TSUKUBA J. MATH.
Vol. 29 No. 1 (2005), 111–145

ENERGY METHOD FOR NUMERICAL ANALYSIS

By

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Introduction

"Energy Inequality" played an essential role in the study of partial differential equations throughout 20-th century. Especially, it is a reliable paradigm that existence of solutions of a problem comes from energy inequality on its adjoint problem.

Recently, we found that the energy method to prove existence of solutions involves the numerical method. In other words, we can say that numerical approximation of solutions comes from energy inequalities on adjoint problems ([1], [2]). We will state its summary in Chapter 1. In Chapter 2, we consider nonlinear problems, where Sobolev's imbedding theorem in general type plays an essential role ([3], [4]). Our proof of existence of solutions suggests a method of numerical approximation of solutions.

Chapter 1. Numerical Approximation in Linear Case

Let us consider a linear boundary value problem as follows.

PROBLEM: To seek a solution $u \in L^2(\Omega)$ satisfying

$$\text{(P)} \qquad \begin{cases} Au = f & \text{in } \Omega \\ B_j u = f_j & \text{on } \Gamma \ \ (j \in J), \end{cases}$$

for given data $\{f, f_j \ (j \in J)\}$, where

(i) A is a linear partial differential operator of order m with smooth coefficients,

(ii) B_j is a linear partial differential operator of order j $(j \in J)$ with smooth coefficients, $(J \subset \{0, 1, \dots, m-1\})$,

(iii) Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary Γ,

(iv) Γ is non-characteristic for $\{A, B_j \ (j \in J)\}$.

Received July 1, 2003.

§ 1. Construction and Approximation of Solutions

We can define the adjoint problem (P*) for the problem (P).

ADJOINT PROBLEM: To seek $v \in L^2(\Omega)$ satisfying

$$\text{(P*)} \qquad \begin{cases} A^* v = g & \text{in } \Omega \\ \mathscr{B}_j^* v = g_j & \text{on } \Gamma \ (j \in J^*) \end{cases}$$

for given data $\{g, g_j \ (j \in J^*)\}$.

Let us assume the energy inequality

$$\text{(E*)} \qquad \|v\| \leqq C\Big(\|A^* v\| + \sum\nolimits_{j \in J^*} \langle \mathscr{B}_j^* v \rangle_{\sigma_j}\Big) \quad (\forall v \in H^M(\Omega))$$

holds. Then *Hilbert Space* $\mathscr{H}$ is defined by the completion of $H^M(\Omega)$ by the norm

$$[v]^2 = \|A^* v\|^2 + \sum\nolimits_{j \in J^*} \langle \mathscr{B}_j^* v \rangle_{\sigma_j}^2$$

with inner product

$$[v, w] = (A^* v, A^* w) + \sum\nolimits_{j \in J^*} \langle \mathscr{B}_j^* v, \mathscr{B}_j^* w \rangle_{\sigma_j}.$$

Then it holds $H^M(\Omega) \subset \mathscr{H} \subset L^2(\Omega)$ and energy inequality (E*) means

$$\|v\| \leqq C[v] \quad (\forall v \in \mathscr{H}).$$

For $f \in L^2(\Omega)$, let

$$f : \mathscr{H} \ni v \mapsto (v, f) \in \mathbf{C}$$

then $f \in \mathscr{H}'$. In fact, we have

$$|(v, f)| \leqq \|v\| \, \|f\| \leqq C\|f\|[v].$$

Owing to Riesz' Theorem in $\mathscr{H}$, there exists $w \in \mathscr{H}$ such that

$$(f, v) = [w, v] \quad (\forall v \in \mathscr{H}).$$

We say that w is a *Riesz' function* of $f \in L^2(\Omega)$.

REMARK. For $f \in L^2(\Omega)$, define

$$\mathscr{H} \ni v \mapsto J[v] = [v]^2 - 2\,\mathrm{Re}(v, f) = [v]^2 - (v, f) - (f, v),$$

then J takes minimal value iff v is a Riesz' function of $f \in L^2(\Omega)$.

THEOREM I. *Assume* (E*). *Let w be a Riesz' function of $f \in L^2(\Omega)$. Set $u = A^* w$, then $u \in L^2(\Omega)$ and u satisfies*

$$\text{(P)} \qquad \begin{cases} Au = f & \text{in } \Omega \\ B_j u = 0 & \text{on } \Gamma \ (j \in J). \end{cases}$$

Moreover, it holds

$$\|u\| \leqq C\|f\|.$$

We say that $u \in L^2(\Omega)$ is a *$\mathscr{H}$-solution* of (P) for $f \in L^2(\Omega)$, iff

$$\begin{cases} u = A^* w & (\exists w \in \mathscr{H}) \\ [w, v] = (f, v) & (\forall v \in \mathscr{H}). \end{cases}$$

We say that a subset $\{v_1, v_2, \ldots\}$ in $\mathscr{H}$ is a *basis* of $\mathscr{H}$, iff any finite subset of $\{v_1, v_2, \ldots\}$ is linearly independent and the space spanned by $\{v_1, v_2, \ldots\}$ is dense in $\mathscr{H}$.

LEMMA 1.1. *Let* $\mathrm{diam}(\Omega) < a\pi$. *Then*

$$\{\exp(ia^{-1}\alpha \cdot x) \mid \alpha \in \mathbf{Z}^n\}$$

is a basis of $\mathscr{H}$.

THEOREM II. *Assume* (E*). *Let $\{v_k \ (k = 1, 2, \ldots)\}$ be a basis of $\mathscr{H}$. Let $u \in L^2(\Omega)$ be a $\mathscr{H}$-solution of* (P). *Set*

$$u_N = ((f, v_1), \ldots, (f, v_N))\Gamma_N^{-1} \begin{pmatrix} A^* v_1 \\ \vdots \\ A^* v_N \end{pmatrix},$$

where

$$\Gamma_N = ([v_k, v_s])_{k,s=1,2,\ldots,N}.$$

Then it holds

$$u_N \to u \quad (N \to \infty) \ \textit{in} \ L^2(\Omega).$$

PROOF. (1) Let $\{v_1^\wedge, v_2^\wedge, \dots\}$ be Schmidt's ortho-normalization of $\{v_1, v_2, \dots\}$ in $\mathscr{H}$. Then, for any $w \in \mathscr{H}$, it holds

$$w_N = \sum_{1 \leqq k \leqq N} [w, v_k^\wedge] v_k^\wedge \to w \quad \text{in } \mathscr{H},$$

owing to the theory of Fourier's series in $\mathscr{H}$, where w_N is represented as

$$w_N = ([w, v_1], \dots, [w, v_N]) \Gamma_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}.$$

(2) Especially, let $w \in \mathscr{H}$ be a Riesz' function of $f \in L^2(\Omega)$, then we have

$$[w, v] = (f, v) \quad (\forall v \in \mathscr{H}),$$

therefore, we have

$$[w, v_k] = (f, v_k) \quad (k = 1, 2, \dots).$$

Hence, we have

$$w_N = ((f, v_1), \dots, (f, v_N)) \Gamma_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

and

$$w_N \to w \quad \text{in } \mathscr{H}.$$

It is remarkable that w_N is represented only by f and $\{v_1, v_2, \dots\}$, without w.

(3) Let u be a $\mathscr{H}$-solution of (P), that is,

$$\begin{cases} u = A^* w \quad w \in \mathscr{H} \\ w \text{ is a Riesz' function of } f \in L^2(\Omega). \end{cases}$$

Then we have

$$w_N \to w \quad \text{in } \mathscr{H}$$

from (2), therefore,

$$u_N = A^* w_N \to A^* w = u \quad \text{in } L^2(\Omega),$$

where

$$u_N = A^* w_N = ((f, v_1), \dots, (f, v_N)) \Gamma_N^{-1} \begin{pmatrix} A^* v_1 \\ \vdots \\ A^* v_N \end{pmatrix}. \qquad \square$$

REMARK. Let $\mathscr{H}_N$ be the space spanned by $\{v_1, v_2, \dots, v_N\}$, then it holds

$$[w_N, v] = (f, v) \quad (\forall v \in \mathscr{H}_N),$$

that is, w_N is a Riesz' function of f in $\mathscr{H}_N$. Therefore

$$\begin{cases} u_N = A^* w_N & (\exists w_N \in \mathscr{H}_N) \\ [w_N, v] = (f, v) & (\forall v \in \mathscr{H}_N), \end{cases}$$

which means that u_N is a $\mathscr{H}_N$-solution of (P).

§ 2. Generalization

In § 2, we assume

$$(\mathrm{E}^*)_\mu \quad \|v\|_\mu \leqq C\Big(\|A^* v\| + \sum\nolimits_{j \in J^*} \langle \mathscr{B}_j^* v \rangle_{\sigma_j}\Big) \quad (\forall v \in H^M(\Omega)) \ (0 \leqq \mu \leqq m).$$

Let $\mathscr{H}$ be the same one defined in § 1. $(\mathrm{E}^*)_\mu$ means

$$\|v\|_\mu \leqq C[v] \quad (\forall v \in H^M(\Omega)),$$

therefore $\mathscr{H} \subset H^\mu(\Omega)$.

Let $f \in H^{-\mu}(\Omega) = (H_0^\mu(\Omega))'$ i.e.

$$f : H_0^\mu(\Omega) \ni v \mapsto (v, f) \in \mathbf{C}$$

and

$$\|f\|_{-\mu} = \sup_{v \in H_0^\mu(\Omega)} \frac{|(v, f)|}{\|v\|_\mu}.$$

Owing to Riesz' Theorem in $H_0^\mu(\Omega)$, there exists $\phi \in H_0^\mu(\Omega)$ uniquely for $f \in H^{-\mu}(\Omega)$:

$$(v, f) = \sum\nolimits_{|\nu| \leqq \mu} (\partial^\nu v, \partial^\nu \phi) \quad (\forall v \in H_0^\mu(\Omega)),$$

$$\|f\|_{-\mu} = \Big(\sum\nolimits_{|\nu| \leqq \mu} \|\partial^\nu \phi\|^2\Big)^{1/2}.$$

Set $f_\nu = (-\partial)^\nu \phi$, then

$$(v, f) = \sum\nolimits_{|\nu| \leqq \mu} ((-\partial)^\nu v, f_\nu) \quad (\forall v \in H_0^\mu(\Omega)),$$

$$\|f\|_{-\mu} = \Big(\sum\nolimits_{|\nu| \leqq \mu} \|f_\nu\|^2\Big)^{1/2}.$$

Let $f \in H^{-\mu}(\Omega)$. Let $f^{\sim}$ be an extension of f defined by

$$f^{\sim} : H^{\mu}(\Omega) \ni v \mapsto (v, f^{\sim}) = \sum_{|\nu| \leqq \mu} ((-\partial)^{\nu} v, f_{\nu}) \in \mathbf{C},$$

then $f^{\sim} \in (H^{\mu}(\Omega))'$. Define

$$|f^{\sim}|_{-\mu} = \sup_{v \in H^{\mu}(\Omega)} \frac{|(v, f^{\sim})|}{\|v\|_{\mu}},$$

then $|f^{\sim}|_{-\mu} = \|f\|_{-\mu}$. In fact, since $H_0^{\mu}(\Omega) \subset H^{\mu}(\Omega)$, we have

$$\|f\|_{-\mu} \leqq |f^{\sim}|_{-\mu}.$$

On the other hand, we have

$$\begin{aligned} |f^{\sim}|_{-\mu} &= \sup_{v \in H^{\mu}(\Omega)} \frac{|\sum_{|\nu| \leqq \mu} ((-\partial)^{\nu} v, f_{\nu})|}{\|v\|_{\mu}} \\ &\leqq \left(\sum_{|\nu| \leqq \mu} \|f_{\nu}\|^2\right)^{1/2} = \|f\|_{-\mu}. \end{aligned}$$

Let $f \in H^{-\mu}(\Omega)$, then $f^{\sim} \in (H^{\mu}(\Omega))'$. Since $\mathscr{H} \subset H^{\mu}(\Omega)$,

$$f^{\sim}|_{\mathscr{H}} : \mathscr{H} \ni v \mapsto (v, f^{\sim}) = \sum_{|\nu| \leqq \mu} ((-\partial)^{\nu} v, f_{\nu}) \in \mathbf{C}$$

belongs to $\mathscr{H}'$. In fact,

$$\begin{aligned} |(v, f^{\sim})| &\leqq \left(\sum_{|\nu| \leqq \mu} \|(-\partial)^{\nu} v\|^2\right)^{1/2} \left(\sum_{|\nu| \leqq \mu} \|f_{\nu}\|^2\right)^{1/2} \\ &= \|v\|_{\mu} \|f\|_{-\mu} \leqq C[v] \|f\|_{-\mu}. \end{aligned}$$

Owing to Riesz' Theorem in $\mathscr{H}$, there exists $w \in \mathscr{H}$ such that

$$\begin{cases} [w, v] = (f^{\sim}, v) \quad (\forall v \in \mathscr{H}) \\ [w] \leqq C \|f\|_{-\mu}, \end{cases}$$

where

$$(f^{\sim}, v) = \overline{(v, f^{\sim})}.$$

We say that w is a Riesz' function of $f \in H^{-\mu}(\Omega)$, iff

$$[w, v] = (f^{\sim}, v) \quad (\forall v \in \mathscr{H})$$

holds. We say that u is a $\mathscr{H}$-solution of

$$\text{(P)} \qquad \begin{cases} Au = f & \text{in } \Omega \\ B_j u = 0 & \text{on } \Gamma \ (j \in J) \end{cases}$$

for $f \in H^{-\mu}(\Omega)$, iff it holds

$$\begin{cases} u = A^* w & (\exists w \in \mathscr{H}) \\ [w, v] = (f^{\sim}, v) & (\forall v \in \mathscr{H}). \end{cases}$$

REMARK. Let u be a $\mathscr{H}$-solution of (P) for $f \in H^{-\mu}(\Omega)$, then

$$[w, v] = (f^{\sim}, v) \quad (\forall v \in \mathscr{D}(\Omega))$$

holds, that is,

$$(A^* w, A^* v) = \sum_{|\nu| \leqq \mu} (f_\nu, (-\partial)^\nu v) \quad (\forall v \in \mathscr{D}(\Omega)),$$

that is,

$$(u, A^* v) = (f, v), \quad f = \sum_{|\nu| \leqq \mu} \partial^\nu f_\nu \quad (\forall v \in \mathscr{D}(\Omega)).$$

The following Theorem I′ and Theorem II′ are the generalizations of Theorem I and Theorem II in §1.

THEOREM I′. *Assume* $(\mathrm{E}^*)_\mu$. *Then, there exists a* $\mathscr{H}$*-solution* u *of* (P) *for* $f \in H^{-\mu}(\Omega)$ *and it holds*

$$\|u\| \leqq C\|f\|_{-\mu}.$$

THEOREM II′. *Assume* $(\mathrm{E}^*)_\mu$. *Let* $\{v_k \ (k = 1, 2, \ldots)\}$ *be a basis of* $\mathscr{H}$. *Let* u *be a* $\mathscr{H}$*-solution of* (P) *for* $f \in H^{-\mu}(\Omega)$. *Set*

$$u_N = ((f^{\sim}, v_1), \ldots, (f^{\sim}, v_N))\Gamma_N^{-1} \begin{pmatrix} A^* v_1 \\ \vdots \\ A^* v_N \end{pmatrix},$$

where

$$\Gamma_N = ([v_k, v_s])_{k,s=1,2,\ldots,N}.$$

Then it holds

$$u_N \to u \quad (N \to \infty) \ \text{in } L^2(\Omega).$$

Chapter 2. Numerical Approximation in Non-liner Case

§3. Sobolev's Imbedding Theorem

SOBOLEV'S IMBEDDING THEOREM (See [3]). *Let*

$$0 < \alpha < \beta < \infty, \quad -\infty < \gamma, \delta < \infty, \quad \gamma - n/\alpha \geqq \delta - n/\beta.$$

Then

$$W^{\gamma,\alpha}(\Omega) \subset W^{\delta,\beta}(\Omega)$$

and

$$\|U\|_{W^{\delta,\beta}(\Omega)} \leqq C\|U\|_{W^{\gamma,\alpha}(\Omega)} \quad (\forall U \in W^{\gamma,\alpha}(\Omega)).$$

COROLLARY. *Let* $(n/2)(p-1) \leqq s$ $(p, s \in \mathbf{N})$. *Then it holds*

$$L^{2/p}(\Omega) \subset H^{-s}(\Omega)$$

and

$$\|U\|_{H^{-s}(\Omega)} \leqq C\|U\|_{L^{2/p}(\Omega)} \quad (\forall U \in L^{2/p}(\Omega)).$$

In fact, set

$$\alpha = 2/p, \quad \beta = 2, \quad \gamma = 0, \quad \delta = -s$$

in Sobolev's Theorem. Since

$$(\gamma - n/\alpha) - (\delta - n/\beta) = (-np/2) - (-s - n/2) = s - (n/2)(p-1) \geqq 0,$$

we have

$$\|U\|_{W^{-s,2}(\Omega)} \leqq C\|U\|_{W^{0,2/p}(\Omega)},$$

that is

$$\|U\|_{H^{-s}(\Omega)} \leqq C\|U\|_{L^{2/p}(\Omega)}.$$

LEMMA 3.1. *Assume that* $(n/2)(p-1) \leqq s$ $(p, s \in \mathbf{N})$. *Then it holds*

$$\|u^p\|_{-s} \leqq C\|u\|^p \quad (\forall u \in L^2(\Omega)).$$

In fact, let $u \in L^2(\Omega)$, then $U = u^p \in L^{2/p}(\Omega)$ and

$$\|u^p\|_{-s} \leqq C\|u\|^p.$$

LEMMA 3.2. *Assume that* $(n/2)(p-1) \leqq s$ $(p, s \in \mathbf{N})$. *Then it holds*

$$\|u^p - v^p\|_{-s} \leqq C(\|u\| + \|v\|)^{p-1}\|u - v\| \quad (\forall u, v \in L^2(\Omega)).$$

PROOF. We have

$$\|u^p - v^p\|_{-s} \leqq C\|u^p - v^p\|_{L^{2/p}}$$

from Corollary. On the other hand, we have

$$\begin{aligned}(\|u^p - v^p\|_{L^{2/p}})^{2/p} &= \int |u^p - v^p|^{2/p}\,dx = \int |(u-v)(u^{p-1} + u^{p-2}v + \cdots + v^{p-1})|^{2/p}\,dx \\ &\leqq \int |u - v|^{2/p}(|u| + |v|)^{2(p-1)/p}\,dx \\ &\leqq \left(\int |u - v|^2\,dx\right)^{1/p}\left(\int (|u| + |v|)^2\,dx\right)^{(p-1)/p} \\ &\leqq C\|u - v\|^{2/p}(\|u\| + \|v\|)^{2(p-1)/p}.\end{aligned}$$

Hence we have

$$\|u^p - v^p\|_{-s} \leqq C\|u - v\|(\|u\| + \|v\|)^{p-1}. \qquad \square$$

Let Q be a linear partial differential operator of order ℓ $(\ell \leqq \mu)$. Let $u \in L^2(\Omega)$ and let p satisfy

$$(n/2)(p-1) + \ell \leqq \mu,$$

then we have $u^p \in H^{-(\mu-\ell)}(\Omega)$ from Lemma 3.1. Let

$$u^p = \sum_{|\nu| \leqq \mu - \ell} \partial^\nu U_\nu$$

$$\|u^p\|_{-(\mu-\ell)} = \left(\sum_{|\nu| \leqq \mu-\ell} \|U_\nu\|^2\right)^{1/2},$$

and set

$$Q(u^p)^\sim : H^\mu(\Omega) \ni v \mapsto (v, Q(u^p)^\sim) = (Q^* v, (u^p)^\sim).$$

Then we have

$$(v, Q(u^p)^{\sim}) = \sum_{|\nu| \leqq \mu-\ell}((-\partial)^{\nu} Q^* v, U_\nu),$$

$$|(v, Q(u^p)^{\sim})| \leqq C\|v\|_\mu \|u\|^p.$$

Hence we have

$$|Q(u^p)^{\sim}|_{-\mu} \leqq C\|u\|^p.$$

§4. Successive Approximation (Case 1)

In this section, we consider

NON-LINEAR PROBLEM: To seek a $\mathscr{H}$-solution $u \in L^2(\Omega)$ of

$$\text{(Q-1)} \qquad \begin{cases} Au = Q(u^p) + f & \text{in } \Omega \\ B_j u = 0 & \text{on } \Gamma \ (j \in J) \end{cases}$$

for given $f \in H^{-\mu}(\Omega)$, where Q is a linear partial differential operator of order ℓ satisfying

$$(n/2)(p-1) + \ell \leqq \mu,$$

where a $\mathscr{H}$-solution $u \in L^2(\Omega)$ of (Q-1) for $f \in H^{-\mu}(\Omega)$ means

$$\begin{cases} u = A^* w & (\exists w \in \mathscr{H}) \\ [w, v] = (Q(u^p)^{\sim} + f^{\sim}, v) & (\forall v \in \mathscr{H}). \end{cases}$$

THEOREM III. *Assume* $(\mathrm{E}^*)_\mu$. *Then there exists a positive number* η *such that there exists a* $\mathscr{H}$*-solution* $u \in L^2(\Omega)$ *of* (Q-1) *for* $f \in H^{-\mu}(\Omega)$ *satisfying*

$$\|f\|_{-\mu} \leqq \eta^p.$$

Let us prove this by two methods, i.e. by a method of simple successive approximation and by a method of double successive approximation.

SIMPLE SUCCESSIVE APPROXIMATION: "Let $u_0 \in L^2(\Omega)$ be given. Let $u_k \in L^2(\Omega)$ be a $\mathscr{H}$-solution of

$$\text{(Q-1)}_k \qquad \begin{cases} Au_k = Q(u_{k-1}^p) + f & \text{in } \Omega \\ B_j u_k = 0 & \text{on } \Gamma \ (j \in J) \end{cases}$$

for $f \in H^{-\mu}(\Omega)$ $(k = 1, 2, \ldots)$. Then $u_k \to u$ in $L^2(\Omega)$ and u is a $\mathscr{H}$-solution of (Q-1) for $f \in H^{-\mu}(\Omega)$."

PROOF ALONG THE LINE OF SIMPLE SUCCESSIVE APPROXIMATION.

(1) Let $u_{k-1} \in L^2(\Omega)$. Then there exists a $\mathscr{H}$-solution $u_k \in L^2(\Omega)$ of $(\text{Q-1})_k$ such that

$$\|u_k\| \leqq C_1(\|u_{k-1}^p\|_{-(\mu-\ell)} + \|f\|_{-\mu}) \quad \cdots ①$$

from Theorem I′. Since

$$\|u_{k-1}^p\|_{-(\mu-\ell)} \leqq C_2\|u_{k-1}\|^p \quad \cdots ②$$

from Lemma 3.1, we have

$$\|u_k\| \leqq C(\|u_{k-1}\|^p + \|f\|_{-\mu}) \quad \cdots ③$$

where $C = \max(C_1, C_1C_2)$.

(2) Let η be a positive number satisfying

$$2C\eta^{p-1} \leqq 1 \quad \ldots (☆).$$

Let

$$\|u_0\| \leqq \eta, \quad \|f\|_{-\mu} \leqq \eta^p,$$

then

$$\|u_k\| \leqq \eta \quad (k = 1, 2, \ldots).$$

In fact, assuming

$$\|u_{k-1}\| \leqq \eta,$$

we have

$$\|u_k\| \leqq 2C\eta^p \leqq \eta$$

from ③.

(3) $\{u_k \ (k = 1, 2, \ldots)\}$ is a Cauchy sequence in $L^2(\Omega)$. In fact, setting $U_k = u_{k+1} - u_k$, U_k is a $\mathscr{H}$-solution of

$$\begin{cases} A(U_k) = Q(u_k^p) - Q(u_{k-1}^p) & \text{in } \Omega \\ B_j(U_k) = 0 & \text{on } \Gamma \ (j \in J). \end{cases}$$

Therefore we have,

$$\|U_k\| \leqq C_1\|u_k^p - u_{k-1}^p\|_{-(\mu-\ell)} \quad \cdots ④$$

from Theorem I′. On the other hand, we have

$$\|u_k^p - u_{k-1}^p\|_{-(\mu-\ell)} \leqq C_3(\|u_k\| + \|u_{k-1}\|)^{p-1}\|u_k - u_{k-1}\| \quad \cdots ⑤$$

from Lemma 3.2. Hence we have

$$\|U_k\| \leqq C_1 C_3(\|u_k\| + \|u_{k-1}\|)^{p-1}\|U_{k-1}\| \leqq C_1 C_3(2\eta)^{p-1}\|U_{k-1}\|.$$

Let η be a positive number satisfying

$$C_1 C_3(2\eta)^{p-1} \leqq 1/2 \quad \ldots (☆☆)$$

in addition to (☆), then we have

$$\|U_k\| \leqq 2^{-1}\|U_{k-1}\|.$$

Hence we have

$$\|U_k\| \leqq 2^{-k}\|U_0\|,$$

which means that $\{u_k\}$ is a Cauchy sequence in $L^2(\Omega)$. Let u be the limit of $\{u_k\}$.

(4) u is $\mathscr{H}$-solution of (Q-1). In fact, since u_k is a $\mathscr{H}$-solution of $(\text{Q-1})_k$, there exists $w_k \in \mathscr{H}$ such that

$$\begin{cases} u_k = A^* w_k \\ [w_k, v] = ((Q(u_{k-1}^p) + f)^\sim, v) \quad (v \in \mathscr{H}). \end{cases}$$

Hence we have

$$[w_{k+1} - w_k, v] = (Q(u_k^p - u_{k-1}^p)^\sim, v) \quad (v \in \mathscr{H}),$$

therefore,

$$[w_{k+1} - w_k]^2 = (Q(u_k^p - u_{k-1}^p)^\sim, w_{k+1} - w_k) \leqq |(u_k^p - u_{k-1}^p)^\sim|_{-(\mu-\ell)}\|w_{k+1} - w_k\|_\mu,$$

that is,

$$[w_{k+1} - w_k] \leqq C_1\|u_k^p - u_{k-1}^p\|_{-(\mu-\ell)} \quad \ldots ⑥.$$

Hence we have

$$[w_{k+1} - w_k] \leqq 2^{-1}\|u_k - u_{k-1}\| \quad (\leqq 2^{-k}\|U_0\|)$$

from ⑤ and ⑥, which means that

$$w_k \longrightarrow w \quad \text{in } \mathscr{H}.$$

Now, let $k \to \infty$ in

$$\begin{cases} u_k = A^* w_k \\ [w_k, v] = ((Q(u_{k-1}^p) + f)^\sim, v) \quad (v \in \mathscr{H}), \end{cases}$$

then we have

$$\begin{cases} u = A^*w \\ [w,v] = ((Q(u^p)+f)^{\sim},v) \quad (v \in \mathscr{H}). \end{cases} \qquad \square$$

Double successive approximation: "Let $\phi_0 \in L^2(\Omega)$ be given. Let $u_1 \in L^2(\Omega)$ be a $\mathscr{H}$-solution of

$$(\mathrm{Q}^{\sim}\text{-}1)_1 \qquad \begin{cases} Au_1 = Q(\phi_0^p) + f & \text{in } \Omega \\ B_j u_1 = 0 & \text{on } \Gamma \ (j \in J). \end{cases}$$

Let $\phi_1 \in L^2(\Omega)$ belong to a neighborhood of u_1 in $L^2(\Omega)$. Successively, let $u_k \in L^2(\Omega)$ be $\mathscr{H}$-solution of

$$(\mathrm{Q}^{\sim}\text{-}1)_k \qquad \begin{cases} Au_k = Q(\phi_{k-1}^p) + f & \text{in } \Omega \\ B_j u_k = 0 & \text{on } \Gamma \ (j \in J), \end{cases}$$

and let $\phi_k \in L^2(\Omega)$ belong to a neighborhood of $u_k \in L^2(\Omega)$. We can choose $\{\phi_k\}$ such that $\phi_k \to u$. Then u is a $\mathscr{H}$-solution of (Q-1)."

Proof along the line of double successive approximation.

(1)′ Let $\phi_{k-1} \in L^2(\Omega)$ be given, then there exists a $\mathscr{H}$-solution $u_k \in L^2(\Omega)$ of $(\mathrm{Q}^{\sim}\text{-}1)_k$ from Theorem I′. Choose ϕ_k satisfying

$$\|u_k - \phi_k\| \leqq 2^{-k}\eta,$$

where η satisfies (☆) and (☆☆).

(2)′ Let η satisfy

$$4C\eta^{p-1} \leqq 1 \quad \ldots (☆☆☆)$$

in addition to (☆) and (☆☆). Let

$$\|\phi_0\| \leqq \eta, \quad \|f\|_{-\mu} \leqq \eta^p.$$

Suppose that $\phi_{k-1} \in L^2(\Omega)$ satisfy

$$\|\phi_{k-1}\| \leqq \eta,$$

then we have

$$\|u_k\| \leqq 2^{-1}\eta.$$

Since

$$\|u_k - \phi_k\| \leqq 2^{-k}\eta \leqq 2^{-1}\eta \quad (k = 1, 2, \ldots),$$

we have

$$\|\phi_k\| \leqq \eta \quad (k = 1, 2, \dots).$$

(3)′ Set $U_k = u_{k+1} - u_k$, U_k is a $\mathscr{H}$-solution of

$$\begin{cases} A(U_k) = Q(\phi_k^p) - Q(\phi_{k-1}^p) & \text{in } \Omega \\ B_j(U_k) = 0 & \text{on } \Gamma \ (j \in J). \end{cases}$$

Hence we have

$$\|U_k\| \leqq C_1 \|\phi_k^p - \phi_{k-1}^p\|_{-(\mu-\ell)} \quad \cdots ④'$$

from Theorem I′. Since

$$\|\phi_k^p - \phi_{k-1}^p\|_{-(\mu-\ell)} \leqq C_3(\|\phi_k\| + \|\phi_{k-1}\|)^{p-1} \|\phi_k - \phi_{k-1}\| \quad \cdots\cdots ⑤'$$

from Lemma 3.2, we have

$$\|U_k\| \leqq 2^{-1} \|\phi_k - \phi_{k-1}\|.$$

Now, since

$$\begin{aligned} \|\phi_{k+1} - \phi_k\| &\leqq \|\phi_{k+1} - u_{k+1}\| + \|u_{k+1} - u_k\| + \|u_k - \phi_k\| \\ &\leqq 2^{-k-1}\eta + 2^{-1}\|\phi_k - \phi_{k-1}\| + 2^{-k}\eta \leqq 2^{-1}\|\phi_k - \phi_{k-1}\| + 2^{-(k-1)}\eta, \end{aligned}$$

we have

$$\|\phi_{k+1} - \phi_k\| \leqq 2^{-k}\|\phi_1 - \phi_0\| + k2^{-(k-1)}\eta,$$

which means that

$$u_k, \phi_k \to u \quad \text{in } L^2(\Omega).$$

It is proved, analogously as in (4), that u is a $\mathscr{H}$-solution of (Q-1). □

§ 5. Successive Approximation (Case 2)

In this section, we consider

NON-LINER PROBLEM: To seek a $\mathscr{H}$-solution $u \in H^{\mu'}(\Omega)$ of

$$\text{(Q-2)} \qquad \begin{cases} Au = Q[u] + f & \text{in } \Omega \\ B_j u = 0 & \text{on } \Gamma \ (j \in J) \end{cases}$$

for given $f \in H^{-(\mu-\mu')}(\Omega)$, where

$$Q[u] = \sum_{|\nu| \leqq \ell} \partial^{\nu} \pi_{\nu}[u],$$

where

$$\pi_{\nu}[u] = \sum_{2 \leqq h \leqq p} \sum_{|\beta_k| \leqq \mu'} C_{\nu h \beta_1 \cdots \beta_h} (\partial^{\beta_1} u) \cdots\cdots (\partial^{\beta_h} u),$$

and $(n/2)(p-1) \leqq \mu - \mu' - \ell$.

Before saying details, we will prepare some lemmas. Let $p, h (\geqq 2)$ be integers, then it is obvious that

$$(u_1 + u_2 + \cdots + u_h)^p = \sum_{p_1 + p_2 + \cdots + p_h = p} \left\{ \frac{p!}{p_1! p_2! \cdots p_h!} (u_1)^{p_1} (u_2)^{p_2} \cdots (u_h)^{p_h} \right\},$$

therefore

$$\begin{aligned}(\alpha_1 u_1 &+ \alpha_2 u_2 + \cdots + \alpha_h u_h)^p \\ &= \sum_{p_1 + p_2 + \cdots + p_h = p} (\alpha_1)^{p_1} (\alpha_2)^{p_2} \cdots (\alpha_h)^{p_h} \left\{ \frac{p!}{p_1! p_2! \cdots p_h!} (u_1)^{p_1} (u_2)^{p_2} \cdots (u_h)^{p_h} \right\}.\end{aligned}$$

Conversely, we have

LEMMA 5.1. *Let $p, h (\geqq 2)$ be integers and*

$$\alpha_1(t) = 1, \quad \alpha_{k+1}(t) = t \alpha_k(t)^p \quad (k = 1, \ldots, h).$$

Then there exist $\{t_1, t_2, \ldots, t_N\}$ and $\{c_1, c_2, \ldots, c_N\}$ such that

$$\left\{ \frac{p!}{p_1! p_2! \cdots p_h!} (u_1)^{p_1} (u_2)^{p_2} \cdots (u_h)^{p_h} \right\} = \sum\nolimits_{1 \leqq k \leqq N} c_k (v_k)^p$$

and

$$v_k = \alpha_1(t_k) u_1 + \alpha_2(t_k) u_2 + \cdots + \alpha_h(t_k) u_h,$$

where N is a number of combination $(p_1, p_2, \ldots, p_h)$ satisfying

$$p_1 + p_2 + \cdots + p_h = p \quad (p_k\text{: non-negative integer}).$$

PROOF. Let us prove that the linear system

$$V_k = \sum_{p_1 + p_2 + \cdots + p_h = p} (\alpha_1(t_k))^{p_1} (\alpha_2(t_k))^{p_2} \cdots (\alpha_h(t_k))^{p_h} U_{p_1 p_2 \cdots p_h} \quad (k = 1, 2, \ldots, N)$$

can be solved with respect to $\{U_{p_1p_2\cdots p_h}(p_1+p_2+\cdots+p_h=p)\}$, choosing $\{t_1,t_2,\dots,t_N\}$ suitably. Since N is the number of combinations $(p_2,\dots,p_h)$ satisfying $p_2+\cdots+p_h\leqq p$, where $p_1=p-(p_2+\cdots+p_h)$. Set

$$W_{p_2\cdots p_h}=U_{p_1p_2\cdots p_h}\quad (p_1=p-(p_2+\cdots+p_h)),$$
$$C_{p_2\cdots p_h}(t)=(\alpha_2(t))^{p_2}\cdots(\alpha_h(t))^{p_h},$$

and moreover

$$Z_1=W_{0,0,\dots,0},\quad Z_2=W_{1,0,\dots,0},\quad Z_3=W_{2,0,\dots,0},\quad \dots,\quad Z_{p+1}=W_{p,0,\dots,0},$$
$$Z_{p+2}=W_{0,1,0,\dots,0},\quad Z_{p+3}=W_{1,1,0,\dots,0},\quad \dots,\quad Z_{2p+1}=W_{p-1,1,0,\dots,0},$$
$$Z_{2p+2}=W_{0,2,0,\dots,0},\quad Z_{2p+3}=W_{1,2,0,\dots,0},\quad \dots,\quad Z_{3p}=W_{p-2,2,0,\dots,0},$$
$$\dots\dots\dots,$$
$$D_1=C_{0,0,\dots,0},\quad D_2=C_{1,0,\dots,0},\quad D_3=C_{2,0,\dots,0},\quad \dots,\quad D_{p+1}=C_{p,0,\dots,0},$$
$$D_{p+2}=C_{0,1,0,\dots,0},\quad D_{p+3}=C_{1,1,0,\dots,0},\quad \dots,\quad D_{2p+1}=C_{p-1,1,0,\dots,0},$$
$$D_{2p+2}=C_{0,2,0,\dots,0},\quad D_{2p+3}=C_{1,2,0,\dots,0},\quad \dots,\quad D_{3p}=C_{p-2,2,0,\dots,0},$$
$$\dots\dots\dots$$

Then the above system become

$$V_k=\sum\nolimits_{1\leqq j\leqq N}D_j(t_k)Z_j\quad (k=1,2,\dots,N).$$

By the definition of $\alpha_k(t)$, we have

$$D_j=t^{s(j)},$$

where

$$0=s(1)<s(2)<\cdots<s(N).$$

Hence, choosing $1<t_1<t_2<\cdots<t_N$ suitably, the above system can be solvable with respect to $\{Z_j\}$. □

LEMMA 5.2. *Let* $(n/2)(p-1)\leqq s$. *Suppose that*

$$u_1,u_2,\dots,u_p\in L^2(\Omega),$$

then

$$\|u_1u_2\cdots u_p\|_{-s}\leqq C(\|u_1\|+\|u_2\|+\cdots+\|u_p\|)^p.$$

PROOF. For $u_1, u_2, \ldots, u_p \in L^2(\Omega)$, set

$$v_k = \alpha_1(t_k)u_1 + \alpha_2(t_k)u_2 + \cdots + \alpha_p(t_k)u_p \in L^2(\Omega),$$

$$\|v_k\| \leqq C(\|u_1\| + \|u_2\| + \cdots + \|u_p\|),$$

where $\{\alpha_j(t_k)\}$ are given in Lemma 5.1. Hence we have

$$v_k^p \in H^{-s}(\Omega), \quad \|v_k^p\|_{-s} \leqq C\|v_k\|^p$$

from Lemma 3.1. Owing to Lemma 5.1, we have

$$u_1 u_2 \cdots u_p \in H^{-s}(\Omega)$$

$$\|u_1 u_2 \cdots u_p\|_{-s} \leqq C \sum\nolimits_k \|v_k^p\|_{-s} \leqq C'\Big(\sum\nolimits_k \|v_k\|\Big)^p \leqq C''\Big(\sum\nolimits_k \|u_k\|\Big)^p. \qquad \square$$

LEMMA 5.3. *Let* $(n/2)(p-1) \leqq s$. *Then*

$$\|u_1 u_2 \cdots u_p - v_1 v_2 \cdots v_p\|_{-s} \leqq C\Big\{1 + \Big(\sum\nolimits_k \|u_k\| + \sum\nolimits_k \|v_k\|\Big)^{p-1}\Big\}\|u_k - v_k\|$$

$$(\forall u_k, v_k \in L^2(\Omega)).$$

PROOF. We have

$$\|u_1 u_2 \cdots u_p - v_1 v_2 \cdots v_p\|_{-s} \leqq C\|u_1 u_2 \cdots u_p - v_1 v_2 \cdots v_p\|_{L^{2/p}}$$

from Corollary. On the other hand, we have

$$\begin{aligned}
&(\|u_1 u_2 \cdots u_p - v_1 v_2 \cdots v_p\|_{L^{2/p}})^{2/p} \\
&\quad = \int |u_1 u_2 \cdots u_p - v_1 v_2 \cdots v_p|^{2/p}\, dx \\
&\quad = \int \Big|\sum\nolimits_k u_1 u_2 \cdots u_{k-1}(u_k - v_k)v_{k+1}v_{k+2}\cdots v_p\Big|^{2/p} dx \\
&\quad \leqq C\sum\nolimits_k \int |u_k - v_k|^{2/p}(|u_1| + \cdots + |u_p| + |v_1| + \cdots + |v_p|)^{2(p-1)/p}\, dx \\
&\quad \leqq C\sum\nolimits_k \Big(\int |u_k - v_k|^2\, dx\Big)^{1/p}\Big\{\int (|u_1| + \cdots + |u_p| + |v_1| + \cdots + |v_p|)^2\, dx\Big\}^{(p-1)/p} \\
&\quad \leqq C'\Big(\sum\nolimits_k \|u_k - v_k\|\Big)^{2/p}\Big(\sum\nolimits_k \|u_k\| + \sum\nolimits_k \|v_k\|\Big)^{2(p-1)/p} \qquad \square
\end{aligned}$$

Now we come back to the non-linear problem (Q-2), under the assumption

$$(\mathrm{E}^*)_\mu \qquad \|v\|_\mu \leqq C\Big(\|A^*v\| + \sum\nolimits_{j\in J^*} \langle \mathscr{B}_j^* v\rangle_{\sigma_j}\Big) \quad (\forall v \in H^M(\Omega))$$

$(0 \leqq \mu \leqq m)$. Moreover, we assume $(\mathrm{A})_{\mu'}$ $(0 \leqq \mu' \leqq \mu)$.

ADDITIONAL ASSUMPTION. A $\mathscr{H}$-solution u of (P) for $f \in H^{-(\mu-\mu')}(\Omega)$ satisfies

$$(\mathrm{A})_{\mu'} \qquad \|u\|_{\mu'} \leqq C\|f\|_{-(\mu-\mu')}.$$

Let us consider a $\mathscr{H}$-solution $u \in H^{\mu'}(\Omega)$ of non-linear problem

$$(\text{Q-2}) \qquad \begin{cases} Au = Q[u] + f & \text{in } \Omega \\ B_j u = 0 & \text{on } \Gamma \ (j \in J) \end{cases}$$

for $f \in H^{-(\mu-\mu')}(\Omega)$, where $(n/2)(p-1) \leqq \mu - \mu' - \ell$.

What is $\mathscr{H}$-solution $u \in H^{\mu'}(\Omega)$ of (Q-2) for $f \in H^{-(\mu-\mu')}(\Omega)$? Since

$$(n/2)(p-1) \leqq \mu - \mu' - \ell,$$

we have

$$(\partial^{\beta_1}u), \dots, (\partial^{\beta_h}u) \in L^2(\Omega) \quad (|\beta_k| \leqq \mu').$$

Hence we have

$$\|(\partial^{\beta_1}u)\cdots(\partial^{\beta_h}u)\|_{-(\mu-\mu'-\ell)} \leqq C(\|u\|_{\mu'})^h$$

from Lemma 5.2, therefore,

$$\|\pi_\nu[u]\|_{-(\mu-\mu'-\ell)} \leqq C\{(\|u\|_{\mu'})^2 + (\|u\|_{\mu'})^p\}.$$

Hence we have

$$\pi_\nu[u]^\sim \in (H^{\mu-\mu'-\ell}(\Omega))'.$$

Moreover, we define

$$Q[u]^\sim : H^{\mu-\mu'}(\Omega) \ni v \mapsto (v, Q[u]^\sim) = \sum_{|\nu|\leqq\ell} ((-\partial)^\nu v, \pi_\nu[u]^\sim)$$

and we define $\mathscr{H}$-solution $u \in H^{\mu'}(\Omega)$ of (Q-2) for $f \in H^{-(\mu-\mu')}(\Omega)$ by

$$\begin{cases} u = A^*w, & w \in \mathscr{H} \\ [w, v] = (Q[u]^\sim + f^\sim, v) & (\forall\, v \in \mathscr{H}). \end{cases}$$

THEOREM IV. *Assume* $(E^*)_\mu$ *and* $(A)_{\mu'}$ $(0 \leqq \mu \leqq m, 0 \leqq \mu' \leqq \mu)$. *Then there exists a positive number* η *such that there exists a* $\mathscr{H}$*-solution* $u \in H^{\mu'}(\Omega)$ *for* $f \in H^{-(\mu-\mu')}(\Omega)$, *satisfying* $\|f\|_{-(\mu-\mu')} \leqq \eta$.

Theorem IV is proved analogously to Theorem III, by using Lemma 5.2 and Lemma 5.3.

Appendix

In §2, we considered a generalized linear boundary value problem:

$$\text{(P)} \qquad \begin{cases} Au = f & \text{in } \Omega \\ B_j u = 0 & (j \in J) \text{ on } \Gamma \end{cases}$$

for $f \in H^{-\mu}(\Omega)$ $(0 \leqq \mu \leqq m)$, where

$$A = \sum\nolimits_{|\nu| \leqq m} a_\nu(x) \partial_x^\nu, \quad a_\nu(x) \in C^\infty(\bar{\Omega}),$$

$$B_j = \sum\nolimits_{|\nu| \leqq j} b_{j\nu}(x) \partial_x^\nu \quad (j \in J), \quad b_{j\nu}(x) \in C^\infty(\Gamma)$$

$(\Gamma$ is non-characteristic for $\{A, B_j\})$.

We proved the existence of a $\mathscr{H}$-solution of (P). We also proved that a $\mathscr{H}$-solution of (P) satisfies $Au = f$ in Ω. Does a $\mathscr{H}$-solution of (P) satisfy $B_j u = 0$ $(j \in J)$ on Γ? Here, in Appendix, we will see "yes" if orders of boundary operators are less than $m - \mu$, that is, $J \subset \{0, 1, \ldots, m-1-\mu\}$ i.e. $\{0, 1, \ldots, \mu - 1\} \subset J^*$, where

$$J^c \cup J = \{0, 1, \ldots, m-1\}, \quad J^c \cap J = \phi,$$

$$J^* = \{j \mid m - 1 - j \in J^c\}.$$

Relating to (P), there exist linear differential operators $\{\mathscr{B}_j^* \ (j = 0, 1, \ldots, m-1)\}$ of order $\{j\}$ such that Γ is non-characteristic for $\mathscr{B}_j^*$ and the following Theorem holds. The adjoint problem is defined by

$$\text{(P*)} \qquad \begin{cases} A^* v = g & \text{in } \Omega \\ \mathscr{B}_j^* v = g_j & (j \in J^*) \text{ on } \Gamma. \end{cases}$$

THEOREM (Green's Theorem). *Let* $u \in L^2(\Omega)$, $Au = f \in H^{-\mu}(\Omega)$ $(0 \leqq \mu \leqq m)$, *then*

$$\langle (d/dn)^k u|_\Gamma \rangle_{-k-m+1/2} \leqq C(\|u\| + \|f\|_{-\mu}) \quad (k = 0, 1, \ldots, m-1-\mu)$$

and

$$(f^{\sim}, v) - (u, A^*v) = -\sum_{j\in J} \langle B_j u|_\Gamma, \mathscr{B}^*_{m-1-j} v|_\Gamma\rangle$$
$$(\forall v \in H^{2m}(\Omega), \mathscr{B}^*_j v|_\Gamma = 0 \ \ (j \in J^*)),$$

where $n = n(x)$ *is the unit outer normal at* x *on* Γ *and*

$$f = \sum_{|\nu|\leq\mu} (\partial)^\nu f_\nu, \quad \|f\|^2_{-\mu} = \sum_{|\nu|\leq\mu} \|f_\nu\|^2,$$

$$(f^{\sim}, v) = \sum_{|\nu|\leq\mu} (f_\nu, (-\partial)^\nu v).$$

Let u be a $\mathscr{H}$-solution, then $u = A^*w$ $(\exists w \in \mathscr{H})$ satisfies

$$(u, A^*v) + \sum_{j\in J^*} \langle \mathscr{B}^*_j w, \mathscr{B}^*_j v\rangle_{\sigma j} = (f^{\sim}, v) \quad (\forall v \in \mathscr{H}),$$

therefore, we have

$$(u, A^*v) = (f^{\sim}, v) \quad (\forall v \in H^{2m}(\Omega), \mathscr{B}^*_j v|_\Gamma = 0 \ \ (j \in J^*)).$$

Hence, from Green's Theorem, we have

$$\sum_{j\in J} \langle B_j u|_\Gamma, \mathscr{B}^*_{m-1-j} v|_\Gamma\rangle = 0 \quad (\forall v \in H^{2m}(\Omega), \mathscr{B}^*_j v|_\Gamma = 0 \ \ (j \in J^*)),$$

which means that $B_j u|_\Gamma = 0$ $(j \in J)$.

Hereafter, we prove Green's Theorem, where

$$\Omega = \mathbf{R}^n_+ = (0, \infty) \times \mathbf{R}^{n-1} = \{(x, y) \mid x > 0, y \in \mathbf{R}^{n-1}\}, \quad \Gamma = \{x = 0\} \times \mathbf{R}^{n-1}.$$

The proof is composed of Lemma A.1, Lemma A.2, and Lemma A.3. We use notations:

$$(1) \qquad A = \sum_{0\leq j\leq m} a_{m-j} (\partial_x)^j, \quad a_j = \sum_{|\nu|\leq j} a_{j\nu}(x, y)(\partial_y)^\nu,$$
$$A' = \sum_{0\leq j\leq m} (-\partial_x)^j a'_{m-j}, \quad a'_j = \sum_{|\nu|\leq j} (-\partial_y)^\nu a_{j\nu}(x, y),$$

where $a_0 = a'_0 = 1$,

$$(2) \quad u \in \mathscr{D}'(\mathbf{R}^n_+) : v \in \mathscr{D}(\mathbf{R}^n_+) \to \langle u, v\rangle, \quad u \in \mathscr{D}'(\mathbf{R}^{n-1}) : v \in \mathscr{D}(\mathbf{R}^{n-1}) \to \langle u, v\rangle_y.$$

Let $u \in L^2(\mathbf{R}^n_+)$. Let $v \in C^\infty([0, \infty) \times \mathbf{R}^{n-1})$ have a bounded support and let $\phi \in \mathscr{D}(0, \infty)$, then $v(x, y)\phi(x) \in \mathscr{D}(\mathbf{R}^n_+)$ and

$$\begin{aligned}\langle Au, v\phi\rangle &= \langle u, A'(v\phi)\rangle \\ &= \sum\nolimits_{0\leqq j\leqq m}\left\langle u, \sum\nolimits_{0\leqq k\leqq j}\binom{j}{k}(-\partial_x)^{j-k}(a'_{m-j}v)(-\partial_x)^k\phi\right\rangle \\ &= \sum\nolimits_{0\leqq k\leqq m}\left\langle\left\langle u, \sum\nolimits_{k\leqq j\leqq m}\binom{j}{k}(-\partial_x)^{j-k}(a'_{m-j}v)\right\rangle_y, (-\partial_x)^k\phi\right\rangle_x \\ &= \sum\nolimits_{0\leqq k\leqq m}\langle\langle u, P_{m-k}v\rangle_y, (-\partial_x)^k\phi\rangle_x,\end{aligned}$$

where

$$P_k = \sum\nolimits_{0\leqq j\leqq k}\binom{m-j}{m-k}(-\partial_x)^{k-j}a'_j, \quad \langle u, P_k v\rangle_y \in L^1(0, \infty).$$

Set

$$\langle Au, v\rangle_y = \sum\nolimits_{0\leqq k\leqq m}(\partial_x)^k\langle u, P_{m-k}v\rangle_y \quad (v \in H^m(\mathbf{R}^n_+)),$$

then we have

$$\langle Au, v\phi\rangle = \langle\langle Au, v\rangle_y, \phi\rangle_x.$$

Let $f \in H^{-\mu}(\mathbf{R}^n_+)$ be

$$f = \sum\nolimits_{0\leqq j\leqq\mu}\sum\nolimits_{|\nu|\leqq\mu-j}(\partial_x)^j(\partial_y)^\nu f_{j\nu}, \quad f_{j\nu} \in L^2(\mathbf{R}^n_+),$$

then

$$\begin{aligned}\langle f, v\phi\rangle &= \sum\nolimits_{0\leqq k\leqq\mu}\left\langle\sum\nolimits_{k\leqq j\leqq\mu}\sum\nolimits_{|\nu|\leqq\mu-j}\binom{j}{k}\langle f_{j\nu}, (-\partial_x)^{j-k}(-\partial_y)^\nu v\rangle_y, (-\partial_x)^k\phi\right\rangle_x \\ &= \sum\nolimits_{0\leqq k\leqq\mu}\langle F_k[v], (-\partial_x)^k\phi\rangle_x,\end{aligned}$$

where

$$F_k[v] = \sum\nolimits_{k\leqq j\leqq\mu}\sum\nolimits_{|\nu|\leqq\mu-j}\binom{j}{k}\langle f_{j\nu}, (-\partial_x)^{j-k}(-\partial_y)^\nu v\rangle_y,$$

$$\|F_k[v]\|_{L^1(0,\infty)} \leqq C\|f\|_{-\mu}\|v\|_\mu.$$

Set

$$\langle f, v\rangle_y = \sum\nolimits_{0\leqq k\leqq\mu}(\partial_x)^k F_k[v],$$

then we have

$$\langle f, v\phi\rangle = \langle\langle f, v\rangle_y, \phi\rangle_x.$$

Hence, if $Au = f$ holds for $u \in L^2(\mathbf{R}^n_+)$ and $f \in H^{-\mu}(\mathbf{R}^n_+)$, it holds

$$(*) \quad \sum_{0\leqq k\leqq m}(\partial_x)^k(\langle u, P_{m-k}v\rangle_y - F_k) = 0 \quad \text{in } \mathscr{D}'(0,\infty) \quad (\forall v \in H^m(\mathbf{R}^n_+)),$$

denoting $F_k[v] = 0$ $(\mu + 1 \leqq k \leqq m)$.

LEMMA A.1. *Let* $u \in L^2(\mathbf{R}^n_+)$ *satisfy* $Au = f \in H^{-\mu}(\mathbf{R}^n_+)$, *where*

$$f = \sum_{0\leqq j\leqq \mu}\sum_{|\nu|\leqq \mu-j}(\partial_x)^j(\partial y)^\nu f_{j\nu}, \quad f_{j\nu} \in L^2(\mathbf{R}^n_+).$$

Let $v \in H^{2m}(\mathbf{R}^n_+)$ *satisfy* $\operatorname{supp}[v] \subset \{x \leqq 1\}$. *Then*

(i) $(\partial_x)^j\langle u, v\rangle_y$ *is absolutely continuous in* $(0,1)$ *and*

$$\sup_{0<x<1}|(\partial_x)^j\langle u, v\rangle_y| + \|(\partial_x)^{j+1}\langle u, v\rangle_y\|_{L^1(0,1)} \leqq C(\|u\| + \|f\|_{-\mu})\|v\|_{m+k}$$

holds for $0 \leqq j \leqq m - 1 - \mu$,

(ii) $(\partial_x)^j\langle u, v\rangle_y - \Phi_j[v]$ *is absolutely continuous in* $(0,1)$ *and*

$$\sup_{0<x<1}|(\partial_x)^j\langle u, v\rangle_y - \Phi_j[v]| \leqq C(\|u\| + \|Au\|_{-\mu})\|v\|_{m+j}$$

holds for $m - \mu \leqq j \leqq m$, *where*

$$\begin{aligned}\Phi_j[v] = &-\Phi_{j-1}[P_1v] - \Phi_{j-2}[P_2v] - \cdots - \Phi_{m-\mu}[P_{j-m+\mu}v]\\ &+ (\partial_x)^{j-m+\mu}F_\mu[v] + \cdots + (\partial_x)F_{-j+m+1}[v] + F_{-j+m}[v] \quad (m - \mu \leqq j \leqq m)\end{aligned}$$

i.e.

$$\Phi_{m-\mu}[v] = F_\mu[v],$$

$$\Phi_{m-\mu+1}[v] = -\Phi_{m-\mu}[P_1v] + (\partial_x)F_\mu[v] + F_{\mu-1}[v],$$

$$\Phi_{m-\mu+2}[v] = -\Phi_{m-\mu+1}[P_1v] - \Phi_{m-\mu}[P_2v] + (\partial_x)^2F_\mu[v] + (\partial_x)F_{\mu-1}[v] + F_{\mu-2}[v],$$

$$\cdots$$

$$\begin{aligned}\Phi_m[v] = &-\Phi_{m-1}[P_1v] - \Phi_{m-2}[P_2v] - \cdots - \Phi_{m-\mu}[P_\mu v],\\ &+ (\partial_x)^\mu F_\mu[v] + (\partial_x)^{\mu-1}F_{\mu-1}[v] + \cdots + (\partial_x)F_1[v] + F_0[v].\end{aligned}$$

Proof. (i) Denoting $F_k = F_k[v]$, set

$$g_s = \sum\nolimits_{0 \leqq k \leqq s} (\partial_x)^k (\langle u, P_{s-k} v\rangle_y - F_{m-s+k}) \quad (s = 0, 1, \ldots, m)$$

i.e.

$$g_0 = \langle u, v\rangle_y - F_m \qquad \cdots\cdots \boxed{0}$$

$$g_1 = (\partial_x)(\langle u, v\rangle_y - F_m) + (\langle u, P_1 v\rangle_y - F_{m-1}) \qquad \cdots\cdots \boxed{1}$$

$$g_2 = (\partial_x)^2(\langle u, v\rangle_y - F_m) + (\partial_x)(\langle u, P_1 v\rangle_y - F_{m-1}) + (\langle u, P_2 v\rangle_y - F_{m-2}) \cdots\cdots \boxed{2}$$

$$\cdots$$

$$g_{m-1} = (\partial_x)^{m-1}(\langle u, v\rangle_y - F_m) + (\partial_x)^{m-2}(\langle u, P_1 v\rangle_y - F_{m-1}) + \cdots + (\langle u, P_{m-1} v\rangle_y - F_1) \qquad \cdots\cdots \boxed{m-1}$$

$$0 = (\partial_x)^m(\langle u, v\rangle_y - F_m) + (\partial_x)^{m-1}(\langle u, P_1 v\rangle_y - F_{m-1}) + \cdots + (\langle u, P_m v\rangle_y - F_0) \qquad \cdots\cdots \boxed{m}$$

i.e.

$$g_0 = \langle u, v\rangle_y - F_m \qquad \cdots\cdots \boxed{0}$$

$$g_1 = (\partial_x) g_0 + (\langle u, P_1 v\rangle_y - F_{m-1}) \qquad \cdots\cdots \boxed{1}'$$

$$g_2 = (\partial_x) g_1 + (\langle u, P_2 v\rangle_y - F_{m-2}) \qquad \cdots\cdots \boxed{2}'$$

$$\cdots$$

$$g_{m-1} = (\partial_x) g_{m-2} + (\langle u, P_{m-1} v\rangle_y - F_1) \qquad \cdots\cdots \boxed{m-1}'$$

$$0 = (\partial_x) g_{m-1} + (\langle u, P_m v\rangle_y - F_0) \qquad \cdots\cdots \boxed{m}'.$$

From $\boxed{m}'$, we have

$$(\partial_x) g_{m-1} = -(\langle u, P_m v\rangle_y - F_0),$$

$$\|\langle u, P_m v\rangle_y - F_0\|_{L^1(0,1)} \leqq C(\|u\| + \|f\|_{-\mu})\|v\|_m,$$

therefore, g_{m-1} is absolutely continuous in $(0, 1)$ and it holds

$$\sup\nolimits_{0<x<1} |g_{m-1}| + \|(\partial_x) g_{m-1}\|_{L^1(0,1)} \leqq C(\|u\| + \|f\|_{-\mu})\|v\|_m.$$

Step by step, from $\boxed{m-1}' \sim \boxed{1}'$, we have "$g_s$ is absolutely continuous in $(0, 1)$ and it holds

$\sup_{0<x<1}|g_s| + \|(\partial_x)g_s\|_{L^1(0,1)} \leqq C(\|u\| + \|f\|_{-\mu})\|v\|_m \quad (s = 0, 1, \ldots, m-1)$"

(i-2) Since $F_s = 0$ $(\mu + 1 \leqq s \leqq m)$, we have

$$g_0 = \langle u, v\rangle_y \qquad \cdots\cdots \boxed{0}$$

$$g_1 = (\partial_x)\langle u, v\rangle_y + \langle u, P_1 v\rangle_y \qquad \cdots\cdots \boxed{1}$$

$$g_2 = (\partial_x)^2\langle u, v\rangle_y + (\partial_x)\langle u, P_1 v\rangle_y + \langle u, P_2 v\rangle_y \qquad \cdots\cdots \boxed{2}$$

$$\cdots$$

$$g_{m-\mu-1} = (\partial_x)^{m-\mu-1}\langle u, v\rangle_y + (\partial_x)^{m-\mu-2}\langle u, P_1 v\rangle_y$$
$$+ \cdots + \langle u, P_{m-\mu-1} v\rangle_y \qquad \cdots\cdots \boxed{m-\mu-1}.$$

From $\boxed{0}$, we have $g_0 = \langle u, v\rangle_y$, therefore we have

$\langle u, v\rangle_y$: abs. cont. in $(0, 1)$,

$\sup_{0<x<1}|\langle u, v\rangle_y| + \|(\partial_x)\langle u, v\rangle_y\|_{L^1(0,1)} \leqq C(\|u\| + \|f\|_{-\mu})\|v\|_m$,

therefore we have

$\langle u, P_j v\rangle_y$: abs. cont. in $(0, 1)$,

$\sup_{0<x<1}|\langle u, P_j v\rangle_y| + \|(\partial_x)\langle u, P_j v\rangle_y\|_{L^1(0,1)} \leqq C(\|u\| + \|f\|_{-\mu})\|v\|_{m+j}$.

Now, we have

$$(\partial_x)\langle u, v\rangle_y = g_1 - \langle u, P_1 v\rangle_y$$

from $\boxed{1}$, we have

$(\partial_x)\langle u, v\rangle_y$: abs. cont. in $(0, 1)$,

$\sup_{0<x<1}|(\partial_x)\langle u, v\rangle_y| + \|(\partial_x)^2\langle u, v\rangle_y\|_{L^1(0,1)} \leqq C(\|u\| + \|f\|_{-\mu})\|v\|_{m+1}$,

therefore we have

$$\sup_{0<x<1}|(\partial_x)\langle u, P_j v\rangle_y| + \|(\partial_x)^2\langle u, P_j v\rangle_y\|_{L^1(0,1)}$$
$$\leqq C(\|u\| + \|f\|_{-\mu})\|v\|_{m+1+j}.$$

Step by step, from $\boxed{2}\sim\boxed{m-\mu-1}$, we have "$(\partial_x)^s\langle u, v\rangle_y$ is absolutely continuous in $(0, 1)$, and it holds

$$\sup_{0<x<1}|(\partial_x)^s\langle u,v\rangle_y| + \|(\partial_x)^{s+1}\langle u,v\rangle_y\|_{L^1(0,1)}$$
$$\leqq C(\|u\| + \|f\|_{-\mu})\|v\|_{m+s} \quad (s = 0, 1, \dots, m-\mu-1)."$$

From $\boxed{m-\mu}$, we have

$$(\partial_x)^{m-\mu}\langle u,v\rangle_y - F_\mu[v]$$
$$= g_{m-\mu} - ((\partial_x)^{m-\mu-1}\langle u, P_1 v\rangle_y + \cdots + (\partial_x)\langle u, P_{m-\mu-1}v\rangle_y + \langle u, P_{m-\mu}v\rangle_y),$$

therefore, we have

$$(\partial_x)^{m-\mu}\langle u,v\rangle_y - \Phi_{m-\mu}[v]\text{: abs. cont. in } (0,1),$$

where

$$\Phi_{m-\mu}[v] = F_\mu[v].$$

From $\boxed{m-\mu+1}$, we have

$$g_{m-\mu+1} = (\partial_x)^{m-\mu+1}\langle u,v\rangle_y + \{(\partial_x)^{m-\mu}\langle u, P_1 v\rangle_y - \Phi_{m-\mu}[P_1 v]\} + \Phi_{m-\mu}[P_1 v]$$
$$+ (\partial_x)^{m-\mu-1}\langle u, P_2 v\rangle_y + \cdots + (\partial_x)^2\langle u, P_{m-\mu-1}v\rangle_y$$
$$+ (\partial_x)(\langle u, P_{m-\mu}v\rangle_y - F_\mu[v]) + (\langle u, P_{m-\mu+1}v\rangle_y - F_{\mu-1}[v]),$$

therefore, we have

$$(\partial_x)^{m-\mu+1}\langle u,v\rangle_y - \Phi_{m-\mu+1}[v]\text{: abs. cont. in } (0,1),$$

where

$$\Phi_{m-\mu+1}[v] = -\Phi_{m-\mu}[P_1 v] + (\partial_x)F_\mu[v] + F_{\mu-1}[v].$$

From $\boxed{m-\mu+2}$, we have

$$g_{m-\mu+2} = (\partial_x)^{m-\mu+2}\langle u,v\rangle_y$$
$$+ \{(\partial_x)^{m-\mu+1}\langle u, P_1 v\rangle_y - \Phi_{m-\mu+1}[P_1 v]\} + \Phi_{m-\mu+1}[P_1 v]$$
$$+ \{(\partial_x)^{m-\mu}\langle u, P_2 v\rangle_y - \Phi_{m-\mu}[P_2 v]\} + \Phi_{m-\mu}[P_2 v]$$
$$+ (\partial_x)^{m-\mu-1}\langle u, P_3 v\rangle_y + \cdots + (\partial_x)^3\langle u, P_{m-\mu-1}v\rangle_y$$
$$+ (\partial_x)^2(\langle u, P_{m-\mu}v\rangle_y - F_\mu[v]) + \cdots + (\langle u, P_{m-\mu+2}v\rangle_y - F_{\mu-2}[v]),$$

therefore, we have

$$(\partial_x)^{m-\mu+2}\langle u,v\rangle_y - \Phi_{m-\mu+2}[v]\text{: abs. cont. in } (0,1),$$

where

$$\Phi_{m-\mu+2}[v] = -\Phi_{m-\mu+1}[P_1 v] - \Phi_{m-\mu}[P_2 v] + (\partial_x)^2 F_\mu[v] + (\partial_x) F_{\mu-1}[v] + F_{\mu-2}[v].$$

In the same way, setting

$$\Phi_j[v] = -\Phi_{j-1}[P_1 v] - \Phi_{j-2}[P_2 v] - \cdots - \Phi_{m-\mu}[P_{j-m+\mu} v]$$
$$+ (\partial_x)^{j-m+\mu} F_\mu[v] + \cdots + (\partial_x) F_{-j+m+1}[v] + F_{-j+m}[v],$$

we have

$$(\partial_x)^j \langle u, v \rangle_y - \Phi_j[v]\text{: abs. cont. in } (0,1) \quad (m-\mu \leqq j \leqq m). \qquad \square$$

LEIBNIZ' FORMULA. Let $u \in L^2(\mathbf{R}^n_+)$ and let $v \in H^{2m}(\mathbf{R}^n_+)$, then it holds

$$\langle (\partial_x)^j u, v \rangle_y = \langle u, (-\partial_x)^j v \rangle_y + \binom{j}{1} (\partial_x) \langle u, (-\partial_x)^{j-1} v \rangle_y$$
$$+ \binom{j}{2} (\partial_x)^2 \langle u, (-\partial_x)^{j-2} v \rangle_y + \cdots + (\partial_x)^j \langle u, v \rangle_y \quad (j = 0, 1, \dots, m),$$

therefore it holds

$$(\partial_x)^j \langle u, v \rangle_y = \langle (\partial_x)^j u, v \rangle_y + \binom{j}{1} \langle (\partial_x)^{j-1} u, (\partial_x) v \rangle_y$$
$$+ \binom{j}{2} \langle (\partial_x)^{j-2} u, (\partial_x)^2 v \rangle_y + \cdots + \langle u, (\partial_x)^j v \rangle_y \quad (j = 0, 1, \dots, m).$$

LEMMA A.2. *Let* $u \in L^2(\mathbf{R}^n_+)$ *satisfy* $Au = f \in H^{-\mu}(\mathbf{R}^n_+)$. *Let* $v(x, y) \in H^{2m}(\mathbf{R}^n_+)$ *satisfy* $\mathrm{supp}[v] \subset \{x \leqq 1\}$. *Then*

(i) $\langle (\partial_x)^j u, v \rangle_y$: *abs. cont. in* $(0,1)$

$$(\langle (\partial_x)^j u|_{x=0}, \beta(y) \rangle_y := \langle (\partial_x)^j u, \beta(y) \rangle_y|_{x=+0}),$$
$$\sup_{0<x<1} |\langle (\partial_x)^j u, v \rangle_y| \leqq C(\|u\| + \|f\|_{-\mu}) \|v\|_{m+j},$$
$$\langle (\partial_x)^j u, v \rangle_y|_{x=0} = \langle (\partial_x)^j u|_{x=0}, v(0, y) \rangle_y,$$
$$\langle (\partial_x)^j u|_{x=0} \rangle_{-m-j+1/2} \leqq C(\|u\| + \|f\|_{-\mu})$$

for $j = 0, 1, \dots, m - \mu - 1$,

(ii) $\langle (\partial_x)^j u, v \rangle_y - \Psi_j[v]$: *abs. cont. in* $(0, 1)$,

$$\{\langle (\partial_x)^j u, v \rangle_y - \Psi_j[v]\}|_{x=0} = \langle (\partial_x)^j u|_{x=0}, v(0, y) \rangle_y - \Psi_j[v(0, y)]$$

for $j = m - \mu, \dots, m$, *where* $\{\Psi_j[v]\ (j = m - \mu, \dots, m)\}$ *are defined by*

$$\Phi_j[v] = \Psi_j[v] + \binom{j}{1}\Psi_{j-1}[(\partial_x)v] + \binom{j}{2}\Psi_{j-2}[(\partial_x)^2 v]$$

$$+ \cdots + \binom{j}{j-m+\mu}\Psi_{m-\mu}[(\partial_x)^{j-m+\mu}v].$$

PROOF. (i) From Leibniz' formula and Lemma A.1, we have

$$\sup_{0<x<1}|\langle(\partial_x)^j u, v\rangle_y| + \|(\partial_x)\langle(\partial_x)^j u, v\rangle_y\|_{L1(0,1)} \leqq C(\|u\| + \|f\|_{-\mu})\|v\|_{m+j}$$

$$(\forall v \in H^{2m}(\mathbf{R}^n_+), \operatorname{supp}[v] \subset \{x \leqq 1\})$$

for $j = 0, 1, \dots, m - \mu - 1$. Set $v(x, y) = v(0, y) + xw(x, y)$, then we have

$$\langle(\partial_x)^j u, v\rangle_y = \langle(\partial_x)^j u, v(0, y)\rangle_y + \langle(\partial_x)^j u, xw(x, y)\rangle_y$$

$$= \langle(\partial_x)^j u, v(0, y)\rangle_y + x\langle(\partial_x)^j u, w(x, y)\rangle_y.$$

Since there exists a map

$$E : H^{m+j-1/2}(\mathbf{R}^{n-1}) \ni \beta \mapsto (E\beta)(x, y) \in H^{m+j}(\mathbf{R}^n_+)$$

satisfying $\operatorname{supp}[E\beta] \subset \{x \leqq 1\}$, $(E\beta)(0, y) = \beta(y)$ and

$$\|E\beta\|_{m+j} \leqq C\langle\beta\rangle_{m+j-1/2},$$

we have

$$|\langle(\partial_x)^j u|_{x=0}, \beta(y)\rangle_y| = |\langle(\partial_x)^j u, E\beta)\rangle_y|_{x=0}|$$

$$\leqq C(\|u\| + \|f\|_{-\mu})\|E\beta\|_{m+k} \leqq C(\|u\| + \|f\|_{-\mu})\langle\beta\rangle_{m+j-1/2}.$$

(ii) From Leibniz' Formula and Lemma A.1, we have

$$\langle(\partial_x)^{m-\mu}u, v\rangle_y - \Phi_{m-\mu}[v]$$

$$= \{(\partial_x)^{m-\mu}\langle u, v\rangle_y - \Phi_{m-\mu}[v]\} - \left\{\binom{m-\mu}{1}\langle(\partial_x)^{m-\mu-1}u, (\partial_x)v\rangle_y\right.$$

$$\left. + \binom{m-\mu}{2}\langle(\partial_x)^{m-\mu-2}u, (\partial_x)^2 v\rangle_y + \cdots + \langle u, (\partial_x)^{m-\mu}v\rangle_y\right\}: \text{abs. cont. in } (0, 1),$$

that is,

$$\langle(\partial_x)^{m-\mu}u, v\rangle_y - \Psi_{m-\mu}[v]: \text{abs. cont. in } (0, 1),$$

where $\Psi_{m-\mu}[v]=\Phi_{m-\mu}[v]$. Next, we have

$$\langle(\partial_x)^{m-\mu+1}u,v\rangle_y-\left\{\Phi_{m-\mu+1}[v]-\binom{m-\mu+1}{1}\Psi_{m-\mu}[(\partial_x)v]\right\}$$

$$=\{(\partial_x)^{m-\mu+1}\langle u,v\rangle_y-\Phi_{m-\mu+1}[v]\}-\binom{m-\mu+1}{1}$$

$$\times\{\langle(\partial_x)^{m-\mu}u,(\partial_x)v\rangle_y-\Psi_{m-\mu}[(\partial_x)v]\}$$

$$-\left\{\binom{m-\mu+1}{2}\langle(\partial_x)^{m-\mu-1}u,(\partial_x)^2v\rangle_y+\binom{m-\mu+1}{3}\langle(\partial_x)^{m-\mu-2}u,(\partial_x)^3v\rangle_y\right.$$

$$\left.+\cdots+\langle u,(\partial_x)^{m-\mu+1}v\rangle_y\right\}:\ \text{abs. cont. in } (0,1),$$

that is,

$$\langle(\partial_x)^{m-\mu+1}u,v\rangle_y-\Psi_{m-\mu+1}[v]:\ \text{abs. cont. in } (0,1),$$

where

$$\Psi_{m-\mu+1}[v]=\Phi_{m-\mu+1}[v]-\binom{m-\mu+1}{1}\Psi_{m-\mu}[(\partial_x)v].$$

In the same way, we have

$$\langle\partial_x^j u,v\rangle_y-\Psi_j[v]:\ \text{abs. cont. in } (0,1),$$

where $\Psi_j[v]$ is defined by

$$\Phi_j[v]=\Psi_j[v]+\binom{j}{1}\Psi_{j-1}[(\partial_x)v]+\binom{j}{2}\Psi_{j-2}[(\partial_x)^2v]$$

$$+\cdots+\binom{j}{j-m+\mu}\Psi_{m-\mu}[(\partial_x)^{j-m+\mu}v]\quad(j=m-\mu,\dots,m).$$

Now, set $v(x,y)=v(0,y)+xw(x,y)$, then we have

$$\langle(\partial_x)^j u,v\rangle_y-\Psi_j[v]=\{\langle(\partial_x)^j u,v(0,y)\rangle_y-\Psi_j[v(0,y)]\}+x\{\langle(\partial_x)^j u,w\rangle_y-\Psi_j[w]\},$$

therefore we have

$$\{\langle(\partial_x)^j u,v\rangle_y-\Psi_j[v]\}|_{x=+0}=\{\langle(\partial_x)^j u,v(0,y)\rangle_y-\Psi_j[v(0,y)]\}|_{x=+0}.\qquad\square$$

$\{A_j'\}$ AND $\{a_j'\}$. Set

$$A_k'=\sum\nolimits_{j\leqq k}(-\partial_x)^{k-j}a_j'\quad(k=0,1,\dots,m),$$

then we have

$$\begin{pmatrix} 1 \\ A_1' \\ A_2' \\ \vdots \\ A_{m-2}' \\ A_{m-1}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ (-\partial_x) & 1 & 0 & \cdots & & 0 \\ (-\partial_x)^2 & (-\partial_x) & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & & & \vdots \\ (-\partial_x)^{m-2} & (-\partial_x)^{m-3} & \cdots & (-\partial_x) & 1 & 0 \\ (-\partial_x)^{m-1} & (-\partial_x)^{m-2} & \cdots & & (-\partial_x) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ a_1' \\ a_2' \\ \vdots \\ a_{m-2}' \\ a_{m-1}' \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ a_1' \\ a_2' \\ \vdots \\ a_{m-2}' \\ a_{m-1}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ (\partial_x) & 1 & 0 & \cdots & & 0 \\ 0 & (\partial_x) & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (\partial_x) & 1 & 0 \\ 0 & \cdots & 0 & & (\partial_x) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ A_1' \\ A_2' \\ \vdots \\ A_{m-2}' \\ A_{m-1}' \end{pmatrix},$$

and

$$A_m' = (-\partial_x)^m a_0' + (-\partial_x)^{m-1} a_1' + \cdots + a_m' = A'.$$

Hence we have

$$\langle Au, v\rangle_y - \langle u, A'v\rangle_y = (\partial_x)\{\langle(\partial_x)^{m-1}u, v\rangle_y + \langle(\partial_x)^{m-2}u, A_1'v\rangle_y + \cdots + \langle u, A_{m-1}'v\rangle_y\}$$

for $u \in L^2(\mathbf{R}_+^n)$ and $v \in H^{2m}(\mathbf{R}_+^n)$.

$\{A_j'\}$ AND $\{P_j\}$. From the definition of $\{P_j\}$, we have

$$\begin{pmatrix} 1 \\ P_1 \\ P_2 \\ \vdots \\ P_{m-2} \\ P_{m-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & & & & 0 \\ \binom{m}{m-1}(-\partial_x) & 1 & 0 & & \cdots & & 0 \\ \binom{m}{m-2}(-\partial_x)^2 & \binom{m-1}{m-2}(-\partial_x) & 1 & 0 & & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \ddots & \vdots \\ \binom{m}{2}(-\partial_x)^{m-2} & \binom{m-1}{2}(-\partial_x)^{m-3} & \cdots & \binom{3}{2}(-\partial_x) & & 1 & 0 \\ \binom{m}{1}(-\partial_x)^{m-1} & \binom{m-1}{1}(-\partial_x)^{m-2} & \cdots & & \binom{2}{1}(-\partial_x) & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ a_1' \\ a_2' \\ \vdots \\ a_{m-2}' \\ a_{m-1}' \end{pmatrix},$$

therefore

$$\begin{pmatrix} 1 \\ P_1 \\ P_2 \\ \vdots \\ P_{m-2} \\ P_{m-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & & & \cdots & 0 \\ \binom{m-1}{m-2}(-\partial_x) & 1 & 0 & & & \cdots & 0 \\ \binom{m-1}{m-3}(-\partial_x)^2 & \binom{m-2}{m-3}(-\partial_x) & 1 & 0 & & \cdots & 0 \\ \vdots & \vdots & \ddots & & \ddots & \ddots & \vdots \\ \binom{m-1}{1}(-\partial_x)^{m-2} & \binom{m-2}{1}(-\partial_x)^{m-3} & \cdots & \binom{2}{1}(-\partial_x) & & 1 & 0 \\ \binom{m-1}{0}(-\partial_x)^{m-1} & \binom{m-2}{0}(-\partial_x)^{m-2} & & \cdots & & \binom{1}{0}(-\partial_x) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ A'_1 \\ A'_2 \\ \vdots \\ A'_{m-2} \\ A'_{m-1} \end{pmatrix},$$

therefore

$$\begin{pmatrix} 1 \\ A'_1 \\ A'_2 \\ \vdots \\ A'_{m-2} \\ A'_{m-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & & & \cdots & 0 \\ \binom{m-1}{m-2}(\partial_x) & 1 & 0 & & & \cdots & 0 \\ \binom{m-1}{m-3}(\partial_x)^2 & \binom{m-2}{m-3}(\partial_x) & 1 & 0 & & \cdots & 0 \\ \vdots & \vdots & \ddots & & \ddots & \ddots & \vdots \\ \binom{m-1}{1}(\partial_x)^{m-2} & \binom{m-2}{1}(\partial_x)^{m-3} & \cdots & \binom{2}{1}(\partial_x) & & 1 & 0 \\ \binom{m-1}{0}(\partial_x)^{m-1} & \binom{m-2}{0}(\partial_x)^{m-2} & & \cdots & & \binom{1}{0}(\partial_x) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ P_1 \\ P_2 \\ \vdots \\ P_{m-2} \\ P_{m-1} \end{pmatrix},$$

that is,

$$A'_i = \sum\nolimits_{0 \leqq j \leqq i} \binom{m-1-j}{m-1-i} (\partial_x)^{i-j} P_j.$$

LEMMA A.3. *Let* $u \in L^2(\mathbf{R}^n_+)$ *satisfy* $Au = f \in H^{-\mu}(\mathbf{R}^n_+)$. *Let* $v \in H^{2m}(\mathbf{R}^n_+)$ *satisfy* $\operatorname{supp}[v] \subset \{x \leqq 1\}$. *Then it holds*

$$\begin{aligned} &\langle f^{\sim}, v\rangle - \langle u, A'v\rangle \\ &\quad = -\{\{\langle(\partial_x)^{m-1}u, v\rangle_y - \Psi_{m-1}[v]\}|_{x=0} + \{\langle(\partial_x)^{m-2}u, A'_1 v\rangle_y - \Psi_{m-2}[A'_1 v]\}|_{x=0} \\ &\qquad + \cdots + \{\langle(\partial_x)^{m-\mu}u, A'_{\mu-1}v\rangle_y - \Psi_{m-\mu}[A'_{\mu-1}v]\}|_{x=0} \\ &\qquad + \{\langle(\partial_x)^{m-\mu-1}u, A'_\mu v\rangle_y|_{x=0} + \cdots + \langle u, A'_{m-1}v\rangle_y|_{x=0}\}\}, \end{aligned}$$

where

$$\langle f^{\sim}, v\rangle = \int_{R_+} F_0[v](x)\, dx = \sum\nolimits_{|\nu| \leqq \mu} \int_{R^n_+} f_\nu(x, y)(-\partial)^\nu v(x, y)\, dxdy.$$

Especially when $v|_{x=0} = (\partial_x)v|_{x=0} = \cdots = (\partial_x)^{\mu-1}v|_{x=0} = 0$, *it holds*

$$\langle f^{\sim}, v\rangle - \langle u, A'v\rangle = -\{\langle(\partial_x)^{m-\mu-1}u, A'_\mu v\rangle_y|_{x=0} + \cdots + \langle u, A'_{m-1}v\rangle_y|_{x=0}\}.$$

PROOF. Since

$$\begin{aligned}\langle Au, v\rangle_y &- \langle u, A'v\rangle_y \\ &= (\partial_x)\{\langle(\partial_x)^{m-1}u, v\rangle_y + \langle(\partial_x)^{m-2}u, A'_1 v\rangle_y + \cdots + \langle u, A'_{m-1}v\rangle_y\}\end{aligned}$$

and

$$\langle Au, v\rangle_y = \langle f, v\rangle_y = (\partial_x)^\mu F_\mu[v] + (\partial_x)^{\mu-1}F_{\mu-1}[v] + \cdots + F_0[v],$$

we have

$$\begin{aligned}F_0[v] &- \langle u, A'v\rangle_y \\ &= (\partial_x)\{\{\langle(\partial_x)^{m-1}u, v\rangle_y - \Psi_{m-1}[v]\} + \{\langle(\partial_x)^{m-2}u, A'_1 v\rangle_y - \Psi_{m-2}[A'_1 v]\} \\ &\quad + \cdots + \{\langle(\partial_x)^{m-\mu}u, A'_{\mu-1}v\rangle_y - \Psi_{m-\mu}[A'_{\mu-1}v]\} \\ &\quad + \{\langle(\partial_x)^{m-\mu-1}u, A'_\mu v\rangle_y + \cdots + \langle u, A'_{m-1}v\rangle_y\} \\ &\quad + \{\Psi_{m-1}[v] + \Psi_{m-2}[A'_1 v] + \cdots + \Psi_{m-\mu}[A'_{\mu-1}v]\} \\ &\quad - \{(\partial_x)^{\mu-1}F_\mu[v] + (\partial_x)^{\mu-2}F_{\mu-1}[v] + \cdots + F_1[v]\}\}.\end{aligned}$$

Since

$$\begin{aligned}\Phi_{m-1}[v] &+ \Phi_{m-2}[P_1 v] + \cdots + \Phi_{m-\mu+1}[P_{\mu-2}v] + \Phi_{m-\mu}[P_{\mu-1}v] \\ &= (\partial_x)^{\mu-1}F_\mu[v] + (\partial_x)^{\mu-2}F_{\mu-1}[v] + \cdots + (\partial_x)F_2[v] + F_1[v]\end{aligned}$$

from the definition of $\{\Phi_j\}$, we have

$$\begin{aligned}F_0[v] &- \langle u, A'v\rangle_y \\ &= (\partial_x)\{\{\langle(\partial_x)^{m-1}u, v\rangle_y - \Psi_{m-1}[v]\} + \{\langle(\partial_x)^{m-2}u, A'_1 v\rangle_y - \Psi_{m-2}[A'_1 v]\} \\ &\quad + \cdots + \{\langle(\partial_x)^{m-\mu}u, A'_{\mu-1}v\rangle_y - \Psi_{m-\mu}[A'_{\mu-1}v]\} \\ &\quad + \{\langle(\partial_x)^{m-\mu-1}u, A'_\mu v\rangle_y + \cdots + \langle u, A'_{m-1}v\rangle_y\} \\ &\quad + \{\Psi_{m-1}[v] + \Psi_{m-2}[A'_1 v] + \cdots + \Psi_{m-\mu}[A'_{\mu-1}v]\} \\ &\quad - \{\Phi_{m-1}[v] + \Phi_{m-2}[P_1 v] + \cdots + \Phi_{m-\mu}[P_{\mu-1}v]\}\}.\end{aligned}$$

From the definition of $\{\Psi_j\}$, we have

$$\begin{aligned}
&\Phi_{m-1}[v] + \Phi_{m-2}[P_1 v] + \cdots + \Phi_{m-\mu}[P_{\mu-1} v] \\
&= \left\{\Psi_{m-1}[v] + \binom{m-1}{1}\Psi_{m-2}[(\partial_x)v] + \binom{m-1}{2}\Psi_{m-3}[(\partial_x)^2 v]\right. \\
&\quad \left. + \cdots + \binom{m-1}{\mu-1}\Psi_{m-\mu}[(\partial_x)^{\mu-1}v]\right\} + \left\{\Psi_{m-2}[P_1 v] + \binom{m-2}{1}\Psi_{m-3}[(\partial_x)P_1 v]\right. \\
&\quad \left. + \binom{m-2}{2}\Psi_{m-4}[(\partial_x)^2 P_1 v] + \cdots + \binom{m-2}{\mu-2}\Psi_{m-\mu}[(\partial_x)^{\mu-2}P_1 v]\right\} \\
&\quad + \cdots + \Psi_{m-\mu}[P_{\mu-1} v] \\
&= \Psi_{m-1}[v] + \Psi_{m-2}[A_1 v] + \Psi_{m-3}[A_2 v] + \cdots + \Psi_{m-\mu}[A_{\mu-1} v].
\end{aligned}$$

Hence we have

$$\begin{aligned}
&F_0[v] - \langle u, A'v\rangle_y \\
&\quad = (\partial_x)\{\{\langle(\partial_x)^{m-1}u, v\rangle_y - \Psi_{m-1}[v]\} + \{\langle(\partial_x)^{m-2}u, A_1'v\rangle_y - \Psi_{m-2}[A_1'v]\} \\
&\qquad + \cdots + \{\langle(\partial_x)^{m-\mu}u, A_{\mu-1}'v\rangle_y - \Psi_{m-\mu}[A_{\mu-1}'v]\} \\
&\qquad + \{\langle(\partial_x)^{m-\mu-1}u, A_\mu'v\rangle_y + \cdots + \langle u, A_{m-1}'v\rangle_y\}\}.
\end{aligned}$$

From Lemma A.2, we have

$$\begin{aligned}
&\langle f^{\sim}, v\rangle - \langle u, A'v\rangle \\
&\quad = -\{\{\langle(\partial_x)^{m-1}u, v\rangle_y - \Psi_{m-1}[v]\}|_{x=0} + \{\langle(\partial_x)^{m-2}u, A_1'v\rangle_y - \Psi_{m-2}[A_1'v]\}|_{x=0} \\
&\qquad + \cdots + \{\langle(\partial_x)^{m-\mu}u, A_{\mu-1}'v\rangle_y - \Psi_{m-\mu}[A_{\mu-1}'v]\}|_{x=0} \\
&\qquad + \{\langle(\partial_x)^{m-\mu-1}u, A_\mu'v\rangle_y|_{x=0} + \cdots + \langle u, A_{m-1}'v\rangle_y|_{x=0}\}\},
\end{aligned}$$

by integrating both sides of the above equality with respect to x in $(0,1)$. Especially when $v|_{x=0} = (\partial_x)v|_{x=0} = \cdots = (\partial_x)^{\mu-1}v|_{x=0} = 0$, we have

$$\langle f^{\sim}, v\rangle - \langle u, A'v\rangle = -\{\langle(\partial_x)^{m-\mu-1}u, A_\mu'v\rangle_y|_{x=0} + \cdots + \langle u, A_{m-1}'v\rangle_y|_{x=0}\}. \qquad \square$$

Adjoint boundary operators.

Adding $\{B_j = (\partial_x)^j \ (j \in J^c)\}$ to boundary operators $\{B_j \ (j \in J)\}$, we have

$$\begin{pmatrix} B_{m-1} \\ B_{m-2} \\ \vdots \\ B_0 \end{pmatrix} = \begin{pmatrix} b_{m-1\ 0} & b_{m-1\ 1} & \cdots\cdots & b_{m-1\ m-1} \\ 0 & b_{m-2\ 0} & \cdots\cdots & b_{m-2\ m-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{00} \end{pmatrix} \begin{pmatrix} (\partial_x)^{m-1} \\ (\partial_x)^{m-2} \\ \vdots \\ 1 \end{pmatrix},$$

where

$$b_{jk} = b_{jk}(y, \partial_y) = \sum\nolimits_{|\nu| \leqq k} b_{jk\nu}(y)(\partial_y)^\nu \quad (b_{j0} = b_{j0}(y) \neq 0).$$

Then we have

$$\begin{pmatrix} (\partial_x)^{m-1} \\ (\partial_x)^{m-2} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} c_{m-1\ 0} & c_{m-1\ 1} & \cdots\cdots & c_{m-1\ m-1} \\ 0 & c_{m-2\ 0} & \cdots\cdots & c_{m-2\ m-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{00} \end{pmatrix} \begin{pmatrix} B_{m-1} \\ B_{m-2} \\ \vdots \\ B_0 \end{pmatrix},$$

where

$$c_{jk} = c_{jk}(y, \partial_y) = \sum\nolimits_{|\nu| \leqq k} c_{jk\nu}(y)(\partial_y)^\nu \quad (c_{j0} = c_{j0}(y) \neq 0).$$

Let us define $\{\mathscr{B}'_j \ (j = 0, 1, \ldots, m-1)\}$ by

$$\begin{pmatrix} \mathscr{B}'_0 \\ \mathscr{B}'_1 \\ \vdots \\ \mathscr{B}'_{m-2} \\ \mathscr{B}'_{m-1} \end{pmatrix} = \begin{pmatrix} (c_{m-1\ 0})' & 0 & \cdots\cdots\cdots & & 0 \\ (c_{m-1\ 1})' & (c_{m-2\ 0})' & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ (c_{m-1\ m-2})' & (c_{m-2\ m-3})' & \cdots & (c_{10})' & 0 \\ (c_{m-1\ m-1})' & (c_{m-2\ m-2})' & \cdots\cdots & & (c_{00})' \end{pmatrix} \begin{pmatrix} 1 \\ A'_1 \\ \vdots \\ A'_{m-2} \\ A'_{m-1} \end{pmatrix},$$

where

$$(c_{jk})' = \sum\nolimits_{|\nu| \leqq k} (\partial_y)^\nu c_{jk\nu}(y).$$

Here we have

$$\left\langle \begin{pmatrix} (\partial_x)^{m-1} \\ (\partial_x)^{m-2} \\ \vdots \\ 1 \end{pmatrix} u, \begin{pmatrix} 1 \\ A'_1 \\ \vdots \\ A'_{m-1} \end{pmatrix} v \right\rangle_y = \left\langle \begin{pmatrix} B_{m-1} \\ B_{m-2} \\ \vdots \\ B_0 \end{pmatrix} u, \begin{pmatrix} \mathscr{B}'_0 \\ \mathscr{B}'_1 \\ \vdots \\ \mathscr{B}'_{m-1} \end{pmatrix} v \right\rangle_y.$$

Since $J \subset \{0, 1, \dots, m-1-\mu\}$, the above equalities are specialized as follows.

$$\begin{pmatrix} B_{m-1} \\ B_{m-2} \\ \vdots \\ B_{m-\mu} \\ B_{m-\mu-1} \\ B_{m-\mu-2} \\ \vdots \\ B_0 \end{pmatrix} = \begin{pmatrix} 1 & & & & 0 & \\ & \ddots & & & & \\ & & b_{m-\mu-1\,0} & \cdots\cdots & & b_{m-\mu-1\;m-\mu-1} \\ & & 0 & b_{m-\mu-2\,0} & \cdots & b_{m-\mu-2\;m-\mu-2} \\ 0 & & \vdots & \ddots & \ddots & \vdots \\ & & 0 & \cdots\cdots & 0 & b_{00} \end{pmatrix} \begin{pmatrix} (\partial_x)^{m-1} \\ (\partial_x)^{m-2} \\ \vdots \\ (\partial_x)^{m-\mu} \\ (\partial_x)^{m-\mu-1} \\ (\partial_x)^{m-\mu-2} \\ \vdots \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} (\partial_x)^{m-1} \\ (\partial_x)^{m-2} \\ \vdots \\ (\partial_x)^{m-\mu} \\ (\partial_x)^{m-\mu-1} \\ (\partial_x)^{m-\mu-2} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & & & & 0 & \\ & \ddots & & & & \\ & & c_{m-\mu-1\,0} & \cdots\cdots & & c_{m-\mu-1\;m-\mu-1} \\ & & 0 & c_{m-\mu-2\,0} & \cdots & c_{m-\mu-2\;m-\mu-2} \\ 0 & & \vdots & \ddots & \ddots & \vdots \\ & & 0 & \cdots\cdots & 0 & c_{00} \end{pmatrix} \begin{pmatrix} B_{m-1} \\ B_{m-2} \\ \vdots \\ B_{m-\mu} \\ B_{m-\mu-1} \\ B_{m-\mu-2} \\ \vdots \\ B_0 \end{pmatrix},$$

$$\begin{pmatrix} \mathscr{B}'_0 \\ \mathscr{B}'_1 \\ \vdots \\ \mathscr{B}'_{\mu-1} \\ \mathscr{B}'_{\mu} \\ \mathscr{B}'_{\mu+1} \\ \vdots \\ \mathscr{B}'_{m-1} \end{pmatrix} = \begin{pmatrix} 1 & & & 0 & \\ & \ddots & & & \\ & (c_{m-\mu-1\,0})' & 0 & \cdots\cdots\cdots & 0 \\ & (c_{m-\mu-1\,1})' & (c_{m-\mu-2\,0})' & 0\cdots\cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ & (c_{m-\mu-1\;m-\mu-1})' & (c_{m-\mu-2\;m-\mu-2})' & \cdots & (c_{00})' \end{pmatrix} \begin{pmatrix} A'_0 \\ A'_1 \\ \vdots \\ A'_{\mu-1} \\ A'_{\mu} \\ A'_{\mu+1} \\ \vdots \\ A'_{m-1} \end{pmatrix}.$$

Here we remark that $\mathscr{B}'_j = A'_j$ $(j = 0, 1, \dots, \mu-1)$.

Now we have Corollary from Lemma A.3, which means Green's Theorem $(\Omega = \mathbf{R}^n_+)$, where $\mathscr{B}^*_j = \overline{\mathscr{B}'_j}$.

COROLLARY. *Let* $u \in L^2(\mathbf{R}^n_+)$ *satisfy* $Au = f \in H^{-\mu}(\mathbf{R}^n_+)$. *Let* $v \in H^{2m}(\mathbf{R}^n_+)$ *satisfy* $\operatorname{supp}[v] \subset \{x \leqq 1\}$. *Then it holds*

$$\langle f^{\sim}, v\rangle - \langle u, A'v\rangle$$
$$= -\{\{\langle(\partial_x)^{m-1}u, v\rangle_y - \Psi_{m-1}[v]\}|_{x=0} + \{\langle(\partial_x)^{m-2}u, A_1'v\rangle_y - \Psi_{m-2}[A_1'v]\}|_{x=0}$$
$$+ \cdots + \{\langle\partial_x)^{m-\mu}u, A_{\mu-1}'v\rangle_y - \Psi_{m-\mu}[A_{\mu-1}'v]\}|_{x=0}$$
$$+ \{\langle B_{m-\mu-1}u, \mathscr{B}_\mu' v\rangle_y|_{x=0} + \cdots + \langle B_0 u, \mathscr{B}_{m-1}'v\rangle_y|_{x=0}\}\}.$$

Especially when $v|_{x=0} = (\partial_x)v|_{x=0} = \cdots = (\partial_x)^{\mu-1}v|_{x=0} = 0$, *it holds*

$$\langle f^{\sim}, v\rangle - \langle u, A'v\rangle = -\{\langle B_{m-\mu-1}u, \mathscr{B}_\mu' v\rangle_y|_{x=0} + \cdots + \langle B_0 u, \mathscr{B}_{m-1}'v\rangle_y|_{x=0}\},$$

that is, it holds

$$\langle f^{\sim}, v\rangle - \langle u, A'v\rangle = -\sum_{j\in J}\langle B_j u|_{x=0}, \mathscr{B}_{m-1-j}'v|_{x=0}\rangle_y,$$

if $\mathscr{B}_j'v|_{x=0} = 0 \ (j \in J^*)$.

References

[1] R. Sakamoto, Numerical approximation of weak solution for boundary value problems, Tsukuba J. Math. **26** (2002), 79–94.

[2] R. Sakamoto, Dirichlet-Neumann problem in a domain with piecewise-smooth boundary, Tsukuba J. Math. **26** (2002), 387–406.

[3] R. Adams, Sobolev Spaces, Pure and applied mathematics series 65, Academic Press, San Diego, 1978.

[4] H. Triebel, Theory of Function Spaces, Monographs in Mathematics 78, Birkhauser, Basel, 1983.

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