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ON A NEW ALGORITHM FOR INHOMOGENEOUS DIOPHANTINE APPROXIMATION

By

Shin-ichi YASUTOMI

Abstract. The inhomogeneous Diophantine approximation algorithm of Nishioka et al., $(X, T_2, c(x), d(x, y))$, was shown by Komatsu to be efficient for inhomogeneous Diophantine approximation, but lacks a properly founded natural extension and not all periodic points about the approximation are determined. A new algorithm, $(X, T, a(x), b(x, y))$, is proposed in this paper as a modification of $(X, T_2, c(x), d(x, y))$, and is shown to be efficient for inhomogeneous Diophantine approximation similar to $(X, T_2, c(x), d(x, y))$ but also to have a natural extension, which allows all periodic points about $(X, T, a(x), b(x, y))$ to be determined and gives $\liminf_{q\to\infty} q \,|\, |q\alpha - \beta - p|$ for the periodic points (α, β).

1. Introduction

It is well known that connections exist between the continued fractions algorithm and the minimization of $|q\alpha - p|$, where q is an natural number, p is an integer, and α is an irrational number. The problem of minimizing $|q\alpha - \beta - p|$, where β is a real number, is called the inhomogeneous Diophantine approximation. This problem has been considered by many authors (e.g., [12, 18, 13, 6, 7, 1, 2, 3, 4, 8, 21, 10, 11, 5, 14, 16, 17], and detailed information can be obtained by a review of the literature. Many algorithms related to the problem have been used. For example, Ito and Kasahara [10] defined the following algorithm, which was implicitly introduced by Morimoto [18]. Let $Z = \{(x, y) \mid 0 \leqq y < 1, -y < x < -y + 1\}$, as shown in Fig. 1.

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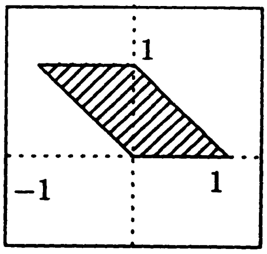

Figure 1.1 Figure of Z

Then for $(x, y) \in Z$:

$$a'(x, y) = \left\lfloor \frac{1-y}{x} \right\rfloor - \left\lfloor \frac{-y}{x} \right\rfloor, \quad b'(x, y) = -\left\lfloor \frac{-y}{x} \right\rfloor.$$

The algorithm T_1 is then defined by the following transformation on Z for $(x, y) \in Z$.

$$T_1(x, y) = \left(\frac{1}{x} - a'(x, y), b'(x, y) - \frac{y}{x}\right).$$

This algorithm $(Z, T_1, a'(x, y), b'(x, y))$ gives the best solution to the inhomogeneous Diophantine approximation. Constructing the natural extension of the algorithm, they determined all the periodic points about the algorithm. Ito [9] was the first to subsequently find that a certain natural extension of the Diophantine algorithm is useful for investigating the algorithm. Komatsu studied the following algorithm, which was introduced by Nishioka et al. [19]. With $X = [0, 1]^2$, T_2 is defined as the following transformation on X for $(x, y) \in X$.

$$T_2(x, y) = \left(\frac{1}{x} - c(x), d(x, y) - \frac{y}{x}\right),$$

where $c(x) = \lfloor \frac{1}{x} \rfloor$ and $d(x, y) = \lceil \frac{y}{x} \rceil$. Using this algorithm, $(X, T_2, c(x), d(x, y))$, Komatsu [14] obtained $\liminf_{q \to \infty} q|q\alpha - \beta - p|$ in some cases.

In this paper, an algorithm $(X, T, a(x), b(x, y))$ is introduces as a modification of $(X, T_2, c(x), d(x, y))$. The new algorithm also gives the best solution for the inhomogeneous Diophantine approximation as does $(X, T_2, c(x), d(x, y))$. However, a natural extension is constructed for $(X, T, a(x), b(x, y))$, which has not been done for $(X, T_2, c(x), d(x, y))$. Using the natural extension of $(X, T, a(x), b(x, y))$, all purely periodic points about the algorithm are determined, and for the purely periodic point (α, β), a relation between $\liminf_{q \to \infty} q \mid |q\alpha - \beta - p|$ and the natural extension of $(X, T, a(x), b(x, y))$ is obtained. Although all eventually periodic points have been determined by Komatsu [15], all purely periodic points have not.

2. Definition and Some Properties of Algorithm

We denote **R**, **Q** and **Z** the set of all real numbers, the set of all rational numbers and the set of all integers respectively. For $(x, y) \in X$ with $x \neq 0$ we define $a(x)$ by $\lfloor \frac{1}{x} \rfloor$ and we define $b(x, y)$ by

$$b(x, y) = \begin{cases} 1 & \text{if } y = 0, \\ \lceil \frac{y}{x} \rceil & \text{if } y > 0 \text{ and } \lfloor \frac{1}{x} \rfloor > \lfloor \frac{y}{x} \rfloor \text{ or } \lfloor \frac{1}{x} \rfloor = \frac{y}{x}, \\ 0 & \text{if } \lfloor \frac{1}{x} \rfloor = \lfloor \frac{y}{x} \rfloor \text{ and } \lfloor \frac{1}{x} \rfloor \neq \frac{y}{x}. \end{cases}$$

We define a transformation T as follows; for $(x, y) \in X$ if $x > 0$, then

$$T(x, y) = \begin{cases} \left(\dfrac{1}{x} - a(x), b(x, y) - \dfrac{y}{x} \right) & \text{if } b(x, y) > 0, \\ \left(\dfrac{1}{x} - a(x), \dfrac{1}{x} - \dfrac{y}{x} \right) & \text{if } b(x, y) = 0, \end{cases}$$

and if $x = 0$, then $T(x, y) = (x, y)$.

We define $a_n(x) = a(T^{n-1}(x, y))$, $b_n(x, y) = b(T^{n-1}(x, y))$ and $(x_n, y_n) = T^{n-1}(x, y)$. It is not difficult to see that if $x \notin \mathbf{Q}$, then for any integer $n > 0$ $a_n(x)$ and $b_n(x, y)$ are defined.

Lemma 2.1 follows from the continued fraction theory.

LEMMA 2.1. *Let $(x, y) \in X$ and $x \notin \mathbf{Q}$. Then, for each integer $n > 0$*

(1) $q_n(x)x - p_n(x) = (-1)^n x_1 \cdots x_{n+1} = \frac{(-1)^n}{q_{n+1}(x) + x_{n+2} q_n(x)}$,

(2)

$$|q_{n-1}(x)x - p_{n-1}(x)| = a_{n+1}(x, y)|q_n(x)x - p_n(x, y)| + |q_{n+1}(x, y)x - p_{n+1}(x, y)|,$$

(3) $|q_n(x)x - p_n(x, y)| > |q_{n+1}(x, y)x - p_{n+1}(x, y)|$,

(4) *for any integer j, k with $q_n(x) < j < q_{n+1}(x, y)$, $|q_n(x)x - p_n(x, y)| < |jx - k|$,*

where $\{p_n(x)\}_{-1 \leq n}$, $\{q_n(x)\}_{-1 \leq n}$ are defined by

$$p_{-1}(x) = 1, \quad p_0(x) = 0,$$

$$q_{-1}(x) = 0, \quad q_0(x) = 1,$$

for $n \geq 1$

$$p_n(x) = a_n(x)p_{n-1}(x) + p_{n-2}(x),$$

$$q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x).$$

LEMMA 2.2. *Let* $(x, y) \in X$. *Then,*
(1) $a_n(x) > 0$ *and* $a_n(x) \geq b_n(x, y) \geq 0$,
(2) *if* $b_n(x, y) = 0$, *then* $b_{n+1}(x, y) = 1$.

PROOF. The proof of (1) is easy. Let us prove (2). We suppose that $b_n(x, y) = 0$. Then, we see that $va(x_n) = \left\lfloor \frac{y_n}{x_n} \right\rfloor$ and $a(x_n) < \frac{y_n}{x_n}$. Since $x_{n+1} = \frac{1}{x_n} - a(x_n)$ and $y_{n+1} = \frac{1}{x_n} - \frac{y_n}{x_n}$, we have $x_{n+1} > y_{n+1}$. Thus, we obtain $b(x_{n+1}, y_{n+1}) = 1$. □

Let $(x, y) \in X$ and $x \notin \mathbf{Q}$. Let us define integers $A_n(x, y)$, $B_n(x, y)$ as follows:

$$A_1(x, y) = \begin{cases} 0 & \text{if } b(x, y) > 0, \\ -1 & \text{if } b(x, y) = 0. \end{cases} \qquad B_1(x, y) = \begin{cases} b_1(x, y) & \text{if } b(x, y) > 0, \\ 0 & \text{if } b(x, y) = 0, \end{cases}$$

For $n > 1$

$$A_n(x, y) = \begin{cases} A_{n-1}(x, y) + b_n(x, y)p_{n-1}(x) & \text{if } b(x, y) > 0, \\ A_{n-1}(x, y) - p_{n-2}(x) & \text{if } b(x, y) = 0, \end{cases}$$

$$B_n(x, y) = \begin{cases} B_{n-1}(x, y) + b_n(x, y)q_{n-1}(x) & \text{if } b(x, y) > 0, \\ B_{n-1}(x, y) - q_{n-2}(x) & \text{if } b(x, y) = 0. \end{cases}$$

We remark that $\{B_n(x, y)\}_{n=1,2,\ldots}$ and $\{A_n(x, y)\}_{n=1,2,\ldots}$ are not increasing sequences generally as $n \to \infty$.

LEMMA 2.3. *Let* $(x, y) \in X$ *and* $x \notin \mathbf{Q}$. *Then, for any* $n > 0$

$$y = B_n(x, y)x - A_n(x, y) + (-1)^n y_{n+1} x_1 \cdots x_n. \tag{1}$$

PROOF. We prove the lemma by the induction on n. Let $n = 1$. First, let $b_1(x, y) > 0$. Then, we see $y_2 = b_1(x, y) - \frac{y_1}{x_1}$. Therefore, we have $y_1 = b_1(x, y)x_1 - y_2 x_1 = B_1(x, y)x - A_1(x, y) - y_2 x_1$. Next, let $b_1(x, y) = 0$. Then, we see $y_2 = \frac{1}{x_1} - \frac{y_1}{x_1}$. Therefore, we have $y_1 = 1 - y_2 x_1 = B_1(x, y)x - A_1(x, y) - y_2 x_1$. Hence, (1) holds for $n = 1$. Secondly, we suppose that (1) holds for $n = k$, that is, $y = B_k(x, y)x - A_k(x, y) + (-1)^{k+1} y_{k+1} x_1 \cdots x_k$. Let $b_{k+1}(x, y) > 0$. Then, we have $y_{k+2} = b_{k+1}(x, y) - \frac{y_{k+1}}{x_{k+1}}$, which implies $y_{k+1} = b_{k+1}(x, y)x_{k+1} - x_{k+1} y_{k+2}$. Therefore, using $x_1 \cdots x_{k+1} = (-1)^k (q_k x - p_k)$, we see

$$\begin{aligned} y &= B_k(x, y)x - A_k(x, y) + (-1)^k y_{k+1} x_1 \cdots x_k, \\ &= B_k(x, y)x - A_k(x, y) + (-1)^k b_{k+1}(x, y) x_1 \cdots x_{k+1} (-1)^{k+1} y_{k+1} x_1 \cdots x_{k+1}, \\ &= B_{k+1}(x, y)x - A_{k+1}(x, y) + (-1)^{k+1} y_{k+2} x_1 \cdots x_{k+1}. \end{aligned}$$

Let $b_{k+1}(x, y) = 0$. Then, we have $y_{k+2} = \frac{1}{x_{k+1}} - \frac{y_{k+1}}{x_{k+1}}$, which implies $y_{k+1} = 1 - x_{k+1}y_{k+2}$. Using $x_1 \cdots x_k = (-1)^{k+1}(q_{k-1}x - p_{k-1})$, we have

$$\begin{aligned} y &= B_k(x, y)x - A_k(x, y) + (-1)^k y_{k+1}x_1 \cdots x_k, \\ &= B_k(x, y)x - A_k(x, y) + (-1)^k x_1 \cdots x_k + (-1)^{k+1} y_{k+2}x_1 \cdots x_{k+1}, \\ &= B_{k+1}(x, y)x - A_{k+1}(x, y) + (-1)^{k+1} y_{k+2}x_1 \cdots x_{k+1}. \end{aligned}$$

Therefore, (1) holds for $n = k + 1$. Thus, we have Lemma. □

LEMMA 2.4. *Let* $(x, y) \in X$ *and* $x \notin \mathbf{Q}$. *Then,* $\lim_{n\to\infty}(B_n(x, y)x - A_n(x, y)) = y$.

PROOF. By Lemma 2.3 $|y - B_n(x, y)x + A_n(x, y)| = y_{n+1}x_1 \cdots x_n$. By Lemma 2.1 we have $x_1 \cdots x_n = |q_{n-1}x - p_{n-1}| < \frac{1}{q_n}$. Thus, we have Lemma. □

We define $\Psi = \{(x, y) \in \mathbf{R}^2 \mid x \notin \mathbf{Q}$ and $y \neq mx + n$ for any $m, n \in \mathbf{Z}\}$.

LEMMA 2.5. *Let* $(x, y), (z, w) \in X$ *and* $x, z \notin \mathbf{Q}$. *If* $a_n(x) = a_n(z)$ *and* $b_n(x, y) = b_n(z, w)$, *for any integer* $n > 0$, *then* $(x, y) = (z, w)$.

PROOF. By continued fraction theory we obtain $x = z$. From Lemma 2.4 we have $y = w$. □

LEMMA 2.6. *Let* $(x, y) \in X \cap \Psi$. *Then, if* $b_n(x, y) = 0$ *for some integer* $n > 0$, *then there exists an integer* $k > 0$ *such that* $b_{n+2k}(x, y) > 0$.

PROOF. We suppose that there exists an integer m such that for any $k \geq 0$ $b_{m+2k}(x, y) = 0$. Then, from Lemma 2.2 we have $b_{m+2k+1}(x, y) = 1$ for any $k \geq 0$. Let $(u, v) = T^{m-1}(x, y)$. Then, $b_{2k}(u, v) = 0$ and $b_{2k+1}(u, v) = 1$ for any $k \geq 0$. We see easily that $b_n(u, 1) = b_n(u, v)$ for any integer $n \geq 1$. From Lemma 2.5 we have $v = 1$. Then, we see $(x, y) \notin \Psi$. But it is a contradiction. Therefore, we have Lemma. □

LEMMA 2.7. *Let* $(x, y) \in X \cap \Psi$. *Then, if* $a_n(x) = b_n(x, y)$ *for some integer* $n > 0$, *then there exists an integer* $k > n$ *such that* $a_k(x) \neq b_k(x, y)$.

PROOF. We suppose that there exists an integer m such that for any $k \geq m$ $a_k(x) = b_k(x, y)$. Let $(u, v) = T^{m-1}(x, y)$. It is not difficult to see that $b_j(u, 1 - u) = b_j(u, v)$ for any integer $j \geq 1$. From Lemma 2.5 we have $v = 1 - u$.

Then, by using the equation $(u, v) = T^{m-1}(x, y)$ we see easily $(x, y) \notin \Psi$. But it is a contradiction. Therefore, we have Lemma. □

LEMMA 2.8. *Let $(x, y) \in X$ and $x \notin \mathbf{Q}$. We suppose that there exist integers e, f such that $y = ex + f$. If $e \geq 0$, then there exists an integer $n \geq 0$ such that $y_n = 0$. If $e < 0$, then there exists an integer $n \geq 0$ such that $y_n = 1 - x_n$.*

PROOF. Let $e \geq 0$. Since $0 \leq ex + f \leq 1$, we see that $-e < f \leq 0$ for $e > 0$ and $f = 0, 1$ for $e = 0$ respectively. If $b_1(x, y) > 0$, then we have

$$y_2 = b_1(x, y) - \frac{y}{x} = -f\left(\frac{1}{x} - a_1(x)\right) - fa_1(x) + b_1(x, y) - e$$

$$= -fx_2 - fa_1(x) + b_1(x, y) - e.$$

If $b_1(x, y) = 0$, then we have $y_2 = \frac{1}{x} - \frac{y}{x} = (1 - f)(\frac{1}{x} - a_1(x)) + (1 - f)a_1(x) - e$. Therefore, by the induction for each integer $n > 0$ there exists integers r_n and s_n such that $y_n = r_n x_n + s_n$, $r_n \geq 0$ and $r_n \geq r_{n+1}$ for $r_n > 0$. We see also that if $r_n > 0$ and $b_1(x, y) > 0$, then $r_n > r_{n+1}$. Since from Lemma 2.2 we see $b_n(x, y) > 0$ for infinitely many n, there exists a integer $m > 0$ such that $r_m = 0$. Therefore, $y_m = 0$ or $y_m = 1$. If $y_m = 1$, then we have $y_{m+1} = 0$. Thus, we have Lemma.

Let $e < 0$. Since $0 \leq ex + f \leq 1$, we see that $0 < f \leq |e|$. We suppose that $b_1(x, y) > 0$. Then, we have $y_2 = -fx_2 - fa_1(x) + b_1(x, y) - e$. We see easily that if $f = -e = 1$, then we have $-fa_1(x) + b_1(x, y) - e = 1$ and if $f = -e > 1$, then we have $-fa_1(x) + b_1(x, y) - e < f$. Next, we suppose that $b_1(x, y) = 0$. Since the fact that $f = 1$ implies $b_1(x, y) > 0$, we see $f > 1$. Then, $y_2 = (1 - f) \cdot (\frac{1}{x} - a_1(x)) + (1 - f)a_1(x) - e$. Therefore, by the induction we see that for each integer $n > 0$ there exists integers r_n and s_n such that $y_n = r_n x_n + s_n$, $r_n < 0$ and $|r_n| \geq |r_{n+1}|$. We see also that if $|r_n| = |r_{n+1}|$ and $|r_n > | > 1$, then $|r_{n+1}| > |r_{n+2}|$. Therefore, there exists an integer $m > 0$ such that $r_m = -1$ and $s_m = 1$. □

LEMMA 2.9. *Let $(x, y) \in X$, $x \notin \mathbf{Q}$ and $(x, y) \notin \Psi$. Then, following* (1) *or* (2) *holds*:

(1) *there exists integer $m > 0$ such that for any integer $k \geq 0$ $b_{m+2k}(x, y) = 0$,*

(2) *there exists integer $m > 0$ such that for any integer $n \geq m$ $a_n(x) = b_n(x, y)$.*

PROOF. From Lemma 2.8 there exists an integer m such that $y_m = 0$ or $y_m = 1 - x_m$. We suppose $y_m = 0$. Then, we see that for each integer $k \geq 0$ $b_{m+1+2k}(x, y) = 0$. Next, we suppose $y_m = 1 - x_m$. Then, we see that for each integer $n \geq m$ $a_n(x) = b_n(x, y)$. □

Lemma 2.10. *Let $\{a_n\}_{n=1,2,\dots}$ and $\{b_n\}_{n=1,2,\dots}$ be integral sequences such that for any integer $n > 0$*

1. *$a_n > 0$ and $a_n \geq b_n \geq 0$,*
2. *if $b_n = 0$, then $b_{n+1} = 1$,*
3. *if $b_n = 0$, then there exists an integer $k > 0$ such that $b_{n+2k} > 0$,*
4. *if $a_n = b_n$, then there exists an integer $k > 0$ such that $a_{n+k} \neq b_{n+k}$.*

Then, there exists $(x, y) \in X \cap \Psi$ such that $a_n = a_n(x)$ and $b_n = b_n(x, y)$.

Proof. We define $\Delta_{m,n}$ for integers m and n with $m > 0$ and $m \geq n \geq 0$ as follows:

$$\pi_{m,n} = \begin{cases} \left\{(x, y) \in [0,1]^2 \,\middle|\, \frac{1}{m+1} \leq x \leq \frac{1}{m}, (n-1)x \leq y \leq nx\right\} & \text{if } n \geq 1, \\ \left\{(x, y) \in [0,1]^2 \,\middle|\, \frac{1}{m+1} \leq x \leq \frac{1}{m}, y \geq mx\right\} & \text{if } m \geq n \text{ and } n = 0. \end{cases}$$

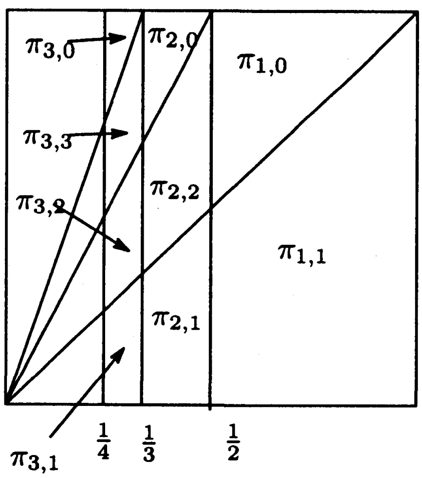

Figure 2.1

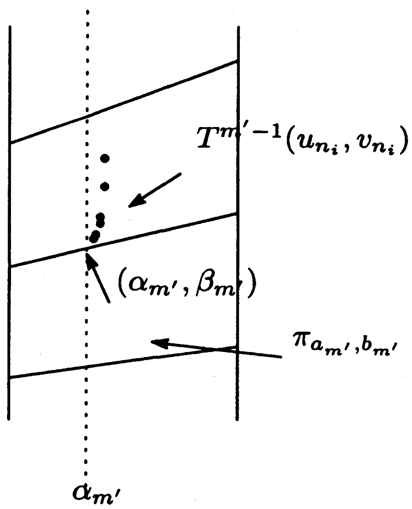

Figure 2.2

We define transformation $T_{(a,b)}$ on $\mathbf{R}^2$ for integers a, b with $a > 0$ and $a \geq b \geq 0$ as follows:

$$T_{(a,b)}(x, y) = \begin{cases} \left(\dfrac{1}{x} - a, b - \dfrac{y}{x}\right) & \text{if } b > 0, \\ \left(\dfrac{1}{x} - a, \dfrac{1}{x} - \dfrac{y}{x}\right) & \text{if } b = 0. \end{cases}$$

Similarly, we define transformation $F_{(a,b)}$ on $\mathbf{R}^2$ for integers a, b with $a > 0$ and $a \geq b \geq 0$ as follows:

$$F_{(a,b)}(x,y) = \begin{cases} \left(\dfrac{1}{x+a}, \dfrac{b-y}{x+a}\right) & \text{if } b > 0, \\ \left(\dfrac{1}{x+a}, 1 - \dfrac{y}{x+a}\right) & \text{if } b = 0. \end{cases}$$

We can easily check $F_{(a,b)} \circ T_{(a,b)} = T_{(a,b)} \circ F_{(a,b)} =$ identity map.

We define $Y = \{(x,y) \in X \mid y \le x\}$. Then, we see that if $b > 0$, then $\pi_{a,b} = F_{(a,b)}(X)$ and $F_{(a,b)} : X \to \pi_{a,b}$ is bijective and if $b = 0$, then $\pi_{a,b} = F_{(a,b)}(Y)$ and $F_{(a,b)} : Y \to \pi_{a,b}$ is bijective. Noting that $F_{(a,1)}(X) \subset Y$, we see that if $b_n > 0$, then $F_{(a_1,b_1)} \cdots F_{(a_{n-1},b_{n-1})} F_{(a_n,b_n)} X$ is included in X and it become a quadrangle with inner points. Similarly, we get that if $b_n = 0$, then $F_{(a_1,b_1)} \cdots F_{(a_{n-1},b_{n-1})} F_{(a_n,b_n)} Y$ is included in X and it become a triangle with inner points. If $b_n > 0$, let (u_n, v_n) be an inner point in $F_{(a_1,b_1)} \cdots F_{(a_{n-1},b_{n-1})} F_{(a_n,b_n)} X$. If $b_n = 0$, let (u_n, v_n) be an inner point in $F_{(a_1,b_1)} \cdots F_{(a_{n-1},b_{n-1})} F_{(a_n,b_n)} Y$. It is not difficult to see that $a_k(u_n) = a_k$ and $b_k(u_n, v_n) = b_k$ for $k = 1, 2, \ldots, n$. Since X is compact, there exist an increasing integral sequence $\{n_i\}$ and $(\alpha, \beta) \in X$ such that $(u_{n_i}, v_{n_i}) \to (\alpha, \beta)$ as $i \to \infty$. Let $(\alpha_n, \beta_n) = T^{n-1}(\alpha, \beta)$. By continued fraction theory $a_k(\alpha) = a_k$ for any integer $k > 0$. We suppose that there exists an integer $m > 0$ such that $b_m(\alpha, \beta) \ne b_m$. Let $m' > 0$ be an integer such that $b'_m(\alpha, \beta) \ne b'_m$. And for any $0 < k < m'$ $b_k(\alpha, \beta) = b_k$. Then, we have $T^{m'-1}(u_{n_i}, v_{n_i}) \to (\alpha_{m'}, \beta_{m'})$ as $i \to \infty$. On the other hand, we see that for large i $T^{m'-1}(u_{n_i}, v_{n_i}) \in \pi_{a_{m'}, b_{m'}}$. Therefore, $(\alpha_{m'}, \beta_{m'})$ is in the boundary set of $\pi_{a_{m'}, b_{m'}}$. Therefore, we see easily that $b(\alpha_{m'}, \beta_{m'})\alpha_{m'} = \beta_{m'}$ and $b(\alpha_{m'}, \beta_{m'}) \ne 0$ (see Figure 2.2). Further more, if $b(\alpha_{m'}, \beta_{m'}) < a(\alpha_{m'}, \beta_{m'})$, then we have $b(\alpha_{m'}, \beta_{m'}) + 1 = b_{m'}$ and if $b(\alpha_{m'}, \beta_{m'}) = a(\alpha_{m'}, \beta_{m'})$, then we have $b_{m'} = 0$. First, we suppose that $b(\alpha_{m'}, \beta_{m'}) + 1 = b_{m'}$. Since $T^{m'-1}(u_{n_i}, v_{n_i}) \to (\alpha_{m'}, b(\alpha_{m'}, \beta_{m'})\alpha_{m'})$, we obtain $T^{m'}(u_{n_i}, v_{n_i}) \to (\alpha_{m'+1}, 1)$ as $i \to \infty$. Then, we have $b_{m'+1} = 0$. By the induction we see $b_{m'+1+j} = 0$ for any even $j > 0$ and $b_{m'+1+j} = 1$ for any odd $j > 0$. But it contradicts the condition of $\{b_n\}_{n=1,2,\ldots}$. Secondly, we suppose that $b_{m'} = 0$. Since $T^{m'-1}(u_{n_i}, v_{n_i}) \to (\alpha_{m'}, a_{m'}\alpha_{m'})$ as $i \to \infty$, we see that $T^{m'}(u_{n_i}, v_{n_i}) \to (\alpha_{m'+1}, \alpha_{m'+1})$ and $b_{m'+1} = 1$. Then, we see easily that $T^{m'+1}(u_{n_i}, v_{n_i}) \to (\alpha_{m'+2}, 0)$ as $i \to \infty$. By the induction we see that $b_{m'+2+j} = 1$ for any even $j > 0$ and $b_{m'+2+j} = 0$ for any odd $j > 0$. But it contradicts the condition of $\{b_n\}_{n=1,2,\ldots}$. Therefore, $b_n(\alpha, \beta) = b_n$ for any integer $n > 0$. From Lemma 2.9 we see $(\alpha, \beta) \in \Psi$. Thus, we have Lemma. □

LEMMA 2.11. *Let* $(x,y) \in X$ *and* $x \notin \mathbf{Q}$. *Then,*

(1) $B_n(x,y) \ge 0$ *for any* $n > 0$ *and* $A_n(x,y) \ge 0$ *for any* $n > 1$,

(2) $\limsup_{n\to\infty} B_n(x,y) = \infty$ *and* $\limsup_{n\to\infty} A_n(x,y) = \infty$,
(3) *if* $(x,y) \in \Psi$, *then* $\lim_{n\to\infty} B_n(x,y) = \infty$ *and* $\lim_{n\to\infty} A_n(x,y) = \infty$.

PROOF OF (1). We suppose that $B_n(x,y) < 0$ for some integer $n > 0$. Without loss of generality we suppose that $B_j(x,y) \ge 0$ for any integer $0 < j < n$. $B_1(x,y) \ge 0$ implies $n > 1$. From the fact that $B_{n-1}(x,y) \ge 0$ and $B_n(x,y) < 0$ we see $b_n(x,y) = 0$. Then, we have $B_n(x,y) = B_{n-1}(x,y) - q_{n-2}(x)$. By Lemma 2.2 we have $b_{n-1}(x,y) > 0$. If $n-1 > 1$, then we have $B_{n-1}(x,y) - q_{n-2}(x) = B_{n-2}(x,y) + (b_{n-1}(x,y) - 1)q_{n-2}(x) \ge 0$. But it is a contradiction. If $n-1 = 1$, then we have $B_{n-1}(x,y) - q_{n-2}(x) = b_1(x,y) - 1 \ge 0$. But it is a contradiction. Similarly, we see $A_n(x,y) \ge 0$ for any $n > 1$.

PROOF OF (2). First, we are proving that $B_{n+2}(x,y) \ge B_n(x,y)$ for any $n \ge 1$ and equation holds iff $b_{n+1}(x,y) = 1$ and $b_{n+2}(x,y) = 0$. If $b_{n+1}(x,y) > 0$ and $b_{n+2}(x,y) > 0$, then the proof is easy. We suppose that $b_{n+1}(x,y) = 0$ and $b_{n+2}(x,y) = 1$. Then, we have $B_{n+1}(x,y) = B_n(x,y) - q_{n-1}(x)$ and $B_{n+2}(x,y) = B_{n+1}(x,y) + b_{n+2}(x,y)q_{n+1}(x,y)$. Therefore, we have $B_{n+2}(x,y) > B_n(x,y)$. Next, we suppose that $b_{n+1}(x,y) > 0$ and $b_{n+2}(x,y) = 0$. Then, we have $B_{n+1}(x,y) = B_n(x,y) + b_{n+1}(x,y)q_n(x)$ and $B_{n+2}(x,y) = B_{n+1}(x,y) - q_n(x)$. Therefore, we see $B_{n+2}(x,y) - B_n(x,y) = (b_{n+1}(x,y) - 1)q_n(x)$, which implies that $B_{n+2}(x,y) \ge B_n(x,y)$ and the equation holds iff $b_{n+1}(x,y) = 1$. Therefore, we see that $\lim_{n\to\infty} B_{2n}(x,y) < \infty$ iff there exists some integer $m > 0$ such that for any $n > m$ $b_{2n}(x,y) = 0$ and $b_{2n-1}(x,y) = 1$. We suppose that for some integer $m > 0$ for any $n > m$ $b_{2n}(x,y) = 0$ and $b_{2n-1}(x,y) = 1$. Then, we obtain $\lim_{n\to\infty} B_{2n+1}(x,y) = \infty$. Thus we have the proof of (2).

PROOF OF (3). From the proof of (2) we see that $\lim_{n\to\infty} B_{2n}(x,y) < \infty$ iff there exists some integer $m > 0$ such that for any $n > m$ $b_{2n}(x,y) = 1$ and $b_{2n-1}(x,y) = 0$. By Lemma 2.6 we see that $\lim_{n\to\infty} B_{2n}(x,y) = \infty$. Similarly, we have $\lim_{n\to\infty} B_{2n+1}(x,y) = \infty$. Thus, we have $\lim_{n\to\infty} B_n(x,y) = \infty$. Similarly, we have $\lim_{n\to\infty} A_n(x,y) = \infty$. □

LEMMA 2.12. *Let* $(x,y) \in X \cap \Psi$. *For any integer* $n \ge 1$, $|B_n(x,y)x - A_n(x,y) - y| \ge |B_{n+2}(x,y)x - A_{n+2}(x,y) - y|$. *The equation holds if and only if* $b_{n+2}(x,y) = 0$ *and* $b_{n+1}(x,y) = 1$ $(B_n(x,y) = B_{n+2}(x,y))$.

PROOF. First, we suppose that $b_{n+1}(x,y) \ge 1$. We also suppose that n is odd. From Lemma 2.1 and Lemma 2.3, we have

$$B_{n+1}(x,y)x - A_{n+1}(x,y) < y < B_{n+1}(x,y)x - A_{n+1}(x,y) - (q_n(x)x - p_n(x))$$
$$\le B_n(x,y)x - A_n(x,y). \tag{2}$$

We suppose $b_{n+2}(x,y)=0$. Then, since $B_{n+2}(x,y)x - A_{n+2}(x,y) = B_{n+1}(x,y)x - A_{n+1}(x,y) - (q_n(x)x - p_n(x))$, by (2) we get $y < B_{n+2}(x,y)x - A_{n+2}(x,y) \leq B_n(x,y)x - A_n(x,y)$, which follows the lemma. We remark that $B_{n+1}(x,y)x - A_{n+1}(x,y) - (q_n(x)x - p_n(x)) = B_n(x,y)x - A_n(x,y)$ if and only if $b_{n+1}(x,y)=1$. We suppose $b_{n+2}(x,y)>0$. Then, from Lemma 2.1 and Lemma 2.3, we have $0 < b_{n+2}(x,y)(q_{n+1}(x)x - p_{n+1}(x)) < -(q_n(x)x - p_n(x))$. Therefore, we get

$$\begin{aligned} B_{n+2}(x,y)x - A_{n+2}(x,y) &< B_{n+1}(x,y)x - A_{n+1}(x,y) - (q_n(x)x - p_n(x)) \\ &\leq B_n(x,y)x - A_n(x,y), \end{aligned}$$

which implies Lemma. We can prove similarly in the case of even n. Next, we suppose that $b_{n+1}(x,y)=0$. Then, from Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} B_{n+1}(x,y)x - A_{n+1}(x,y) < y &< B_{n+1}(x,y)x - A_{n+1}(x,y) + (q_{n-1}(x)x - p_{n-1}(x)) \\ &= B_n(x,y)x - A_n(x,y). \end{aligned} \tag{3}$$

Using $b_{n+2}(x,y)=1$, we get $B_{n+2}(x,y)x - A_{n+2}(x,y) = B_{n+1}(x,y)x - A_{n+1}(x,y) + (q_{n+1}(x)x - p_{n+1}(x)) < B_n(x,y)x - A_n(x,y)$, which implies Lemma. We can prove similarly in the case of even n. □

LEMMA 2.13. *Let $(x,y) \in X \cap \Psi$. If $n>0$ is odd, then $B_n(x,y)x - A_n(x,y) - y > 0$ and for any integers m, j with $0<m<B_n(x,y)$, if $mx - j - y > 0$, then*

$$B_n(x,y)x - A_n(x,y) - y < mx - j - y.$$

If $n>0$ is even, then $B_n(x,y)x - A_n(x,y) - y < 0$ and for any integers m, j with $0<m<B_n(x,y)$, if $mx - y - j < 0$, then

$$B_n(x,y)x - A_n(x,y) - y > mx - y - j.$$

PROOF. We are proving the lemma by using the induction on n. Let $n=1$. From Lemma 2.3 we have $B_1(x,y)x - A_1(x,y) - y = x_1y_2 > 0$. We suppose that there exist integers m, k with $0<m<B_1(x,y)$ such that $mx - j - y > 0$ and $B_1(x,y)x - A_1(x,y) - y \geq mx - j - y$. Let $b_1(x,y)=0$. Then, from the fact $B_1(x,y)=0$ we have a contradiction. Let $b_1(x,y)>0$. Then, we have $B_1(x,y) = b_1(x,y)$ and $A_1(x,y)=0$. We see that $mx - y = B_1(x,y)x - y + (m - B_1(x,y))x = x_1y_2 + (m - B_1(x,y))x < 0$. Therefore, $mx - j - y > 0$ implies $j<0$. On the other hand, we have $B_1(x,y)x - mx = y + x_1y_2 - mx < 1$. By the assumption, we see $0 < B_1(x,y)x - y - (mx - j - y) = B_1(x,y)x - mx + j$. On the other hand, $B_1(x,y)x - mx < 1$ and $j<0$ implies $B_1(x,y)x - mx + j < 0$. This is a contradiction. Thus we have the proof for $n=1$. We suppose that the lemma holds for any n with $1 \le n \le k$. Let $n = k+1$. We suppose that $k+1$ is odd. From Lemma 2.3 we have $B_{k+1}(x,y)x - A_{k+1}(x,y) - y > 0$. We suppose that there exist integers m, j with $0 < m < B_{k+1}(x,y)$ such that $B_{k+1}(x,y)x - A_{k+1}(x,y) - y > mx - j - y > 0$. We suppose $b_{k+1}(x,y) > 0$. First, we suppose $m \ge B_k(x,y)$. Since $B_{k+1}(x,y) - m \le B_{k+1}(x,y) - B_k(x,y) = b_{k+1}(x,y)q_k(x) < q_{k+1}(x)$, from Lemma 2.1 we obtain $|(B_{k+1}(x,y) - m)x - A_{k+1}(x,y) + j| \ge |q_k(x)x - p_k(x)|$. On the other hand, by using Lemma 2.3 we have

$$\begin{aligned}
&|(B_{k+1}(x,y) - m)x - A_{k+1}(x,y) + j| \\
&\quad = B_{k+1}(x,y)x - A_{k+1}(x,y) - y - (mx - j - y) \\
&\quad < B_{k+1}(x,y)x - A_{k+1}(x,y) - y < |q_k(x)x - p_k(x)|.
\end{aligned}$$

But it is a contradiction. Secondly, we suppose $m < B_k(x,y)$. If $m \le B_{k-1}(x,y)$, using Lemma 2.12 we have a contradiction from the assumption of the induction. Therefore, we have $m > B_{k-1}(x,y)$. We suppose $b_k(x,y) > 0$. Since $B_k(x,y) - m \le B_k(x,y) - B_{k-1}(x,y) = b_k(x,y)q_{k-1}(x) < q_k(x)$, from Lemma 2.1 we have $|(B_k(x,y) - m)x - A_k(x,y) + j| \ge |q_{k-1}(x)x - p_{k-1}(x)|$. On the other hand, we obtain

$$\begin{aligned}
&|(B_k(x,y) - m)x - A_k(x,y) + j| \\
&\quad = mx - j - y - (B_k(x,y)x - A_k(x,y) - y) \\
&\quad < B_{k+1}(x,y)x - A_{k+1}(x,y) - y - (B_k(x,y)x - A_k(x,y) - y) \\
&\quad = b_{k+1}(x,y)|q_k(x)x - p_k(x)|.
\end{aligned}$$

From Lemma 2.1 we have $b_{k+1}(x,y)|q_k(x)x - p_k(x)| < |q_{k-1}(x)x - p_{k-1}(x)|$. But it is a contradiction. Next, we suppose $b_k(x,y) = 0$. Then, since $B_{k-1}(x,y) > B_k(x,y)$, the fact $m > B_{k-1}(x,y)$ contradicts the assumption $m < B_k(x,y)$. Secondly, we suppose $b_{k+1}(x,y) = 0$. If $m \le B_{k-1}(x,y)$, then it contradicts the assumption of the induction. Therefore, we have $m > B_{k-1}(x,y)$ by using Lemma 2.12. Since $B_{k+1}(x,y) - m < B_{k+1}(x,y) - B_{k-1}(x,y) = (b_k(x,y) - 1)q_{k-1}(x) < q_k(x)$, by using Lemma 2.1 we have $|(B_{k+1}(x,y) - m)x - A_k(x,y) + j| \ge |q_{k-1}(x)x - p_{k-1}(x)|$. On the other hand, we see

$$\begin{aligned}
&|(B_{k+1}(x,y) - m)x - A_k(x,y) + j| \\
&\quad = B_{k+1}(x,y)x - A_{k+1}(x,y) - y - (mx - j - y) \\
&\quad < B_{k+1}(x,y)x - A_{k+1}(x,y) - y - (B_k(x,y)x - A_k(x,y) - y) \\
&\quad = |q_{k-1}(x)x - p_{k-1}(x)|.
\end{aligned}$$

But it is a contradiction. For even $k+1$ we have a proof similarly. Therefore, we have the proof for $n = k+1$. Thus, we obtain the lemma. □

LEMMA 2.14. *Let* $(x, y) \in X \cap \Psi$. *Let* $n > 0$ *be an integer. Then,* $B_n(x, y) \le q_n(x) + q_{n-1}(x)$. *If* $b_n(x, y) > 0$, *then* $B_n(x, y) \ge q_{n-1}(x)$. *If* $b_n(x, y) = 0$, *then* $B_n(x, y) \le q_{n-1}(x)$. *Furthermore,*

$$\lim_{\substack{n\to\infty \\ b_n(x,y)>0}} (B_n(x, y) - q_{n-1}(x)) = \infty.$$

PROOF. Let $n > 0$ be an integer. Using the induction on n it is not difficult to see that $B_n(x, y) \le q_n(x) + q_{n-1}(x)$. We suppose $b_n(x, y) > 0$. Then, we have $B_n(x, y) - q_{n-1}(x) = B_{n-1}(x, y) + (b_n(x, y) - 1)q_{n-1}(x) \ge B_{n-1}(x, y)$. Therefore, using Lemma 2.11, we have $B_n(x, y) - q_{n-1}(x) \ge 0$ and

$$\lim_{\substack{n\to\infty \\ b_n(x,y)>0}} (B_n(x, y) - q_{n-1}(x)) = \infty.$$

Let $n > 0$ be an integer with $b_n(x, y) = 0$. If $n = 1$, then we see easily $B_n(x, y) \le q_{n-1}(x)$. Let $n > 1$. Then, we have $B_n(x, y) = B_{n-1}(x, y) - q_{n-2}(x) \le q_{n-1}(x)$. □

Following Theorem is a analogous to the result by Komatsu [14].

THEOREM 2.15. *Let* $(x, y) \in X \cap \Psi$.

$$\begin{aligned}
&\liminf_{q\to\infty} q\|qx - y\| \\
&\quad = \liminf_{n\to\infty} \min\{B_n(x, y)|B_n(x, y)x - A_n(x, y) - y|, \\
&\qquad \tau(B_n(x, y) - q_{n-1}(x))|(B_n(x, y) - q_{n-1}(x))x - (A_n(x, y) - p_{n-1}(x)) - y|\},
\end{aligned}$$

where $q \in \mathbf{Z}$ *and for* $z \in \mathbf{R}$ $\|z\| = \min\{|z - m| \mid m \in \mathbf{Z}\}$ *and* $\tau(u) = u$ *for* $u > 0$ *and* $\tau(u) = \infty$ *for* $u \le 0$.

PROOF. We are proving that for each $n > 1$ with $b_n > 0$ if for an integer q $B_{n-1}(x, y) < q < B_n(x, y)$, then

$$\begin{aligned}
&q\|qx - y\| \\
&\quad \ge \min_{j=n,n-1}\{B_j(x, y)|B_j(x, y)x - A_j(x, y) - y|, \\
&\qquad \tau(B_j(x, y) - q_{j-1}(x))|(B_j(x, y) - q_{j-1}(x))x - (A_{j-1}(x, y) - p_j(x)) - y|\}. \quad (4)
\end{aligned}$$

It follows Theorem 2.15. Let $n > 1$ and $b_n(x, y) > 0$. Let $B_{n-1}(x, y) < q < B_n(x, y)$. We suppose that n is odd. If $qx - q' < B_{n-1}(x, y)x - A_{n-1}(x, y)$ for an integer q', then from Lemma 2.3 we have $|q(qx - q' - y)| > |B_{n-1}(x, y)(B_{n-1}(x, y)x - A_{n-1}(x, y) - y)|$. We suppose that $B_{n-1}(x, y)x - A_{n-1}(x, y) < qx - q' < B_n(x, y)x - A_n(x, y)$ for an integer q'. From Lemma 2.13, we have $qx - q' < y$. Since $B_n(x, y)x - A_n(x, y) - (B_{n-1}(x, y)x - A_{n-1}(x, y)) = b_n(x, y)(q_{n-1}(x)x - p_{n-1}(x))$, there exists an integer j such that $0 \leq j < b_n(x, y)$ and

$$\begin{aligned} j(q_{n-1}(x)x - p_{n-1}(x)) &\leq qx - q' - (B_{n-1}(x, y)x - A_{n-1}(x, y)) \\ &< (j+1)(q_{n-1}(x)x - p_{n-1}(x)). \end{aligned}$$

Then, we have $|(q - B_{n-1}(x, y) - jq_{n-1}(x))x - q' + A_{n-1}(x, y) + jp_{n-1}(x)| < |q_{n-1}(x)x - p_{n-1}(x)|$. On the other hand, we have $|q - B_{n-1}(x, y) - jq_{n-1}(x)| < b_n(x, y)q_{n-1}(x) < q_n(x)$. Using Lemma 2.1 we have $q - B_{n-1}(x, y) - jq_{n-1}(x) = 0$. We see easily that $q' - A_{n-1}(x, y) - jp_{n-1}(x) = 0$. Then, we have

$$\begin{aligned} q|qx - q' - y| &= (B_{n-1}(x, y) + jq_{n-1}(x))|(B_{n-1}(x, y) + jq_{n-1}(x))x \\ &\qquad - (A_{n-1}(x, y) + jp_{n-1}(x)) - y| \\ &\geq \min_{0 \leq l \leq b_n(x,y)-1} \{(B_{n-1}(x, y) + lq_{n-1}(x))|(B_{n-1}(x, y) + lq_{n-1}(x))x \\ &\qquad - (A_{n-1}(x, y) + lp_{n-1}(x)) - y|\}. \end{aligned}$$

On the other hand, Lemma 2.3 implies

$$\begin{aligned} &|(B_{n-1}(x, y) + lq_{n-1}(x))x - (A_{n-1}(x, y) + lp_{n-1}(x)) - y| \\ &\quad = y - B_{n-1}(x, y)x + A_{n-1}(x, y) - l(q_{n-1}(x)x - p_{n-1}(x)) \end{aligned}$$

for each integer l with $0 \leq l \leq b_n(x, y) - 1$. Since

$$\begin{aligned} &\min_{0 \leq l \leq b_n(x,y)-1} \{(B_{n-1}(x, y) + lq_{n-1}(x))(y - B_{n-1}(x, y)x + A_{n-1}(x, y) \\ &\qquad - l(q_{n-1}(x)x - p_{n-1}(x)))\} \\ &= \min_{l=0,\, b_n(x,y)-1} \{(B_{n-1}(x, y) + lq_{n-1}(x))(y - B_{n-1}(x, y)x + A_{n-1}(x, y) \\ &\qquad - l(q_{n-1}(x)x - p_{n-1}(x)))\}, \end{aligned}$$

we have

$$q|qx - q' - y|$$
$$\geq \min\{B_{n-1}(x,y)|B_{n-1}(x,y)x - A_{n-1}(x,y) - y|,$$
$$(B_n(x,y) - q_{n-1}(x))|(B_n(x,y) - q_{n-1}(x))x - (A_n(x,y) - p_{n-1}(x)) - y|\}.$$

We suppose that $B_n(x,y)x - A_n(x,y) < qx - q'$ for an integer q'. We consider the case of $b_{n-1}(x,y) > 0$. We suppose $B_n(x,y)x - A_n(x,y) - (q_{n-2}(x)x - p_{n-2}(x)) \leq qx - q'$. Then, we have $y < B_{n-1}(x,y)x - A_{n-1}(x,y) - (q_{n-2}(x)x - p_{n-2}(x)) \leq qx - q'$. Therefore, noting $B_{n-1}(x,y) - q_{n-2}(x) \geq 0$ from Lemma 2.14, we have

$$q|qx - q' - y| \geq (B_{n-1}(x,y) - q_{n-2}(x))$$
$$\times |(B_{n-1}(x,y) - q_{n-2}(x))x - (A_{n-1}(x,y) - p_{n-2}(x)) - y|.$$

Next, we suppose $B_n(x,y)x - A_n - (q_{n-2}(x)x - p_{n-2}(x)) > qx - q'$. Then, we have $0 < qx - q' - (B_n(x,y)x - A_n(x,y)) < -(q_{n-2}(x)x - p_{n-2}(x))$. Noting $0 < B_n(x,y) - q < b_n(x,y)q_{n-1}(x)$, similarly to the previous argument, we see that there exists an integer j' such that $0 \leq j' < b_n(x,y)$ and $(B_n(x,y)x - A_n(x,y)) - (qx - q') = q_{n-2}(x)x - p_{n-2}(x) + j'(q_{n-1}(x)x - p_{n-1}(x))$. Therefore, we have

$$qx - q' = B_n(x,y)x - A_n(x,y) - (q_{n-2}(x)x - p_{n-2}(x)) - j'(q_{n-1}(x)x - p_{n-1}(x)$$
$$= B_{n-1}(x,y)x - A_{n-1}(x,y) - (q_{n-2}(x)x - p_{n-2}(x))$$
$$+ (b_n(x) - j')(q_{n-1}(x)x - p_{n-1}(x)). \tag{5}$$

Using (5) and $B_{n-1}(x,y)x - A_{n-1}(x,y) - q_{n-2}(x)x - p_{n-2}(x) > y$, we see $0 < B_{n-1}(x,y)x - A_{n-1}(x,y) - (q_{n-2}(x)x - p_{n-2}(x)) - y < qx - q' - y$. Therefore,

$$q|qx - q' - y| > (B_{n-1}(x,y) - q_{n-2}(x))$$
$$\times |B_{n-1}(x,y)x - A_{n-1}(x,y) - (q_{n-2}(x)x - p_{n-2}(x)) - y|.$$

We consider the case of $b_{n-1}(x,y) = 0$. We suppose that $B_n(x,y)x - A_n(x,y) - (q_{n-2}(x)x - p_{n-2}(x)) \leq qx - q'$. Since $B_n(x,y)x - A_n(x,y) - (B_{n-1}(x,y)x - A_{n-1}(x,y)) = q_{n-1}(x)x - p_{n-1}(x)$, we have $0 < y - (B_{n-1}(x,y)x - A_{n-1}(x,y)) < q_{n-1}(x)x - p_{n-1}(x)$. On the other hand, we obtain $qx - q' - y > qx - q' - (B_n(x,y)x - A_n(x,y)) \geq -(q_{n-2}(x)x - p_{n-2}(x))$. Therefore, $q|qx - q' - y| > B_{n-1}(x,y)|B_{n-1}(x,y)x - A_{n-1}(x,y) - y|$. Secondly, we suppose $B_n(x,y)x - A_n(x,y) - (q_{n-2}(x)x - p_{n-2}(x)) > qx - q'$. Then, $0 < qx - q' - (B_n(x,y)x - A_n(x,y)) < -(q_{n-2}(x)x - p_{n-2}(x))$. Using $0 < B_n(x,y) - q < q_{n-1}(x)$ and Lemma

2.1, we have a contradiction. Therefore, we have the inequality (4). Thus, we have Lemma. □

LEMMA 2.16. *Let* $(x, y) \in X \cap \Psi$. *For any integer* $n > 0$,

$$\liminf_{q \to \infty} q\|qx - y\| = \liminf_{q \to \infty} q\|qx_n - y_n\|,$$

where $(x_n, y_n) = T^{n-1}(x, y)$.

PROOF. We are proving that $\liminf_{q\to\infty} q\|qx - y\| = \liminf_{q\to\infty} \cdot q\|qx_2 - y_2\|$. It follows the lemma. Let $e = \liminf_{q\to\infty} q\|qx - y\|$ and $f = \liminf_{q\to\infty} q\|qx_2 - y_2\|$. Then, there exist an increasing positive integral sequences $\{p'_k\}_{k=1,2,\ldots}$ and $\{q'_k\}_{k=1,2,\ldots}$ such that $f = \liminf_{k\to\infty} q'_k|q'_k x_2 - y_2 - p'_k|$. We suppose that $b_1(x, y) > 0$. Then, for $k > 0$ we have

$$\begin{aligned} q'_k|q'_k x_2 - y_2 - p'_k| &= q'_k\left|q'_k\left(\frac{1}{x_1} - a_1(x)\right) - \left(b_1(x, y) - \frac{y_1}{x_1}\right) - p'_k\right| \\ &= \frac{q'_k}{x_1}|(q'_k a_1(x) + p'_k + b_1(x, y))x_1 - y_1 - q'_k| \\ &= (q'_k a_1(x) + p'_k + b_1(x, y))|(q'_k a_1(x) + p'_k + b_1(x, y))x_1 \\ &\quad - y_1 - q'_k|\frac{q'_k}{x_1(q'_k a_1(x) + p'_k + b_1(x, y))}. \end{aligned}$$

Since $\frac{p'_k}{q'_k} \to x_2$ as $k \to \infty$, we see that $\lim_{k\to\infty} \frac{q'_k}{x_1(q'_k a_1(x)+p'_k+b_1(x,y))} = \lim_{k\to\infty} \frac{1}{x_1\left(a_1(x)+\frac{p'_k}{q'_k}+\frac{b_1(x,y)}{q'_k}\right)} = 1$. Thus, $e \le f$. If $b_1(x,y) = 0$, we have $e \le f$ by the same manner. Similarly, we have $e \ge f$. Thus, we have the lemma. □

3. Natural Extension

$\mathbf{Z}_+$ denotes the the set of all positive integers. We define Ω_1, Ω_2, Ω'_1 and Ω'_2 as follows:

$$\Omega_1 = \{(x, y) \in [0, 1]^2 \mid (x, y) \in \Psi, y \le x\},$$

$$\Omega_2 = \{(x, y) \in [0, 1]^2 \mid (x, y) \in \Psi, y > x\},$$

$$\Omega'_1 = \{(x, y) \mid (x, y) \in \Psi, y > 1, x \le -1, y \le -x + 1\},$$

$$\Omega'_2 = \{(x, y) \mid (x, y) \in \Psi, 0 \le y \le 1, x \le -1\}.$$

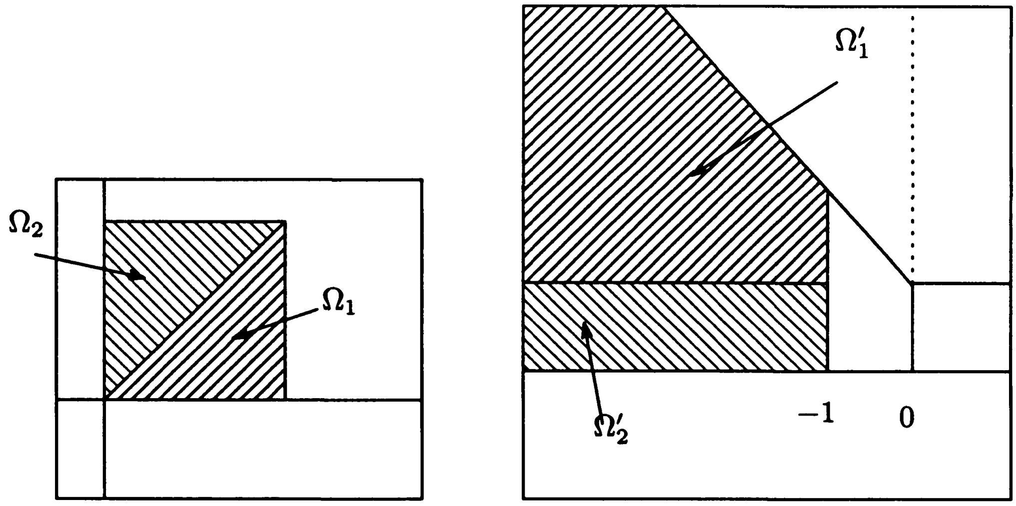

Figure 3.1

Let $\Omega = \{\Omega_1 \times (\Omega_1' \cup \Omega_2')\} \cup (\Omega_2 \times \Omega_1')$.

We define a transformation $\bar{T}$ on Ω as follows: for $(x, y, z, w) \in \Omega$

$$\bar{T}(x, y, z, w) = \begin{cases} \left(\frac{1}{x} - a(x), b(x, y) - \frac{y}{x}, \frac{1}{z} - a(x), b(z, w) - \frac{w}{z}\right) & \text{if } b(x, y) > 0, \\ \left(\frac{1}{x} - a(x), \frac{1}{x} - \frac{y}{x}, \frac{1}{z} - a(x), \frac{1}{z} - \frac{w}{z}\right) & \text{if } b(x, y) = 0. \end{cases}$$

We see easily that $\bar{T}$ is well defined.

THEOREM 3.1. *$\bar{T}$ is bijective.*

PROOF. We define $\Delta_{m,n}$ for $m \in \mathbf{Z}_+$ and $n \in \mathbf{Z}_+ \cup \{0\}$ with $m \geq n$ as follows;

$$\Delta_{m,n} = \begin{cases} \left\{(x, y) \in X \cap \Psi \,\middle|\, \frac{1}{m+1} < x < \frac{1}{m}, (n-1)x < y < nx\right\} & \text{if } n \geq 1, \\ \left\{(x, y) \in X \cap \Psi \,\middle|\, \frac{1}{m+1} < x < \frac{1}{m}, y > mx\right\} & \text{if } m \geq n \text{ and } n = 0. \end{cases}$$

Then, we see easily that $T : \Delta_{m,n} \to X \cap \Psi$ is bijective for $n > 0$ and $T : \Delta_{m,0} \to \Omega_1$ is bijective. We define $\Delta'_{m,n}$ for $m \in \mathbf{Z}_+$ and $n \in \mathbf{Z}_+ \cup \{0\}$ with $m \geq n$ as follows; if $n = 1$, then we see $\Delta'_{m,n} = \{(x, y) \in \Omega_1' \mid -(m+1) < x < -m, 1 < y < -x - m + 2\}$ and if $n > 1$, then we see $\Delta'_{m,n} = \{(x, y) \in \Omega_1' \mid -(m+1) < x < -m, -x - m + n < y < -x - m + n + 1\}$ and if $n = 0$, then we see $\Delta'_{m,n} = \{(x, y) \in \Omega_2' \mid -(m+1) < x < -m\}$.

We see that for $m \in \mathbf{Z}_+$ and $n \in \mathbf{Z}_+ \cup \{0\}$ with $m \geq n$ and $n \neq 1$ $(T_{(m,n)})_{\Omega_1'}\Omega_1' \to \Delta'_{m,n}$ is bijective and $(T_{(m,1)})_{\Omega_1' \cup \Omega_2'}\Omega_1' \cup \Omega_2' \to \Delta'_{m,1}$ is bijective, where $T_{(m,n)}$ is defined in Section 2. On the other hand, we have

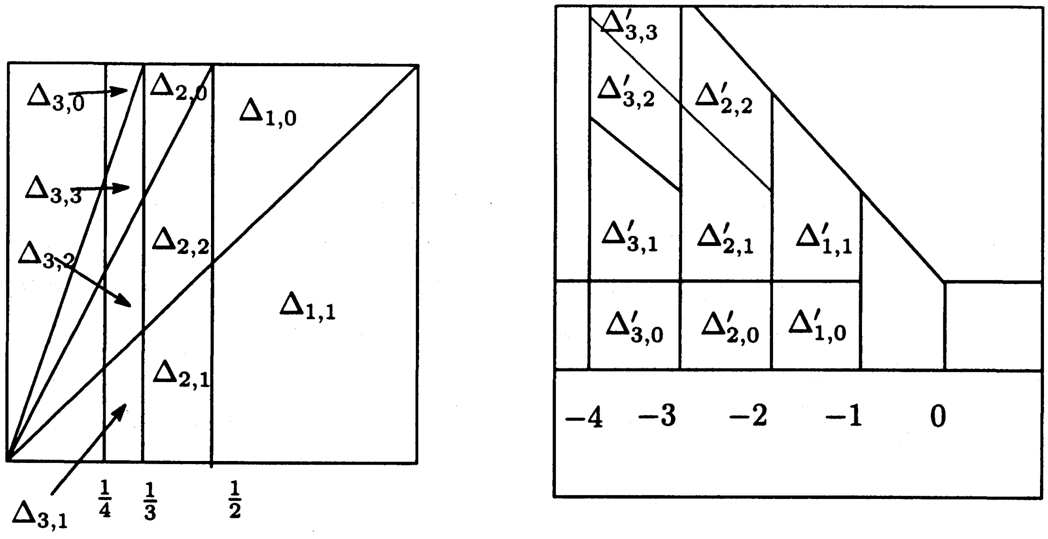

Figure 3.2

$$\begin{aligned}\Omega = & \bigcup_{(m,n)\in \mathbf{Z}_+\times\mathbf{Z}_+, m\geq n, n\neq 1} \Delta_{m,n}\times\Omega'_1 \cup \bigcup_{m\in\mathbf{Z}_+}\Delta_{m,1} \\ & \times(\Omega'_1\cup\Omega'_2)\cup\bigcup_{m\in\mathbf{Z}_+}\Delta_{m,0}\times\Omega'_1 \quad \text{(disjoint)} \\ = & \bigcup_{(m,n)\in \mathbf{Z}_+\times\mathbf{Z}_+, m\geq n, n\neq 1}(X\cap\Psi)\times\Delta'_{m,n}\cup\bigcup_{m\in\mathbf{Z}_+}(X\cap\Psi) \\ & \times\Delta'_{m,1}\cup\bigcup_{m\in\mathbf{Z}_+}\Omega_1\times\Delta'_{m,0} \quad \text{(disjoint).}\end{aligned}$$

We see that $\bar{T}_{\Delta_{m,n}\times\Omega'_1}\Delta_{m,n}\times\Omega'_1 \to (X\cap\Psi)\times\Delta'_{m,n}$ is bijective for $(m,n)\in \mathbf{Z}_+\times\mathbf{Z}_+$ with $n\neq 1$ and $\bar{T}_{\Delta_{m,1}\times(\Omega'_1\cup\Omega'_2)}\Delta_{m,1}\times(\Omega'_1\cup\Omega'_2)\to(X\cap\Psi)\times\Delta'_{m,1}$ for $m\in\mathbf{Z}_+$ is bijective and $\bar{T}_{\Delta_{m,0}\times\Omega'_1}\Delta_{m,0}\times\Omega'_1\to\Omega_1\times\Delta'_{m,0}$ is bijective for $m\in\mathbf{Z}_+$. Therefore, $\bar{T}$ is bijective. □

Following Lemma 3.2 is easily proved.

LEMMA 3.2. *Let K be a real quadratic field over $\mathbf{Q}$. Let $(x,y)\in K^2\cap X\cap\Psi$. Then, if $(x,y,\bar{x},\bar{y})\in\Omega$, then $(T(x,y),\overline{T(x,y)})=\bar{T}(x,y,\bar{x},\bar{y})$, where for $z\in K$ $\bar{z}$ is the algebraic conjugate of z related to $K/\mathbf{Q}$.*

Komatsu [15] determine the all eventually periodic points in (X,T_2). Following Lemma is the similar result.

LEMMA 3.3. *Let $(x, y) \in X \cap \Psi$, x be a quadratic irrational number and $y \in \mathbf{Q}(x)$. Then, $(x, y, \bar{x}, \bar{y})$ is a eventually periodic point related to $\bar{T}$, where for $z \in \mathbf{Q}(x)$ $\bar{z}$ is an algebraic conjugate of z related to $\mathbf{Q}(x)/\mathbf{Q}$.*

PROOF. Since $y \in \mathbf{Q}(x)$, there exist $r_n, s_n \in \mathbf{Q}$ such that $y_n = r_n + s_n x_n$. Let d_n be the denominator of r_n, s_n. By using induction, we see $d_0 = d_n$ for all n. From the well known fact about continued fraction of quadratic irrational numbers, there exists an integer m such that $\{x_m, x_{m+1}, \dots\}$ is purely periodic. It is known that $\overline{x_n} < -1$ for each $n \geq m$. We define a constant c_1 by $c_1 = \min\{|\overline{x_n}| \mid n \geq m\}$. Let $c_2 = \max\{a_n(x) \mid n = 1, \dots\}$. Let $r = \frac{c_1(c_2+1)}{c_1-1}$. Then, if $n > m$ and $|\overline{y_n}| > r$, we have

$$|\overline{y_{n+1}}| < c_2 + \left|\frac{\overline{y_n}}{\overline{x_n}}\right| < c_2 + \left|\frac{\overline{y_n}}{c_1}\right| = |\overline{y_n}| - \frac{|\overline{y_n}|(c_1 - 1)}{c_1} + c_2 < |\overline{y_n}| - 1.$$

Therefore, there exists n_1 such that $n_1 > m$ and $|\overline{y_{n_1}}| \leq r$. On the other hand, if $n > m$ and $|\overline{y_n}| \leq r$, then we have

$$|\overline{y_{n+1}}| < c_2 + \left|\frac{\overline{y_n}}{\overline{x_n}}\right| < 2r.$$

We suppose that $\limsup_{n\to\infty} |\overline{y_n}| = \infty$. Let $n_2 = \min\{k \mid k > n_1, |\overline{y_k}| > 3r\}$. We assume $|\overline{y_{n_2-1}}| > r$. Then, we have $|\overline{y_{n_2}}| < |\overline{y_{n_2-1}}| - 1$. Therefore, we have $|\overline{y_{n_2-1}}| > 3r$. But it is a contradiction. Next, we assume $|\overline{y_{n_2-1}}| \leq r$. Then, by using previous argument, we have $|\overline{y_{n_2}}| \leq 3r$. But it is a contradiction. Thus, there exists $c > 0$ such that $|\overline{y_n}| < c$ for all n. From the facts that $|\overline{y_n}| < c$ and $|y_n| < 1$ for all n, we see that there exits c_3 such that $|r_n|, |s_n| < c_3$ for all n. Using the fact $d_0 = d_n$ for all n, we see that $\{y_n \mid n = 0, 1, \dots\}$ has finitely many numbers. Thus, $(x, y, \bar{x}, \bar{y})$ is a eventually periodic point related to T. □

LEMMA 3.4. *Let $(x, y) \in X \cap \Psi$, x be a quadratic irrational number and $y \in \mathbf{Q}(x)$, where for $z \in \mathbf{Q}(x)$ $\bar{z}$ is an algebraic conjugate of z related to $\mathbf{Q}(x)/\mathbf{Q}$. Then, there exists an integer $n > 0$ such that $(x_n, y_n, \overline{x_n}, \overline{y_n}) \in \Omega$.*

PROOF. By Lemma 3.3 $\{(x_n, y_n)\}_{n=0,1,\dots}$ is eventually periodic. Therefore, there exist integers $m_1, m_2 > 0$ such that for any $n \geq m_1$ $(x_{n+m_2}, y_{n+m_2}) = (x_n, y_n)$. We define m_3 as follows. If $b_n > 0$ for any $n \geq m_1$, then we set $m_3 = m_1$. If there exists $m' \geq m_1$ such that $b_{m'}(x, y) = 0$, then we set $m_3 = m'$. If for integers a, b $b > 0$ and $a \geq b$, then it is not difficult to see that $T_{(a,b)}(cl(\Omega_1')) \subset \{(x, y) \in$

$cl(\Omega_1') \mid -a-1 \le x \le -a\}$, where $cl(\Omega_1')$ is the closure of Ω_1'. Therefore, if $b_n(x, y) > 0$ for any $n \ge m_1$, then we have

$$T_{(a_{m_3+m_2-1}(x), b_{m_3+m_2-1}(x,y))} \cdots T_{(a_{m_3}(x), b_{m_3}(x,y))}\eta \subset \eta,$$

where $\eta = \{(x, y) \in cl(\Omega_1') \mid -a_{m_3+m_2-1}(x) - 1 \le x \le -a_{m_3+m_2-1}(x)\}$. It is not difficult to see that for integers $a, a' \ge 1$ $T_{(a,1)}T_{(a',0)}\ cl(\Omega_1') \subset \{(x, y) \in cl(\Omega_1') \mid -a-1 \le x \le -a\}$. By lemma 2.2 $m_2 > 1$ and $b_{m_3+m_2-1}(x, y) \ne 0$. Thus, we have

$$T_{(a_{m_3+m_2-1}(x), b_{m_3+m_2-1}(x,y))} \cdots T_{(a_{m_3}(x), b_{m_3}(x,y))}\eta \subset \eta.$$

By Bronwell's fixed point theorem there exists $(x', y') \in \{(x, y) \in cl(\Omega_1') \mid -a_{m_3+m_2-1}(x) - 1 \le x \le -a_{m_3+m_2-1}(x)\}$ such that $T_{(a_{m_3+m_2-1}(x), b_{m_3+m_2-1}(x,y))} \cdots T_{(a_{m_3}(x), b_{m_3}(x,y))}(x', y') = (x', y')$. we see easily that $(x', y') = (\overline{x_{m_3}}, \overline{y_{m_3}})$. Therefore, we have $(x_{m_3}, y_{m_3}, \overline{x_{m_3}}, \overline{y_{m_3}}) \in \Omega$. □

LEMMA 3.5. *Let $(x, y) \in X \cap \Psi$, x be a quadratic irrational number and $y \in \mathbf{Q}(x)$. Let $(x, y, \bar{x}, \bar{y}) \in \Omega$, where for $z \in \mathbf{Q}(x)$ $\bar{z}$ is an algebraic conjugate of z related to $\mathbf{Q}(x)/\mathbf{Q}$. Then, $(x, y, \bar{x}, \bar{y})$ is a purely periodic point related to $\bar{T}$.*

PROOF. By Lemma 3.3 there exist integers $m, m_1 \ge 1$ such that for any integer $n > m$ $(x_n, y_n) = (x_{n+m_1}, y_{n+m_1})$. Since $(x_1, y_1, \overline{x_1}, \overline{y_1}) \in \Omega$, by Lemma 3.2 we have $(x_n, y_n, \overline{x_n}, \overline{y_n}) \in \Omega$ for any integer $n > 0$. Since $\bar{T}$ is bijective on Ω, for each integer $n > m$ we have $(x_{n-1}, y_{n-1}, \overline{x_{n-1}}, \overline{y_{n-1}}) = (x_{n+m_1-1}, y_{n+m_1-1}, \overline{x_{n+m_1-1}}, \overline{y_{n+m_1-1}})$. By using the induction we have $(x_1, y_1, \overline{x_1}, \overline{y_1}) = (x_{1+m_1}, y_{1+m_1}, \overline{x_{1+m_1}}, \overline{y_{1+m_1}})$. Thus, $(x, y, \bar{x}, \bar{y})$ is a purely periodic point related to $\bar{T}$. □

THEOREM 3.6. *Let $(x, y) \in X \cap \Psi$. x is a quadratic irrational number, $y \in \mathbf{Q}(x)$ and $(x, y, \bar{x}, \bar{y}) \in \Omega$ if and only if (x, y) is a purely periodic point related to T, where for $z \in \mathbf{Q}(x)$ $\bar{z}$ is an algebraic conjugate of z related to $\mathbf{Q}(x)/\mathbf{Q}$.*

PROOF. The necessary condition of the theorem is proved in Lemma 3.5. Let us prove the sufficient condition. We assume that $(x, y) \in X \cap \Psi$ and (x, y) is a purely periodic point related to T. Then, it is not difficult to see that x is a quadratic irrational number and $y \in \mathbf{Q}(x)$. Using Theorem 3.1 and Lemma 3.4, we see that $(x, y, \bar{x}, \bar{y}) \in \Omega$. □

Following Lemma 3.7 is a well known result.

LEMMA 3.7 (É. Galois). *Let $0 < x < 1$ be a quadratic irrational number and let x have purely periodic continued fraction expansion. Then,*

$\lim_{n\to\infty}\left(\frac{q_n(x)}{q_{n-1}(x)}+\overline{x_{n+1}}\right)=0$, *where for* $z\in\mathbf{Q}(x)$ $\bar{z}$ *is an algebraic conjugate of* z *related to* $\mathbf{Q}(x)/\mathbf{Q}$.

PROOF. Let $W=[0,1]\times(-\infty,-1]$. We define a transformation ρ on W as follows: for $(x,y)\in W$

$$\rho(x,y)=\begin{cases}\left(\frac{1}{x}-a(x),\frac{1}{y}-a(x)\right) & \text{if } x\neq 0,\\ (x,y) & \text{if } x=0.\end{cases}$$

We see easily that ρ is well defined. Since x is reduced, $\bar{x}<-1$ (see [20]). Therefore, $(x,\bar{x})\in W$. We see easily that $\rho^n(x,\bar{x})=(x_{n+1},\overline{x_{n+1}})$. On the other hand, for each integer $n>0$ $\left(x_{n+1},-\frac{q_n(x)}{q_{n-1}(x)}\right)\in W$. We see for each integer $n>0$

$$\rho\left(x_{n+1},-\frac{q_n(x)}{q_{n-1}(x)}\right)=\left(x_{n+2},-\frac{q_{n-1}(x)}{q_n(x)}-a_{n+1}(x)\right)$$
$$=\left(x_{n+2},-\frac{q_{n+1}(x)}{q_n(x)}\right).$$

Therefore, we have $\rho^{n-1}\left(x_2,-\frac{q_1(x)}{q_0(x)}\right)=\left(x_{n+1},-\frac{q_n(x)}{q_{n-1}(x)}\right)$. We denote $u_n=-\frac{q_n(x)}{q_{n-1}(x)}$ for each integer $n>0$. Then, we have

$$|\overline{x_{n+2}}-u_{n+1}|=\frac{|\overline{x_{n+1}}-u_n|}{|\overline{x_{n+1}}u_n|}\leq\frac{|\overline{x_{n+1}}-u_n|}{C},$$

where $C=\min\{|\overline{x_j}|\mid j=1,2,\dots\}$. Therefore, we have $|\overline{x_{n+1}}-u_n|\leq\frac{|\overline{x_2}-u_1|}{C^{n-1}}$ for each $n>0$. Since $C>1$, we obtain the lemma. □

LEMMA 3.8. *Let* $(x,y)\in X\cap\Psi$ *and let* (x,y) *be a purely periodic point related to* T. *Then,* $\lim_{n\to\infty}\left(\frac{B_n(x,y)}{q_{n-1}(x)}-\overline{y_{n+1}}\right)=0$.

PROOF. We see easily that $\bar{T}$ is naturally extended to $\Omega_{\#}=\{\Omega_1\times cl(\Omega_1'\cup\Omega_2')\}\cup(\Omega_2\times cl(\Omega_1'))$. We also denote it $\bar{T}$. For each integer $k\geq 1$ u_k denotes $-\frac{q_k(x)}{q_{k-1}(x)}$ and v_k denotes $\frac{B_k(x,y)}{q_{k-1}(x)}$. First, we show that $(x_2,y_2,u_1,v_1)\in\Omega_{\#}$ and for $n\geq 1$ $\bar{T}^{n-1}(x_2,y_2,u_1,v_1)=(x_{n+1},y_{n+1},u_n,v_n)$. We suppose $b_1(x,y)>0$. Then, we see that $-\frac{q_1(x)}{q_0(x)}=-a_1(x)$ and $\frac{B_1(x)}{q_0(x)}=b_1(x,y)$. Since $0<b_1(x,y)\leq a_1(x,y)$, we have $\left(x_2,y_2,-\frac{q_1(x)}{q_0(x)},\frac{B_1(x,y)}{q_0(x)}\right)\in\Omega_{\#}$. We suppose $b_1(x,y)=0$. Then, we see that $\frac{B_1(x,y)}{q_0(x)}=0$ and $y_2=\frac{1}{x_1}-\frac{y_1}{x_1}$. From the fact that $a_1=\lfloor\frac{y}{x}\rfloor$, we have $\frac{1}{x_1}-a_1\geq\frac{1}{x_1}-\frac{y_1}{x_1}$. Therefore, we have $\left(x_2,y_2,-\frac{q_1(x)}{q_0(x)},\frac{B_1(x,y)}{q_0(x)}\right)\in\Omega_{\#}$. Secondly, we suppose that for an integer $k>0$ $\bar{T}^{k-1}(x_2,y_2,u_1,v_1)=(x_{k+1},y_{k+1},u_k,v_k)$. Then, we have $\frac{1}{u_k} - a_{k+1}(x) = -\frac{q_{k-1}(x)}{q_k(x)} - a_{k+1}(x) = u_{k+1}$. We suppose that $b_{k+1}(x,y) > 0$. Then, we have $b_{k+1}(x,y) - \frac{v_k}{u_k} = b_{k+1}(x,y) + \frac{B_k(x,y)}{q_k(x)} = v_{k+1}$. Therefore, we have $\bar{T}(x_{k+1}, y_{k+1}, u_k, v_k) = (x_{k+2}, y_{k+2}, u_{k+1}, v_{k+1})$. We suppose that $b_{k+1}(x,y) = 0$. Then, we have $\frac{1-v_k}{u_k} = \frac{B_k(x,y) - q_{k-1}(x)}{q_k(x)} = \frac{B_{k+1}(x,y)}{q_k(x)} = v_{k+1}$. Therefore, we have $\bar{T}(x_{k+1}, y_{k+1}, u_k, v_k) = (x_{k+2}, y_{k+2}, u_{k+1}, v_{k+1})$. Thus, we have the proof of that for $n \geq 1$ $\bar{T}^{n-1}(x_2, y_2, u_1, v_1) = (x_{n+1}, y_{n+1}, u_n, v_n)$. Since for $n \geq 1$ $\bar{T}^{n-1}(x_2, y_2, \overline{x_2}, \overline{y_2}) = (x_{n+1}, y_{n+1}, \overline{x_{n+1}}, \overline{y_{n+1}})$. If $b_{n+1}(x,y) > 0$, then we obtain

$$|v_{n+1} - \overline{y_{n+2}}| = \left|\frac{v_n}{u_n} - \frac{\overline{y_{n+1}}}{\overline{x_{n+1}}}\right| = \left|\frac{v_n}{u_n} - \frac{v_n}{\overline{x_{n+1}}} + \frac{v_n}{\overline{x_{n+1}}} - \frac{\overline{y_{n+1}}}{\overline{x_{n+1}}}\right|$$
$$\leq \left|\frac{v_n}{u_n}\right| \left|\frac{\overline{x_{n+1}} - u_n}{\overline{x_{n+1}}}\right| + \frac{|v_n - \overline{y_{n+1}}|}{|\overline{x_{n+1}}|}, \tag{6}$$

and if $b_{n+1}(x,y) = 0$, then we obtain

$$|v_{n+1} - \overline{y_{n+2}}| = \left|\frac{1}{u_n} - \frac{v_n}{u_n} - \frac{1}{\overline{x_{n+1}}} + \frac{\overline{y_{n+1}}}{\overline{x_{n+1}}}\right|$$
$$\leq \left(1 + \left|\frac{v_n}{u_n}\right|\right) \left|\frac{\overline{x_{n+1}} - u_n}{\overline{x_{n+1}}}\right| + \frac{|v_n - \overline{y_{n+1}}|}{|\overline{x_{n+1}}|}. \tag{7}$$

Since $(u_n, v_n) \in cl(\Omega_1' \cup \Omega_2')$, $\left|\frac{v_n}{u_n}\right| \leq 2$ for each integer $n > 0$. From the proof of Lemma 3.7, (6) and (7) we see that

$$|v_{n+1} - \overline{y_{n+2}}| \leq 3(n-1)\frac{|\overline{x_2} - u_1|}{C^{n-1}} + \frac{|v_1 - \overline{y_2}|}{C^{n-1}},$$

where $C = \min\{|\overline{x_j}| \mid j = 1, 2, \ldots\}$. Thus, we have the lemma. □

THEOREM 3.9. *Let* $(x,y) \in [0,1]^2$ *be a periodic point of* $\bar{T}$. *Then,*

$$\lim_{q \to \infty} q\|qx - y\| = \min\left\{\frac{y_n \overline{y_n}}{x_n - \overline{x_n}}, \frac{\tau(\overline{y_n} - 1)(1 - y_n)}{x_n - \overline{x_n}}; n = 0, 1, 2, \ldots\right\},$$

where $\|x\| = \min\{|m - x| \mid m \in \mathbf{Z}\}$ *and* $\tau(u) = u$ *for* $u > 0$ *and* $\tau(u) = \infty$ *for* $u \leq 0$.

PROOF. From Theorem 2.15 we have

$$\liminf_{q \to \infty} q\|qx - y\|$$
$$= \liminf_{n \to \infty} \min\{B_n(x,y)|B_n(x,y)x - A_n(x,y) - y|, \tau(B_n(x,y) - q_{n-1}(x))$$
$$\times |(B_n(x,y) - q_{n-1}(x))x - (A_n(x,y) - p_{n-1}(x,y)) - y|\}.$$

Using Lemma 2.1 and Lemma 2.3

$$\begin{aligned} B_n(x,y)|B_n(x,y)x - A_n(x,y) - y| &= B_n(x,y)y_{n+1}x_1\cdots x_n \\ &= B_n(x,y)y_{n+1}|q_{n-1}(x)x - p_{n-1}(x)| \\ &= \frac{B_n(x,y)y_{n+1}}{q_{n-1}(x)\left(\frac{q_n(x)}{q_{n-1}(x)} + x_{n+1}\right)}. \end{aligned}$$

If $b_n(x,y) > 0$, we have similarly

$$\begin{aligned} &(B_n(x,y) - q_{n-1}(x))|(B_n(x,y) - q_{n-1}(x))x - (A_n(x,y) - p_{n-1}(x)) - y| \\ &\quad = (B_n(x,y) - q_{n-1}(x))|(-1)^n y_{n+1}x_1\cdots x_n - (q_{n-1}(x)x - p_{n-1}(x))| \\ &\quad = (B_n(x,y) - q_{n-1}(x))|q_{n-1}(x)x - p_{n-1}(x)|\,|1 - y_{n+1}| \\ &\quad = \frac{(B_n(x,y) - q_{n-1}(x))|1 - y_{n+1}|}{q_{n-1}(x)} \times \frac{1}{\frac{q_n(x)}{q_{n-1}(x)} + x_{n+1}}. \end{aligned}$$

From Lemma 2.14 we note that if $b_n(x,y) > 0$, $B_n(x,y) - q_{n-1}(x) \leq 0$ and $0 < \overline{y_{n+1}} < 1$. Using Lemma 3.7 and Lemma 3.8, we have Theorem 3.9. □

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General Education Suzuka College of Technology
Shiroko, Suzuka, Mie, 510-0294, Japan
yasutomi@genl.suzuka-ct.ac.jp