{U}$ be a normal open cover of A. There exists a locally finite partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A subordinated to $\mathscr{U}$. Since $\{f_\alpha : \alpha \in \Omega\}$ is pointwise bounded equicontinuous, there exists a pointwise bounded equicontinuous subset $\{g_\alpha : \alpha \in \Omega\}$ of $C(X)$ such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$. By Lemma 3.6, there exists a σ-discrete cozero-set collection $\mathscr{V}$ of X such that $\mathscr{V}$ is a partial refinement of $\{g_\alpha^{-1}((0,+\infty)) : \alpha \in \Omega\}$ and

$$A = \bigcup\{f_\alpha^{-1}((0,1]) : \alpha \in \Omega\} \subset \bigcup\{g_\alpha^{-1}((0,+\infty)) : \alpha \in \Omega\} = \bigcup \mathscr{V}.$$

Since A is C-embedded in X, there exists a cozero-set W of X such that $(\bigcup \mathscr{V}) \cup W = X$ and $W \cap A = \emptyset$. Hence, $\mathscr{V} \cup \{W\}$ is a normal open cover of X and $(\mathscr{V} \cup \{W\}) \wedge A$ refines $\mathscr{U}$, this proves that A is P-embedded in X.

(3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4): These follow from Lemma 3.5.

(4) $\Rightarrow$ (3): Obvious. This completes the proof. □

COROLLARY 3.7. *A space X is collectionwise normal if and only if for any closed subspace A of X, every pointwise bounded equicontinuous subset of $C^*(A)$ can be extended to a pointwise bounded equicontinuous subset of $C^*(X)$.*

PROOF. To prove the "if" part, assume that for any closed subspace A of X, every pointwise bounded equicontinuous subset of $C^*(A)$ can be extended to a pointwise bounded equicontinuous subset of $C^*(X)$. Since every closed subspace of X is C^*-embedded in X, it follows that X is normal. A similar proof to that of "(2) $\Rightarrow$ (3)" of Theorem 3.1 shows that A is P-embedded in X for every closed subspace A of X. Hence, X is collectionwise normal.

To prove the "only if" part, use (i) and (ii) in Lemma 3.5. This completes the proof. □

By using the Ascoli's technique (see [14]), we have the following:

THEOREM 3.8. *Let X be a space and A a subspace of X. Assume X and A are Hausdorff k-spaces. Then, the following statements are equivalent:*

(1) *every compact subspace $\mathscr{F}$ of $C_k(A)$ can be extended to a compact subspace $\mathscr{G}$ of $C_k(X)$;*

(2) *for any Fréchet space Y, every compact subspace $\mathscr{F}$ of $C_k(A, Y)$ can be extended to a compact subspace $\mathscr{G}$ of $C_k(X, Y)$;*

(3) *A is P-embedded in X.*

PROOF. (1) ⇒ (3): Assume (1). By Theorem 3.1, it suffices to show that every pointwise bounded equicontinuous subset of $C(A)$ can be extended to a pointwise bounded equicontinuous subset of $C(X)$. To prove this, let $\mathscr{F}$ be a pointwise bounded equicontinuous subset of $C(A)$. Since $\overline{\mathscr{F}}^{C_k(A)}$ is compact subspace of $C_k(A)$, by the assumption, this can be extended to a compact subspace $\mathscr{G}$ of $C_k(X)$. Since X is Hausdorff k, $\mathscr{G}$ is pointwise bounded equicontinuous. Hence, $\{g \in \mathscr{G} : g|A \in \mathscr{F}\}$ is also pointwise bounded equicontinuous. So, (3) holds.

(3) ⇒ (2): Use (iii) in Lemma 3.5.

(2) ⇒ (1): Obvious. This completes the proof. □

COROLLARY 3.9. *For a Hausdorff k-space X, X is collectionwise normal if and only if for any closed subspace A of X, every compact subspace $\mathscr{F}$ of $C_k^*(A)$ can be extended to a compact subspace $\mathscr{G}$ of $C_k^*(X)$.*

PROOF. Let X be a Hausdorff k-space. To prove the "if" part, the similar proof of Theorem 3.8 works by applying Corollary 3.7.

To prove the "only if" part, use (i) and (ii) in Lemma 3.5. This completes the proof. □

In particular, from Corollary 3.9, we have the following:

THEOREM 3.10 (Heath-Lutzer-Zenor [10]). *Let X be a Hausdorff k-space. Assume that for every closed subspace A of X, there exists a continuous extender $e : C_k^*(A) \to C_k^*(X)$. Then, X is collectionwise normal.*

Finally, we give an application of Lemma 3.3. A *Dieudonné complete* space is a space having a complete uniformity. For a Tychonoff space X, γX is the Dieudonné completion of X (see [6]). We have:

THEOREM 3.11. *Let X be a Tychonoff space and Y a Dieudonné complete TV-space. Then, every equicontinuous subset $\{f_\alpha : \alpha \in \Omega\}$ of $C(X, Y)$ can be extended to an equicontinuous subset $\{g_\alpha : \alpha \in \Omega\}$ of $C(\gamma X, Y)$. If in addition $\{f_\alpha : \alpha \in \Omega\}$ is assumed to be pointwise bounded (resp. pointwise totally bounded), then $\{g_\alpha : \alpha \in \Omega\}$ is also pointwise bounded (resp. pointwise totally bounded).*

Theorem 3.11 slightly improves the theorem of Sanchis [15] that: *Let X be a Tychonoff space and Y a Dieudonné complete TV-space. Then, every pointwise totally bounded equicontinuous subset of $C(X, Y)$ can be extended to a pointwise totally bounded equicontinuous subset of $C(\gamma X, Y)$.*

PROOF OF THEOREM 3.11. Let $\{f_\alpha : \alpha \in \Omega\}$ be an equicontinuous subset of $C(X, Y)$. First, notice that for every $\alpha \in \Omega$, f_α can be extended to some $g_\alpha \in C(\gamma X, Y)$. We shall prove that the collection $\{g_\alpha : \alpha \in \Omega\}$ is the required one. For every neighborhood V of 0 in Y and every $x \in X$, let $O_x(V) = \mathrm{Int}_X(\bigcap\{f_\alpha^{-1}(f_\alpha(x) + V) : \alpha \in \Omega\})$ like as in Lemma 3.2. By Lemma 3.3, $\{O_x(V) : x \in X\}$ is a normal open cover of X. Hence, there exists a normal open cover $\mathscr{U}_V$ of γX such that $\mathscr{U}_V \wedge X$ refines $\{O_x(V) : x \in X\}$. We may assume that $\mathscr{U}_V = \{U_x(V) : x \in X\}$ and $U_x(V) \cap X \subset O_x(V)$ for every $x \in X$.

CLAIM. $g_\alpha(U_x(V)) \subset f_\alpha(x) + \bar{V}$ for every $\alpha \in \Omega$ and every $x \in X$.

The proof of Claim is straightforward.

To prove $\{g_\alpha : \alpha \in \Omega\}$ is equicontinuous, let $x \in \gamma X$ and W a neighborhood of 0 in Y. Take a symmetric neighborhood V of 0 in Y with $\bar{V} + \bar{V} \subset W$. Since $\mathscr{U}_V$ is an open cover of γX, there exists $x_0 \in X$ such that $x \in U_{x_0}(V)$. For every $y \in U_{x_0}(V)$ and every $\alpha \in \Omega$, it follows from Claim that

$$g_\alpha(x) - g_\alpha(y) \in (f_\alpha(x_0) + \bar{V}) - (f_\alpha(x_0) + \bar{V}) = \bar{V} + \bar{V} \subset W.$$

Hence, $\{g_\alpha : \alpha \in \Omega\}$ is equicontinuous.

Assume further that $\{f_\alpha : \alpha \in \Omega\}$ is pointwise bounded. Let $x \in \gamma X$ and W a neighborhood of 0 in Y. Let V be a circled neighborhood of 0 in Y with $\bar{V} + \bar{V} \subset W$. Since $\mathscr{U}_V$ is an open cover of γX, there exists $x_0 \in X$ such that $x \in U_{x_0}(V)$. From Claim, we have $g_\alpha(x) \in f_\alpha(x_0) + \bar{V}$ for every $\alpha \in \Omega$. Since

$\{f_\alpha(x_0) : \alpha \in \Omega\}$ is bounded, there exists $e > 0$ such that $r \cdot \{f_\alpha(x_0) : \alpha \in \Omega\} \subset V$ for every $r \in \mathbf{R}$ with $|r| < e$. Hence, for every $r \in \mathbf{R}$ with $|r| < e \wedge 1$, we have

$$r \cdot \{g_\alpha(x) : \alpha \in \Omega\} \subset r \cdot \{f_\alpha(x) + \bar{V} : \alpha \in \Omega\} \subset V + \bar{V} \subset W.$$

This shows that $\{g_\alpha : \alpha \in \Omega\}$ is pointwise bounded.

The case of the pointwise total boundedness is left to the reader. This completes the proof. $\square$

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