TSUKUBA J. MATH.
Vol. 29 No. 1 (2005), 79–109

A CONSTRUCTION OF COMPACT PSEUDO-KÄHLER SOLVMANIFOLDS WITH NO KÄHLER STRUCTURES

By

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Abstract. In this paper we investigate the Hard Lefschetz property on certain compact symplectic solvmanifolds and construct compact pseudo-Kähler solvmanifolds which do not have the Hard Lefschetz property. We also construct holomorphic symplectic structures, hypercomplex structures and pseudo-hyperkähler structures on certain compact solvmanifolds.

Introduction

Let (M^{2m}, ω) be a compact symplectic manifold. We say that (M^{2m}, ω) has the Hard Lefschetz property, if the Lefschetz mapping $L^k : H_{DR}^{m-k}(M) \to H_{DR}^{m+k}(M)$ defined by $L^k([\alpha]) = [\alpha \wedge \omega^k]$ is an isomorphism for any $k \leq m$. It is well known that the Hard Lefschetz property is a necessary condition for the existence of a Kähler structure. Benson and Gordon [2] proved that non-toral compact nilmanifolds do not have the Hard Lefschetz property. They also conjecture the following:

BENSON-GORDON CONJECTURE [3]. *Let G be a simply-connected completely solvable Lie group and Γ a lattice of G. Then G/Γ has a Kähler structure if and only if it is a torus.*

Moreover, since a hyperelliptic surface has a Kähler structure and a structure of solvmanifold (not completely solvable solvmanifold), there exists the following generalized conjecture (see [6] or [12]): A compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus, which has also a structure of complex torus bundle over a complex torus. A solvable Lie algebra $\mathfrak{g}$

2000 Mathematics Subject Classification. 53D05 (53C55).
Key words: symplectic solvmanifold, pseudo-Kähler structure, holomorphic symplectic structure.
Received June 19, 2003.
Revised November 20, 2003.

is called completely solvable if $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues for each $X \in \mathfrak{g}$. By investigating the properties of the Lefschetz mapping, Benson and Gordon [3] have several necessary conditions for the existence of a Kähler structure. On the other hand, de Andrés, Fernández, de León and Mencía [1] have constructed examples of 6-dimensional non-toral compact pseudo-Kähler solvmanifolds which have the Hard Lefschetz property (See Example 5.1). We do not know whether any of these solvmanifolds admit Kähler structures. Ibáñez [14] has constructed 6-dimensional pseudo-Kähler nilmanifolds. Kodaira-Thurston manifold, which is a compact 4-dimensional nilmanifold, also admits a pseudo-Kähler structure (see [5]).

In the previous paper [21], we constructed completely solvable Lie groups which have a lattice. Let A_i, B_j be the matrices given by

$$A_i = \sum_{k=1}^{m} a_i^k (E_{2k-1,2k-1} - E_{2k,2k}) \quad i = 1, \dots, l,$$

$$B_j = \sum_{k<h} b_j^{kh} (E_{2k-1,2h-1} + E_{2k,2h}) \quad j = 1, \dots, n,$$

where $a_i^k, b_j^{kh} \in \mathbf{Q}$ and $E_{i,j}$ is a matrix unit. We assume that $[A_i, B_j] = [B_i, B_j] = 0$. We define a map

$$\varphi_* : \mathbf{R}^{n+l} \to End(\mathbf{R}^{2m})$$

by

$$\varphi_*(t_1, \dots, t_l, x_1, \dots, x_n) = \sum_{i=1}^{l} t_i A_i + \sum_{i=1}^{n} x_i B_i.$$

Let $\varphi(\mathbf{t}, \mathbf{x}) = \exp(\varphi_*(\mathbf{t}, \mathbf{x}))$ and we define a group structure of $\mathbf{R}^{n+l} \times \mathbf{R}^{2m}$ by

$$(\mathbf{t}_1, \mathbf{x}_1, \mathbf{y}_1) * (\mathbf{t}_2, \mathbf{x}_2, \mathbf{y}_2) = (\mathbf{t}_1 + \mathbf{t}_2, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \varphi(\mathbf{t}_1, \mathbf{x}_1)\mathbf{y}_2)$$

for $\mathbf{t}_i \in \mathbf{R}^l$, $\mathbf{x}_i \in \mathbf{R}^n$ and $\mathbf{y}_i \in \mathbf{R}^{2m}$. We denote the Lie group $(\mathbf{R}^{n+l} \times \mathbf{R}^{2m}, *)$ by $G = \mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$.

In the previous paper [21], we proved the following:

PROPOSITION 1. *A Lie group $G = \mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$ is a completely solvable Lie group which has a lattice Γ.*

The main purpose of this paper is to investigate the properties of the Lefschetz mapping on the compact symplectic solvmanifolds constructed in

Proposition 1 and to construct examples of compact pseudo-Kähler solvmanifolds without the Hard Lefschetz property.

In section 2, 3 and 4 we always assume that for each k, there exists an i such that $a_i^k \neq 0$ and $l+n$ are even numbers. A solvable Lie group $G = \mathbf{R}^{l+n} \ltimes_\varphi \mathbf{R}^{2m}$ constructed above is called A-type if $B_j = 0$ for each j. In section 4 we prove the following:

THEOREM 2. *Let $M = G/\Gamma$ be a compact solvmanifold constructed in Proposition 1 and assume that M has a symplectic structure. Then M has the Hard Lefschetz property if and only if M is a compact A-type solvmanifold.*

PROPOSITION 3. *The minimal model of a compact A-type solvmanifold $M = G/\Gamma$ is formal.*

It is known that formality is also a necessary condition for the existence of a Kähler structure and it is conjectured that if a closed symplectic manifold has the Hard Lefschetz property, then its minimal model is formal (see Tralle [19]). In the paper [1], de Andrés, Fernández, de León and Mencía proved that the minimal models of 6-dimensional compact A-type solvmanifolds are formal.

Next, let $\varphi(\mathbf{t},\mathbf{x})$ $(\mathbf{t} \in \mathbf{R}^l, \mathbf{x} \in \mathbf{R}^n)$ be an automorphism of $\mathbf{R}^{2m}$ constructed above. We consider a solvable Lie group $\tilde{G} = \mathbf{R}^{2n+2l} \ltimes_{\tilde{\varphi}} \mathbf{R}^{4m}$, where $\tilde{\varphi}(\mathbf{t},\mathbf{x}) = \varphi(\mathbf{t},\mathbf{x}) \oplus \varphi(\mathbf{t},\mathbf{x})$, that is, the group structure of $\tilde{G}$ is defined by

$$(\mathbf{s}_1,\mathbf{t}_1,\mathbf{x}_1,\mathbf{r}_1,\mathbf{y}_1,\mathbf{z}_1) * (\mathbf{s}_2,\mathbf{t}_2,\mathbf{x}_2,\mathbf{r}_2,\mathbf{y}_2,\mathbf{z}_2)$$
$$= (\mathbf{s}_1+\mathbf{s}_2,\mathbf{t}_1+\mathbf{t}_2,\mathbf{x}_1+\mathbf{x}_2,\mathbf{r}_1+\mathbf{r}_2,\mathbf{y}_1+\varphi(\mathbf{t}_1,\mathbf{x}_1)\mathbf{y}_2,\mathbf{z}_1+\varphi(\mathbf{t}_1,\mathbf{x}_1)\mathbf{z}_2)$$

for $\mathbf{s}_i,\mathbf{t}_i \in \mathbf{R}^l$, $\mathbf{x}_i,\mathbf{r}_i \in \mathbf{R}^n$ and $\mathbf{y}_i,\mathbf{z}_i \in \mathbf{R}^{2m}$. Then the matrix form of $\tilde{G}$ is given by

$$\tilde{G} = \left\{ \left(\begin{array}{ccccccc} \varphi(\mathbf{t},\mathbf{x}) & 0 & 0 & 0 & 0 & 0 & \mathbf{y} \\ 0 & \varphi(\mathbf{t},\mathbf{x}) & 0 & 0 & 0 & 0 & \mathbf{z} \\ 0 & 0 & 1 & 0 & 0 & 0 & \mathbf{x} \\ 0 & 0 & 0 & 1 & 0 & 0 & \mathbf{t} \\ 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{r} \\ 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{s} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \middle| \, \mathbf{s},\mathbf{t} \in \mathbf{R}^l, \mathbf{r},\mathbf{x} \in \mathbf{R}^n, \mathbf{y},\mathbf{z} \in \mathbf{R}^{2m} \right\}.$$

Note that $\tilde{G}$ is a completely solvable Lie group which has a lattice.

In section 6 we prove the following:

PROPOSITION 4. *A solvable Lie group* $\tilde{G} = \mathbf{R}^{2n+2l} \ltimes_{\tilde{\varphi}} \mathbf{R}^{4m}$ *has a left invariant complex structure.*

PROPOSITION 5. *If a solvable Lie group* $G = \mathbf{R}^{n+l} \ltimes_{\varphi} \mathbf{R}^{2m}$ *has a symplectic structure, then* $\tilde{G} = \mathbf{R}^{2n+2l} \ltimes_{\tilde{\varphi}} \mathbf{R}^{4m}$ *has a pseudo-Kähler structure.*

Using Theorem 2 and Proposition 5, we can construct compact pseudo-Kähler solvmanifolds which do not have the Hard Lefschetz property.

Consider the direct product $G' = \tilde{G} \times \mathbf{C}^{n+l}$. Note that G' also has a lattice and a complex structure. Let M^{2n} be a $2n$-dimensional complex manifold. A holomorphic 2-form $\Omega \in \Omega^{2,0}(M)$ is called a holomorphic symplectic structure on M if it satisfies $d\Omega = 0$ and $\Omega^n \neq 0$ at each point of M. Todorov conjectured that any holomorphic symplectic manifold admits a Kähler structure (See [4], [8]). However, Guan has constructed non-simply-connected holomorphic symplectic non-Kähler manifolds and simply-connected holomorphic symplectic non-Kähler manifolds ([8], [9], [10]). He also consider a deformation of holomorphic symplectic manifolds. However the examples of compact holomorphic symplectic non-Kähler manifolds are not so much (In the non-compact case, many examples are known, say, complex cotangent bundle $M = \bigwedge^{1,0} N$ of a complex manifold N). We prove the following:

PROPOSITION 6. *If a solvable Lie group* $G = \mathbf{R}^{n+l} \ltimes_{\varphi} \mathbf{R}^{2m}$ *has a left G-invariant symplectic form, then* $G'/\Gamma' = \tilde{G}/\tilde{\Gamma} \times \mathbf{C}^{n+l}/\Gamma$ *has a holomorphic symplectic structure.*

In section 6, we also construct hypercomplex structures on certain compact solvmanifolds. We give some examples in section 5 and 7. In section 8 and 9, we construct solvable Lie groups with parameterized lattices and holomorphic symplectic structures. As a consequence, we get families of compact holomorphic symplectic non-Kähler solvmanifolds.

The author would like to express his deep appreciation to Professor Yusuke Sakane for his thoughtful guidance and encouragement given during the completion of this paper. The author also thanks Professor Ryushi Goto for several advice.

1. Definitions and Nomizu-Hattori Theorem

Let (M, ω) be a compact symplectic manifold and $\Omega^k(M)$ the space of all differential k-forms. We define a linear mapping $L : \Omega^k(M) \to \Omega^{k+2}(M)$ by

$L(\alpha) = \alpha \wedge \omega$. Since ω is closed, we have $Ld = dL$. Hence, the mapping L induces a linear mapping $L : H^k_{DR}(M) \to H^{k+2}_{DR}(M)$ by $L([\alpha]) = [L(\alpha)]$.

Definition 1.1. Let (M^{2m}, ω) be a compact symplectic manifold.

(1) If the Lefschetz mapping $L^{m-1} : H^1_{DR}(M) \to H^{2m-1}_{DR}(M)$ is an isomorphism, then (M^{2m}, ω) is called a Lefschetz manifold.

(2) If the Lefschetz mapping $L^k : H^{m-k}_{DR}(M) \to H^{m+k}_{DR}(M)$ is an isomorphism for any $k \leq m$, then we say that (M^{2m}, ω) has the Hard Lefschetz property.

Note that compact Kähler manifolds have the Hard Lefschetz property.

Let $\mathfrak{g}$ be a Lie algebra and put $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}_i]$. A Lie algebra $\mathfrak{g}$ is called solvable if $\mathfrak{g}_{r+1} = (0)$ for some r. A Lie group G is called solvable if its Lie algebra $\mathfrak{g}$ is solvable.

Definition 1.2. A solvable Lie algebra $\mathfrak{g}$ is called completely solvable if $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues for each $X \in \mathfrak{g}$. A solvable Lie group G is called completely solvable if its Lie algebra is completely solvable.

By a compact solvmanifold G/Γ, we mean a right coset space of G modulo Γ, where G is a simply-connected completely solvable Lie group and Γ a lattice, that is, a discrete co-compact subgroup of G.

We denote the Lie algebra of G by $\mathfrak{g}$. We identify $\bigwedge^* \mathfrak{g}^*$ with the space of all left G-invariant forms on G/Γ. Then Hattori [13] proved the following:

Nomizu-Hattori Theorem. *The inclusion map* $i : \bigwedge^k \mathfrak{g}^* \to \Omega^k(G/\Gamma)$ *induces an isomorphism* $H^k(\mathfrak{g}) \to H^k_{DR}(G/\Gamma)$ *for each* k.

Let $(G/\Gamma, \omega)$ be a compact symplectic solvmanifold. By Nomizu-Hattori Theorem, there exists a left G-invariant closed 2-form ω_0 on G/Γ such that $\omega - \omega_0 = d\gamma$. Note that ω_0 is also a symplectic structure. Therefore we may assume that a symplectic structure on $M = G/\Gamma$ is left G-invariant to investigate the Hard Lefschetz property.

2. Closed Forms on Certain Solvable Lie Algebras

In this section we consider left G-invariant closed forms on G constructed in Proposition 1.

The Lie algebra $\mathfrak{g}$ of G constructed in Proposition 1 can be written as follows.

$$\mathfrak{g} = \mathrm{span}\{A_1, \dots, A_l, B_1, \dots, B_n, Y_1, \dots, Y_{2m}\}$$

with

$$\begin{aligned} [A_i, Y_{2k-1}] &= a_i^k Y_{2k-1}, & [A_i, Y_{2k}] &= -a_i^k Y_{2k}, \\ [B_j, Y_{2h-1}] &= \sum_{k<h} b_j^{kh} Y_{2k-1}, & [B_j, Y_{2h}] &= \sum_{k<h} b_j^{kh} Y_{2k} \end{aligned} \tag{2.1}$$

for $i = 1, \dots, l$, $j = 1, \dots, n$ and $1 \le k < h \le m$. We assume that for each k, there exists an i such that $a_i^k \neq 0$. Let $\{\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n, \omega_1, \dots, \omega_{2m}\}$ be the dual basis corresponding to $\{A_1, \dots, A_l, B_1, \dots, B_n, Y_1, \dots, Y_{2m}\}$. We write $\omega_{k_1} \wedge \cdots \wedge \omega_{k_p}$ simply as ω_K and set $\#K = p$ for $K = (k_1, \dots, k_p)$. Note that $d\omega_K$ can be written as follows:

$$d\omega_K = -\sum_{i=1}^{l} a_i^K \alpha_i \wedge \omega_K - \sum_{j=1}^{n} \sum_H b_j^{KH} \beta_j \wedge \omega_H.$$

LEMMA 2.1 ([21]). *Let $\gamma = \sum_{IJK} c_{IJK} \alpha_I \wedge \beta_J \wedge \omega_K$ be a closed form such that $\#I + \#J$ and $\#K$ are constant. If for each K, there exists an i such that $a_i^K \neq 0$, then γ is an exact form.*

PROOF. See [21]. □

We set

$$\begin{aligned} \mathfrak{a} &= \mathrm{span}\{A_1, \dots, A_l\}, \\ \mathfrak{b} &= \mathrm{span}\{B_1, \dots, B_n\}, \\ \mathfrak{m} &= \mathrm{span}\{Y_1, \dots, Y_{2m}\}. \end{aligned}$$

For simplicity, we denote $\bigwedge^i (\mathfrak{a} \times \mathfrak{b})^* \wedge \bigwedge^j \mathfrak{m}^*$ by $\bigwedge^{i,j}$.

LEMMA 2.2 ([21]).

(1) *If $\alpha = \alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2} \in Z^2(\mathfrak{g})$, where $\alpha_{i,j} \in \bigwedge^{i,j}$, then $d\alpha_{2,0} = d\alpha_{1,1} = d\alpha_{0,2} = 0$.*

(2) $\bigwedge^{1,1} \cap Z^2(\mathfrak{g}) \subset B^2(\mathfrak{g})$.

PROOF. Since

$$d\omega_{2k-1} = -\sum_i a_i^k \alpha_i \wedge \omega_{2k-1} - \sum_{k<h}\sum_{j=1}^n b_j^{kh}\beta_j \wedge \omega_{2h-1},$$

$$d\omega_{2k} = \sum_i a_i^k \alpha_i \wedge \omega_{2k} - \sum_{k<h}\sum_{j=1}^n b_j^{kh}\beta_j \wedge \omega_{2h},$$

we have

$$\bigwedge^{0,2} \xrightarrow{d} \bigwedge^{1,2},$$

$$\bigwedge^{2,0} \xrightarrow{d} 0,$$

$$\bigwedge^{1,1} \xrightarrow{d} \bigwedge^{2,1}.$$

Since we assume that for each k, there exists an i such that $a_i^k \neq 0$, we have Lemma 2.2 using Lemma 2.1. □

3. Closed Forms on Nilpotent Lie Algebras

We use the same notations as in section 2.

By Lemma 2.2, we may assume that a symplectic structure ω on a solvable Lie group G constructed above is an element of $\bigwedge^{2,0} + \bigwedge^{0,2}$ to study the Hard Lefschetz property. Thus we write $\omega = \omega_{2,0} + \omega_{0,2}$, where $\omega_{2,0} \in \bigwedge^{2,0}$, $\omega_{0,2} \in \bigwedge^{0,2}$. Note that $\omega_{2,0}$ and $\omega_{0,2}$ are symplectic structures on $\mathfrak{a} \times \mathfrak{b}$, $\mathfrak{m}$ respectively.

Let $\mathfrak{n}$ be a Lie algebra. Put $\mathfrak{n}^{(0)} = \mathfrak{n}$ and $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}]$ for $i \geq 0$. We say that the Lie algebra $\mathfrak{n}$ is $(r+1)$-step nilpotent if $\mathfrak{n}^{(r)} \neq (0)$ and $\mathfrak{n}^{(r+1)} = (0)$. A Lie group N is called $(r+1)$-step nilpotent if its Lie algebra $\mathfrak{n}$ is $(r+1)$-step nilpotent.

Note that $\mathfrak{n} = \mathfrak{b} \ltimes \mathfrak{m}$ is a nilpotent Lie algebra and $\omega_{0,2}$ can be considered as a closed form on the simply-connected nilpotent Lie group N corresponding to $\mathfrak{n}$. Thus we consider left N-invariant closed forms on a nilpotent Lie group N.

Let $\mathfrak{n}$ be an $(r+1)$-step nilpotent Lie algebra. Consider the descending central series $\{\mathfrak{n}^{(i)}\}$ of $\mathfrak{n}$. Let $\mathfrak{u}^{(i)}$ be a vector subspace of $\mathfrak{n}^{(i)}$ such that

$$\mathfrak{n}^{(i)} = \mathfrak{n}^{(i+1)} + \mathfrak{u}^{(i)}$$

for $i = 0, 1, \ldots, r-1$ and define $n_i = \dim \mathfrak{u}^{(i)}$. For simplicity, let $\bigwedge^{i_0} \mathfrak{u}^{(0)^*} \wedge \cdots \wedge \bigwedge^{i_r} \mathfrak{u}^{(r)^*} = \bigwedge^{i_0,\ldots,i_r}$. Then

$$\bigwedge^s \mathfrak{n}^* = \sum_{i_0+\cdots+i_r=s} \bigwedge^{i_0,\dots,i_r}.$$

For an $(r+1)$-step nilpotent Lie algebra $\mathfrak{n}$, we have the following:

LEMMA 3.1 ([2]). *Any closed 2-form* $\sigma \in \bigwedge^2 \mathfrak{n}^*$ *belongs to* $\bigwedge^{1,0,\dots,0,1} + \sum \bigwedge^{i_0,\dots,i_{r-1},0}$.

Let $\zeta_1, \dots, \zeta_{n_r}$ be a basis of $\bigwedge^{0,\dots,0,1}$. By Lemma 3.1, a left N-invariant symplectic form ω on a nilpotent Lie group N can be written as

$$\omega = \gamma_1 \wedge \zeta_1 + \cdots + \gamma_{n_r} \wedge \zeta_{n_r} \quad \text{modulo} \quad \sum \bigwedge^{i_0,\dots,i_{r-1},0},$$

where $\gamma_1, \dots, \gamma_{n_r}$ are elements of $\bigwedge^{1,0,\dots,0}$. Since ω is non-degenerate, $\gamma_1, \dots, \gamma_{n_r}$ are linearly independent and we extend these to a basis

$$\gamma_1, \dots, \gamma_{n_r}, \dots, \gamma_{n_0}$$

of $\bigwedge^{1,0,\dots,0}$.

LEMMA 3.2 ([2]). *Let* $\mathfrak{n}$ *be an* $(r+1)$*-step nilpotent Lie algebra of dimension* $2m$. *Then we have*

(1) $\bigwedge^{2m-1}(\mathfrak{n}^*) = Z^{2m-1}(\mathfrak{n})$,

(2) $\sum \bigwedge^{n_0,i_1,\dots,i_r} = B^{2m-1}(\mathfrak{n})$.

4. The Lefschetz Mapping on Certain Compact Symplectic Solvmanifolds

In this section we prove Theorem 2 and Proposition 3. We assume that a symplectic structure ω is left G-invariant. We use the same notations as in sections 2 and 3.

Put

$$\mathfrak{m}_1 = \text{span}\{Y_1, Y_3, \dots, Y_{2m-1}\},$$

$$\mathfrak{m}_2 = \text{span}\{Y_2, Y_4, \dots, Y_{2m}\},$$

$$\mathfrak{m} = \text{span}\{Y_1, Y_2, \dots, Y_{2m}\}.$$

Then Lie algebras $\mathfrak{n}_1 = \mathfrak{b} \ltimes \mathfrak{m}_1$, $\mathfrak{n}_2 = \mathfrak{b} \ltimes \mathfrak{m}_2$ and $\mathfrak{n} = \mathfrak{b} \ltimes \mathfrak{m}$ are $(r+1)$-step nilpotent.

Since $\mathfrak{n}_1$ is a nilpotent Lie algebra, there exists a basis

$$\mathfrak{n}_1^* = \mathrm{span}\{\beta_1, \dots, \beta_n, \zeta_1^{(0)}, \dots, \zeta_{n_0}^{(0)}, \dots, \zeta_1^{(r)}, \dots, \zeta_{n_r}^{(r)}\},$$

which satisfies for each $\rho = 0, \dots, r-1$,

$$d_{\mathfrak{n}_1}\zeta_k^{(\rho+1)} \in \bigwedge\{\beta_1, \dots, \beta_n, \zeta_1^{(0)}, \dots, \zeta_{n_0}^{(0)}, \dots, \zeta_1^{(\rho)}, \dots, \zeta_{n_\rho}^{(\rho)}\},$$
$$d_{\mathfrak{n}_1}\beta_j = d_{\mathfrak{n}_1}\zeta_k^{(0)} = 0,$$

where $d_{\mathfrak{n}_1}$ is the exterior differential on $\bigwedge^* \mathfrak{n}_1$ (cf. [2, the proof of Lemma 2.1]). Put

$$\mathfrak{u}_1^{(\rho)^*} = \mathrm{span}\{\zeta_1^{(\rho)}, \dots, \zeta_{n_\rho}^{(\rho)}\}.$$

Then we have $\mathfrak{n}_1^* = \mathfrak{b}^* \oplus \mathfrak{u}_1^{(0)^*} \oplus \cdots \oplus \mathfrak{u}_1^{(r)^*}$. For simplicity, let

$$\bigwedge^j \mathfrak{b}^* \wedge \bigwedge^{i_0} \mathfrak{u}_1^{(0)^*} \wedge \cdots \wedge \bigwedge^{i_r} \mathfrak{u}_1^{(r)^*} = \bigwedge^{(j, i_0), i_1, \dots i_r} \mathfrak{n}_1^*.$$

Since $\{\beta_1, \dots, \beta_n, \omega_1, \omega_3, \dots, \omega_{2m-1}\}$ is also a basis of $\mathfrak{n}_1^*$, we can write $\zeta_k^{(\rho)}$ as

$$\zeta_k^{(\rho)} = \sum_h c_{kh}^{(\rho)} \omega_{2h-1}.$$

Then we define 1-forms $\eta_k^{(\rho)}$ by $\eta_k^{(\rho)} = \sum_h c_{kh}^{(\rho)} \omega_{2h}$. It is obvious from (2.1) that

$$d_{\mathfrak{n}_2}\eta_i^{(\rho+1)} \in \bigwedge\{\beta_1, \dots, \beta_n, \eta_1^{(0)}, \dots, \eta_{n_0}^{(0)}, \dots, \eta_1^{(\rho)}, \dots, \eta_{n_\rho}^{(\rho)}\}.$$

Now consider $\zeta_k^{(\rho)}$, $\eta_k^{(\rho)}$ as left G-invariant 1-forms on G. Then we see

$$\begin{aligned} d\zeta_k^{(\rho)} &= -\sum_{i=1}^{l} a_{ik}^{(\rho)} \alpha_i \wedge \zeta_k^{(\rho)} + \sum_{j=1}^{n} \sum_{h=1}^{n_0+\cdots+n_{\rho-1}} b_{kjh}^{(\rho)} \beta_j \wedge \zeta_h, \\ d\eta_k^{(\rho)} &= \sum_{i=1}^{l} a_{ik}^{(\rho)} \alpha_i \wedge \eta_k^{(\rho)} + \sum_{j=1}^{n} \sum_{h=1}^{n_0+\cdots+n_{\rho-1}} b_{kjh}^{(\rho)} \beta_j \wedge \eta_h, \end{aligned} \tag{4.1}$$

where

$$\{\zeta_1, \dots, \zeta_m\} = \{\zeta_1^{(0)}, \dots, \zeta_{n_0}^{(0)}, \dots, \zeta_1^{(r)}, \dots, \zeta_{n_r}^{(r)}\}.$$

Similarly, we write

$$\bigwedge^j \mathfrak{b}^* \wedge \bigwedge^{i_0} \mathfrak{u}_2^{(0)^*} \wedge \cdots \wedge \bigwedge^{i_r} \mathfrak{u}_2^{(r)^*} = \bigwedge^{(j, i_0), i_1, \dots i_r} \mathfrak{n}_2^*,$$

$$\bigwedge^j \mathfrak{b}^* \wedge \bigwedge^{i_0} \mathfrak{u}^{(0)^*} \wedge \cdots \wedge \bigwedge^{i_r} \mathfrak{u}^{(r)^*} = \bigwedge^{(j, i_0), i_1, \dots i_r} \mathfrak{n}^*,$$

where $\mathfrak{u}^{(\rho)^*} = \mathfrak{u}_1^{(\rho)^*} + \mathfrak{u}_2^{(\rho)^*}$.

THEOREM 4.1. *Let $M = G/\Gamma$ be a compact solvmanifold constructed in Proposition* 1 *and assume that M has a symplectic structure. Then M has the Hard Lefschetz property if and only if M is a compact A-type solvmanifold.*

PROOF. By Lemma 2.2 and Lemma 3.1, $\omega_{0,2}$ can be written as

$$\omega_{0,2} = \gamma_1 \wedge \zeta_1^{(r)} + \cdots + \gamma_{n_r} \wedge \zeta_{n_r}^{(r)} + \lambda_1 \wedge \eta_1^{(r)} + \cdots + \lambda_{n_r} \wedge \eta_{n_r}^{(r)} + \tau,$$

where $\gamma_k, \lambda_k \in \bigwedge^{(0,1),0,\dots 0} \mathfrak{n}^*$ and $\tau \in \bigwedge^{(0,i_0),i_1,\dots,i_{r-1},0} \mathfrak{n}^*$. It is obvious from (4.1) that $\gamma_k \wedge \lambda_k$ is a non-exact closed 2-form for each $k = 1, \dots, n_r$ (Note that $d_{\mathfrak{n}} \bigwedge^{(0,1),0,\dots 0} \mathfrak{n}^* = 0$). Then

$$\begin{aligned}\gamma_k \wedge \lambda_k \xrightarrow{L^{(1/2)(n+l)+m-2}} & a_1 \cdot \omega_{2,0}^{(1/2)(n+l)} \wedge \zeta_1^{(0)} \wedge \zeta_2^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \\ & \wedge \eta_1^{(0)} \wedge \eta_2^{(0)} \wedge \cdots \wedge \hat{\eta}_k^{(r)} \wedge \cdots \wedge \eta_{n_r}^{(r)} \\ & = a_2 \cdot \alpha_1 \wedge \cdots \wedge \alpha_l \wedge \zeta_1^{(0)} \wedge \zeta_2^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \\ & \wedge \beta_1 \wedge \cdots \wedge \beta_n \wedge \eta_1^{(0)} \wedge \eta_2^{(0)} \wedge \cdots \wedge \hat{\eta}_k^{(r)} \wedge \cdots \wedge \eta_{n_r}^{(r)} \\ & = a_2 \cdot \alpha_1 \wedge \cdots \wedge \alpha_l \wedge \zeta_1^{(0)} \wedge \zeta_2^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \wedge d_{\mathfrak{n}_2}\theta \\ & = a_2 \cdot \alpha_1 \wedge \cdots \wedge \alpha_l \wedge \zeta_1^{(0)} \wedge \zeta_2^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \wedge d\theta \\ & = (-1)^{m+l-1} a_2 \cdot d(\alpha_1 \wedge \cdots \wedge \alpha_l \wedge \zeta_1^{(0)} \wedge \cdots \wedge \hat{\zeta}_k^{(r)} \wedge \cdots \wedge \zeta_{n_r}^{(r)} \wedge \theta),\end{aligned}$$

where $a_1, a_2 \in \mathbf{R}$, $\theta \in \bigwedge \mathfrak{n}_2^*$ and $d_{\mathfrak{n}_2}$ is the exterior differential on $\bigwedge \mathfrak{n}_2^*$. The second equality holds by Lemma 3.2. The third and fourth equalities hold by the following fact:

$$d\zeta_k^{(r)} = -\sum_{i=1}^{n} a_{ik}^{(r)} \alpha_i \wedge \zeta_k^{(r)} + \sum \bigwedge^{(1,i_0),i_1,\dots i_{r-1},0} \mathfrak{n}_1^*,$$

$$d\eta_k^{(r)} = \sum_{i=1}^{n} a_{ik}^{(r)} \alpha_i \wedge \eta_k^{(r)} + \sum \bigwedge^{(1,i_0),i_1,\dots i_{r-1},0} \mathfrak{n}_2^*.$$

Then

$$L^{(1/2)(n+l)+m-2} : H^2(\mathfrak{g}) \to H^{n+l+2m-2}(\mathfrak{g})$$

is not an isomorphism if M is not a compact A-type solvmanifold.

Conversely, let M be a compact A-type solvmanifold. Since $d\omega_K = -\sum_{i=1}^{l} a_i^K \alpha_i \wedge \omega_K$, if $\sum_{\#I+\#K=p+q=r} c_{IK} \alpha_I \wedge \omega_K$ is a closed form, then $\sum_{\#I=p} c_{IK} \alpha_I \wedge \omega_K$ is also a closed form. Moreover, it is obvious that if $d\omega_K = 0$, then $\sum_{\#I=p} c_{IK}\alpha_I \wedge \omega_K$ is a non-exact closed form. By Lemma 2.1, if $d\omega_K \neq 0$, then a closed form $\sum_{\#I=p} c_{IK}\alpha_I \wedge \omega_K$ is exact. Then for each de Rham cohomology class, we can choose a representation $\alpha = \sum_{I,K} c_{IK}\alpha_I \wedge \omega_K$ such that $d\omega_K = 0$.

On the other hand, we can assume that a symplectic form ω on M can be written as

$$\omega = \omega_{2,0} + \sum_{k,h} P_{kh}\omega_k \wedge \omega_h,$$

where $\omega_{2,0}$ is a non-degenerate closed form on $\bigwedge^{2,0}$. Since $d\omega_K = -\sum_{i=1}^{l} a_i^K \alpha_i \wedge \omega_K$, $\omega_k \wedge \omega_h$ is closed for each k, h such that $P_{kh} \neq 0$. Then we have

$$L^k\alpha = \sum_{I',K'} c_{I'K'}\alpha_{I'} \wedge \omega_{K'} \quad d\omega_{K'} = 0,$$

which implies $L^k\alpha$ is not exact by the above argument. Then A-type has the Hard Lefschetz property. □

REMARK. In the paper [21], we showed that a compact symplectic solvmanifold constructed in Proposition 1 is a compact Lefschetz manifold.

Using the notion of differential graded algebra (or, briefly, D.G.A.), we define the minimal model of M. $\mathscr{A} = (\mathscr{A}, d)$ is called a D.G.A. if $\mathscr{A}$ is a graded algebra $\mathscr{A} = \bigoplus_{i \geq 0} \mathscr{A}^i$ with the commutativity $a \cdot b = (-1)^{pq} b \cdot a$ for $a \in \mathscr{A}^p$, $b \in \mathscr{A}^q$ and d an antiderivation of degree 1 as follows:

$$d^2 = 0,$$
$$d(a \cdot b) = da \cdot b + (-1)^p a \cdot db$$

for $a \in \mathscr{A}^p$, $b \in \mathscr{A}^q$.

DEFINITION 4.2. Let $\mathscr{A}$, $\mathscr{B}$ be D.G.A., $\mathscr{B}$ is a Hirsch extension of degree n of $\mathscr{A}$, if $\mathscr{B}$ is of the following form:

$$\mathscr{B} = \mathscr{A} \otimes \bigwedge\nolimits_n \langle x_1, \ldots, x_k \rangle$$
$$\deg x_i = n, \quad dx_i \in \mathscr{A} \quad \text{for } i = 1, \ldots, k,$$

where $\bigwedge_n \langle x_1, \ldots, x_k \rangle$ is the free graded commutative algebra with unit generated by $\{x_1, \ldots, x_k\}$.

DEFINITION 4.3. A D.G.A. $\mathscr{A}$ is said to be minimal if $\mathscr{A}$ satisfies the following:

(i) $\mathscr{A} = \bigcup_{i \geq 0} \mathscr{A}_i$, where $\mathscr{A}_0 = \mathbf{R}$ and $\mathscr{A}_{i+1}$ is a Hirsch extension of $\mathscr{A}_i$ for $i \geq 0$.

(ii) $dx \in \mathscr{A}_+ \cdot \mathscr{A}_+$, where $x \in \mathscr{A}$ and $\mathscr{A}_+ = \bigoplus_{i \geq 1} \mathscr{A}^i$.

DEFINITION 4.4. Let $(\mathscr{M}, d_{\mathscr{M}})$, $(\mathscr{A}, d_{\mathscr{A}})$ be D.G.A.. $(\mathscr{M}, d_{\mathscr{M}})$ is called a model for $(\mathscr{A}, d_{\mathscr{A}})$ if there exists a D.G.A.-morphism

$$\rho : (\mathscr{M}, d_{\mathscr{M}}) \to (\mathscr{A}, d_{\mathscr{A}})$$

which induces an isomorphism on cohomology. Moreover, if $(\mathscr{M}, d_{\mathscr{M}})$ is minimal, then $(\mathscr{M}, d_{\mathscr{M}})$ is called a minimal model for $(\mathscr{A}, d_{\mathscr{A}})$.

By the minimal model of M, we mean the minimal model of de Rham cohomology complex $(\Omega^*(M), d)$ of M.

DEFINITION 4.5. A manifold M is called formal if $(\Omega^*(M), d)$ and $(H^*_{DR}(M), d = 0)$ have the same minimal model.

PROPOSITION 4.6. *The minimal model of a compact A-type solvmanifold is formal.*

PROOF. We define a mapping of cochain complex $f : (H^*(\mathfrak{g}), d = 0) \to (\bigwedge^*(\mathfrak{g}^*), d)$ by

$$\left[\sum_{\#I + \#K = p+q=r} c_{IK} \alpha_I \wedge \omega_K \right] \xrightarrow{f} \sum_{\#I + \#K = p+q=r} c_{IK} \alpha_I \wedge \omega_K,$$

where each ω_K is closed. It is obvious from the proof of Theorem 4.1 that the mapping is multiplicative, that is, f satisfies $f([a] \wedge [b]) = f([a]) \wedge f([b])$. Then the minimal model of A-type is formal (See [7], p. 158 and [1]). □

5. Examples Related to the Hard Lefschetz Property

EXAMPLE 5.1 ([1]). We consider the following matrices:

$$A = \sum_{k=1}^{m} a(E_{2k-1,2k-1} - E_{2k,2k})$$

$$B = 0.$$

We denote by $\mathfrak{g}$ the Lie algebra constructed by using A and B in Proposition 1. By the proof of Theorem 4.1, it is easy to verify that

$$H^{2q-1}(\mathfrak{g}) = \mathrm{span}\{[\alpha \wedge \zeta_I \wedge \eta_J], [\beta \wedge \zeta_I \wedge \eta_J] \ (\#I = \#J = q-1)\},$$

$$H^{2q}(\mathfrak{g}) = \mathrm{span}\{[\alpha \wedge \beta \wedge \zeta_I \wedge \eta_J] \ (\#I = \#J = q-1), [\zeta_I \wedge \eta_J] \ (\#I = \#J = q)\},$$

where $\zeta_I = \omega_{2i_1-1} \wedge \cdots \wedge \omega_{2i_r-1}$ for $I = (i_1, \dots, i_r)$ and $\eta_J = \omega_{2j_1} \wedge \cdots \wedge \omega_{2j_r}$ for $J = (j_1, \dots, j_r)$. In particular, we see that the odd betti numbers $b_{2i-1}(M)$ are even and $b_i(M) \geq b_{i-2}(M)$ $(i \leq m+1)$. $M(a) = \mathbf{R}^2 \ltimes \mathbf{R}^{2m}/\Gamma$ has a symplectic structure. For example,

$$\omega = \alpha \wedge \beta + \omega_1 \wedge \omega_2 + \cdots + \omega_{2m-1} \wedge \omega_{2m}.$$

By Theorem 4.1, $M(a)$ has the Hard Lefschetz property for any symplectic structure. Moreover, if $M = \mathbf{R}^2 \ltimes \mathbf{R}^{4m}/\Gamma$, then M admits a pseudo-Kähler structure (See Section 6 and Example 7.1).

EXAMPLE 5.2. We consider the following automorphism:

$$\varphi(t_1, t_2, x_1, x_2) = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

where

$$P_i = \begin{pmatrix} e^{t_i} & 0 & x_i e^{t_i} & 0 \\ 0 & e^{-t_i} & 0 & x_i e^{-t_i} \\ 0 & 0 & e^{t_i} & 0 \\ 0 & 0 & 0 & e^{-t_i} \end{pmatrix}$$

for $i = 1, 2$. Then $G = \mathbf{R}^4 \ltimes_\varphi \mathbf{R}^8$ has a symplectic structure. For example,

$$\omega = \alpha_1 \wedge \alpha_2 + \beta_1 \wedge \beta_2 + \omega_1 \wedge \omega_4 - \omega_3 \wedge \omega_2 + \omega_5 \wedge \omega_8 - \omega_7 \wedge \omega_6.$$

Now $\omega_3 \wedge \omega_4$, $\omega_7 \wedge \omega_8$ are non-exact closed 2-forms. As in Theorem 4.1, we see

$$\begin{aligned} \omega_3 \wedge \omega_4 &\overset{L_\omega^4}{\longrightarrow} a \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5 \wedge \omega_6 \wedge \omega_7 \wedge \omega_8 \\ &= \pm a \cdot d(\alpha_1 \wedge \alpha_2 \wedge \beta_2 \wedge \omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6 \wedge \omega_7 \wedge \omega_8), \end{aligned}$$

$$\begin{aligned} \omega_7 \wedge \omega_8 &\overset{L_\omega^4}{\longrightarrow} b \cdot \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_7 \wedge \omega_8 \\ &= \pm b \cdot d(\alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_5 \wedge \omega_6 \wedge \omega_7 \wedge \omega_8). \end{aligned}$$

Similarly, G/Γ does not have the Hard Lefschetz property for any symplectic structure.

6. A Construction of Compact Holomorphic Symplectic Solvmanifolds

In this section we construct pseudo-Kähler Lie groups and holomorphic symplectic Lie groups from certain Lie groups. As an application, we have Propositions 4, 5 and 6. We also construct a compact solvmanifold which have a hypercomplex structure and a pseudo-hyperkähler structure.

DEFINITION 6.1. Let M be a complex manifold of dimension $2m$. A holomorphic symplectic structure is a closed holomorphic 2-form Ω on M of maximal rank, i.e. $\Omega^m \neq 0$ at each point of M.

DEFINITION 6.2. Let M be a manifold. A set of complex structures $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ which satisfies $\mathbf{IJ} = -\mathbf{JI} = \mathbf{K}$ is called a hypercomplex structure. Let (M, g) be a pseudo-Riemannian manifold which carries a hypercomplex structure $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$. Then M is called a pseudo-hyperkähler manifold if $\omega_{\mathbf{I}}$, $\omega_{\mathbf{J}}$ and $\omega_{\mathbf{K}}$ are pseudo-Kähler forms with respect to $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ respectively, where $\omega_{\mathbf{I}}(X, Y) = g(\mathbf{I}X, Y)$, $\omega_{\mathbf{J}}(X, Y) = g(\mathbf{J}X, Y)$ and $\omega_{\mathbf{K}}(X, Y) = g(\mathbf{K}X, Y)$.

We consider the following Lie algebra over $\mathbf{R}$:

$$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b},$$

where $\mathfrak{a}$ is abelian and $\mathfrak{b}$ is an ideal. Assume that

$$\mathfrak{a} = \mathrm{span}_{\mathbf{R}}\{U_1^1, \dots, U_p^1\},$$

$$\mathfrak{b} = \mathrm{span}_{\mathbf{R}}\{V_1^1, \dots, V_q^1\}.$$

Consider the complexification $\mathfrak{g}^{\mathbf{C}}$. Since $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$, ${}_{\mathbf{R}}(\mathfrak{g}^{\mathbf{C}})$ has the following basis:

$${}_{\mathbf{R}}(\mathfrak{g}^{\mathbf{C}}) = \mathrm{span}_{\mathbf{R}}\{U_1^1, \dots, U_p^1, \sqrt{-1}U_1^1, \dots, \sqrt{-1}U_p^1, V_1^1, \dots, V_q^1, V_1^2, \dots, V_q^2\},$$

where $V_j^2 = \sqrt{-1}V_j^1$. Let $\mathfrak{h}$ be the following Lie subalgebra of $\mathfrak{g}$:

$$\mathfrak{h} = \mathfrak{a} + \mathfrak{b} + \sqrt{-1}\mathfrak{b} = \mathrm{span}_{\mathbf{R}}\{U_1^1, \dots, U_p^1, V_1^1, \dots, V_q^1, V_1^2, \dots, V_q^2\}.$$

Consider a direct product

$$\mathfrak{h} \times \mathbf{R}^p = \mathrm{span}_{\mathbf{R}}\{U_1^1, \dots, U_p^1, U_1^2, \dots, U_p^2, V_1^1, \dots, V_q^1, V_1^2, \dots, V_q^2\}.$$

We define a complex structure on $\mathfrak{h} \times \mathbf{R}^p$ by the following:

$$\begin{cases} \mathbf{I}U_i^1 = U_i^2 \ (\mathbf{I}U_i^2 = -U_i^1) & i = 1, \dots, p \\ \mathbf{I}V_j^1 = V_j^2 \ (\mathbf{I}V_j^2 = -V_j^1) & j = 1, \dots, q \end{cases}$$

Note that $\mathfrak{h} \times \mathbf{R}^p$ is a Lie algebra. We use the notation $\Psi_{\mathbf{I}}(\mathfrak{g}) = (\mathfrak{h} \times \mathbf{R}^p, \mathbf{I})$ and let $\Psi_{\mathbf{I}}(G)$ be the simply-connected Lie group corresponding to $\Psi_{\mathbf{I}}(\mathfrak{g})$. Then we have the following:

PROPOSITION 6.3. **I** *is integrable on* $\Psi_{\mathbf{I}}(G)$.

PROOF. We show that the Nijenhuis tensor $N_{\mathbf{I}}(X, Y)$ vanishes. By definition of the almost complex structure **I** and $[\mathfrak{a}, \mathfrak{a}] = 0$, it is obvious that the Nijenhuis tensor $N_{\mathbf{I}}(X, Y)$ vanishes except for the case when $X = U_i^2$, $Y = V_j^1$ or V_j^2. Let $X = U_i^2$, $Y = V_j^1$. Then

$$\begin{aligned} N_{\mathbf{I}}(U_i^2, V_j^1) &= \mathbf{I}[\mathbf{I}U_i^2, V_j^1] - [\mathbf{I}U_i^2, \mathbf{I}V_j^1] \\ &= -\mathbf{I}[U_i^1, V_j^1] + [U_i^1, \mathbf{I}V_j^1] \\ &= -\sqrt{-1}[U_i^1, V_j^1] + [U_i^1, \sqrt{-1}V_j^1] = 0. \end{aligned}$$

Note that $\mathbf{I}[U_i^1, V_j^1]$ and $[U_i^1, \mathbf{I}V_j^1]$ can be considered as elements of $\mathfrak{g}^{\mathbf{C}}$. The other case is similar and hence omitted. □

Let $\{\xi_1^1, \dots, \xi_p^1, \xi_1^2, \dots, \xi_p^2, \omega_1^1, \dots, \omega_q^1, \omega_1^2, \dots, \omega_q^2\}$ be the dual basis of $\{U_1^1, \dots, U_p^1, U_1^2, \dots, U_p^2, V_1^1, \dots, V_q^1, V_1^2, \dots, V_q^2\}$. Thus as a basis for (1,0)-type we can take

$$\begin{cases} \mu_i = \xi_i^1 + \sqrt{-1}\xi_i^2 & i = 1, \dots, p \\ \lambda_j = \omega_j^1 + \sqrt{-1}\omega_j^2 & j = 1, \dots, q \end{cases}$$

THEOREM 6.4. *If* $\mathfrak{b}$ *has a non-degenerate 2-form which is closed on* $\mathfrak{g}$, *then* $\Psi_{\mathbf{I}}(G) \times \mathbf{C}^p$ *has a holomorphic symplectic structure. Moreover, if* $[\mathfrak{b}, \mathfrak{b}] = 0$, *then the solvable Lie group* $\Psi_{\mathbf{I}}(G)$ *has a pseudo-Kähler structure.*

PROOF. Let $\omega_{\mathfrak{b}} = \sum_{k<h} P_{kh}\omega_k^1 \wedge \omega_h^1$ be a non-degenerate 2-form which is closed on $\mathfrak{g}$. It is obvious that if $\tau = \sum_{k<h} P_{kh}(\lambda_k \wedge \lambda_h + \bar{\lambda}_k \wedge \bar{\lambda}_h) = 2\sum_{k<h} P_{kh}(\omega_k^1 \wedge \omega_h^1 - \omega_k^2 \wedge \omega_h^2)$ is a closed 2-form, then $\sum_{k<h} P_{kh}\lambda_k \wedge \lambda_h$ is also closed. Since $d\omega_{\mathfrak{b}} = 0$ and $\tau(X, \mathbf{I}Y) = 0$ for $X, Y \in \mathfrak{g} \subset \mathfrak{h}$, it is easy to check

that $d\tau(X, Y, Z) = d\tau(\mathbf{I}X, \mathbf{I}Y, \mathbf{I}Z) = d\tau(\mathbf{I}X, Y, Z) = 0$ for $X, Y, Z \in \mathfrak{g}$. Since $\tau(X, Y) = -\tau(\mathbf{I}X, \mathbf{I}Y) = \omega_{\mathfrak{b}}(X, Y)$ for $X, Y \in \mathfrak{g}$, we see

$$\begin{aligned} d\tau(\mathbf{I}X, \mathbf{I}Y, Z) &= -\tau([\mathbf{I}X, \mathbf{I}Y], Z) + \tau([\mathbf{I}X, Z], \mathbf{I}Y) - \tau([\mathbf{I}Y, Z], \mathbf{I}X) \\ &= +\tau([X, Y], Z) - \tau([X, Z], Y) + \tau([Y, Z], X) \\ &= +\omega_{\mathfrak{b}}([X, Y], Z) - \omega_{\mathfrak{b}}([X, Z], Y) + \omega_{\mathfrak{b}}([Y, Z], X) \\ &= -d\omega_{\mathfrak{b}}(X, Y, Z) = 0, \end{aligned}$$

where $X, Y \in \mathfrak{b} \subset \mathfrak{h}$, $Z \in \mathfrak{g}$. If $X \in \mathfrak{a}$ or $Y \in \mathfrak{a}$, then it is obvious that $d\tau(\mathbf{I}X, \mathbf{I}Y, Z) = 0$. Thus $\sum_{k<h} P_{kh}\lambda_k \wedge \lambda_h$ is closed. Hence

$$\Omega = \sum_{i=1}^{p} \mu_i \wedge \mu_i' + \sum_{k<h} P_{kh}\lambda_k \wedge \lambda_h,$$

where $\{\mu_i'\}_{i=1,\dots,p}$ is a basis of $\Omega^{1,0}(\mathbf{C}^p)$, is a holomorphic symplectic structure on $\Psi_{\mathbf{I}}(G) \times \mathbf{C}^p$.

Next assume that $[\mathfrak{b}, \mathfrak{b}] = 0$ and consider $\theta = \sum_{k<h} P_{kh}(\lambda_k \wedge \bar{\lambda}_h + \bar{\lambda}_k \wedge \lambda_h) = 2\sum_{k<h} P_{kh}(\omega_k^1 \wedge \omega_h^1 + \omega_k^2 \wedge \omega_h^2)$. Note that $\theta(X, Y) = \theta(\mathbf{I}X, \mathbf{I}Y) = \omega_{\mathfrak{b}}(X, Y)$ and $\theta(\mathbf{I}X, Y) = 0$ for $X, Y \in \mathfrak{g} \subset \mathfrak{h}$. Since $\theta([X, Y], Z) = \omega_{\mathfrak{b}}([X, Y], Z) = 0$ for $X, Y \in \mathfrak{b} \subset \mathfrak{h}$, $Z \in \mathfrak{g}$, we have

$$\begin{aligned} d\theta(\mathbf{I}X, \mathbf{I}Y, Z) &= -\theta([\mathbf{I}X, \mathbf{I}Y], Z) + \theta([\mathbf{I}X, Z], \mathbf{I}Y) - \theta([\mathbf{I}Y, Z], \mathbf{I}X) \\ &= -\omega_{\mathfrak{b}}([X, Y], Z) + \omega_{\mathfrak{b}}([X, Z], Y) - \omega_{\mathfrak{b}}([Y, Z], X) \\ &= d\omega_{\mathfrak{b}}(X, Y, Z) = 0, \end{aligned}$$

where $X, Y \in \mathfrak{b} \subset \mathfrak{h}$, $Z \in \mathfrak{g}$. If $X \in \mathfrak{a}$ or $Y \in \mathfrak{a}$, then it is obvious that $d\theta(\mathbf{I}X, \mathbf{I}Y, Z) = 0$. The other cases are similar to the case of a holomorphic symplectic structure. Thus θ is closed. Hence,

$$\omega = \sqrt{-1}\sum_{i=1}^{p} \mu_i \wedge \bar{\mu}_i + \sum_{k<h} P_{kh}(\lambda_k \wedge \bar{\lambda}_h + \bar{\lambda}_k \wedge \lambda_h)$$

is a pseudo-Kähler form on $(\Psi_{\mathbf{I}}(G), \mathbf{I})$. □

REMARK. The signature of the pseudo-Kähler metric constructed above is $(p+q, q)$.

Let $[\mathfrak{b}, \mathfrak{b}] = 0$ and consider $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(\mathfrak{g}))$. Then the solvable Lie group $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(\mathfrak{g}))$ can be written as follows.

$$\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(\mathfrak{g})) = \mathrm{span}_{\mathbf{R}}\{U_1^1, \dots, U_p^1, \dots, U_1^4, \dots, U_p^4, V_1^1, \dots, V_q^1, \dots, V_1^4, \dots, V_q^4\},$$

where the bracket products are

$$[U_i^1, V_j^h] = \sum_{k=1}^{q} c_{ij}^k V_k^h$$

for $i = 1, \dots, p,\ j = 1, \dots, q,\ h = 1, 2, 3, 4.$

Then we have the following:

Proposition 6.5. *The simply-connected solvable Lie group $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(G))$ corresponding to $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(\mathfrak{g}))$ has a hypercomplex structure.*

Proof. Let $\{W_1^h, \dots, W_{p+q}^h\} = \{U_1^h, \dots, U_p^h, V_1^h, \dots, V_q^h\}$ for each $h = 1, 2, 3, 4$. We define almost complex structures **I**, **J**, **K** which satisfy $\mathbf{IJ} = -\mathbf{JI} = \mathbf{K}$ by

$$\begin{cases} \mathbf{I}W_i^1 = W_i^2, \\ \mathbf{I}W_i^4 = W_i^3, \end{cases} \qquad \begin{cases} \mathbf{J}W_i^1 = W_i^3, \\ \mathbf{J}W_i^2 = W_i^4, \end{cases} \qquad \begin{cases} \mathbf{K}W_i^4 = W_i^1, \\ \mathbf{K}W_i^2 = W_i^3. \end{cases}$$

It is easy to check that the Nijenhuis tensor $N(X, Y)$ vanish for each **I**, **J**, **K**. By the construction, **J** is integrable on $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(G))$. Thus we only check the case of **I** and $X = U_i^h,\ Y = V_j^h$. Let $X = U_i^1,\ Y = V_j^4$. We see

$$\begin{aligned} N_{\mathbf{I}}(U_i^1, V_j^4) &= [U_i^1, V_j^4] + \mathbf{I}[\mathbf{I}U_i^1, V_j^4] + \mathbf{I}[U_i^1, \mathbf{I}V_j^4] - \mathbf{I}[\mathbf{I}U_i^1, \mathbf{I}V_j^4] \\ &= [U_i^1, V_j^4] + \mathbf{I}[U_i^1, V_j^3] \\ &= \sum c_{ij}^k V_k^4 + \mathbf{I}\sum c_{ij}^k V_k^3 \\ &= \sum c_{ij}^k V_k^4 - \sum c_{ij}^k V_k^4 = 0. \end{aligned}$$

The other cases are similar and hence omitted. Then $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ is a hypercomplex structure on $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(G))$. □

Let $\{\xi_i^h, \omega_j^h\}_{i,j,h}$ be the dual basis of $\{U_i^h, V_j^h\}_{i,j,h}$. Then we have the following:

Theorem 6.6. *If $\mathfrak{h}$ has a non-degenerate 2-form which is closed on $\mathfrak{g}$, then the solvable Lie group $\Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(G))$ has a pseudo-hyperkähler structure.*

PROOF. Let $\omega_{\mathfrak{b}} = \sum_{k,h} P_{kh}\omega_k \wedge \omega_h$, where $P_{kh} = -P_{hk}$, be a non-degenerate 2-form which is closed on $\mathfrak{g}$. Consider the following pseudo-Riemannian metric of signature $(4p+2q, 2q)$:

$$g = \sum_{i=1}^{4}\sum_{k=1}^{p} \xi_k^i \otimes \xi_k^i + \sum P_{kh}(\omega_k^1 \otimes \omega_h^2 + \omega_k^3 \otimes \omega_h^4) - \sum P_{hk}(\omega_h^2 \otimes \omega_k^1 + \omega_h^4 \otimes \omega_k^3)$$

Then $\omega_{\mathbf{I}}$, $\omega_{\mathbf{J}}$ and $\omega_{\mathbf{K}}$ are pseudo-Kähler forms with respect to $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$. By a straightforward computation, we see

$$\omega_{\mathbf{I}} = 2\sum_{k=1}^{p}(\xi_k^1 \wedge \xi_k^2 - \xi_k^3 \wedge \xi_k^4) - \sum P_{kh}(\omega_k^1 \wedge \omega_h^1 + \omega_k^2 \wedge \omega_h^2 - \omega_k^3 \wedge \omega_h^3 - \omega_k^4 \wedge \omega_h^4),$$

$$\omega_{\mathbf{J}} = 2\sum_{k=1}^{p}(\xi_k^1 \wedge \xi_k^3 + \xi_k^2 \wedge \xi_k^4) + 2\sum P_{kh}(\omega_k^1 \wedge \omega_h^4 - \omega_k^3 \wedge \omega_h^2),$$

$$\omega_{\mathbf{K}} = -2\sum_{k=1}^{p}(\xi_k^1 \wedge \xi_k^4 - \xi_k^2 \wedge \xi_k^3) + 2\sum P_{kh}(\omega_k^1 \wedge \omega_h^3 + \omega_k^4 \wedge \omega_h^2).$$

Moreover we see $\sum_{k,h} P_{kh}\omega_k \wedge \omega_h \xrightarrow{d} -\sum_{k,h,i,j}(P_{jh}c_{ik}^j + P_{kj}c_{ih}^j)\xi_i \wedge \omega_k \wedge \omega_h$. Hence, $2\sum_j(P_{jh}c_{ik}^j + P_{kj}c_{ih}^j) = 0$. Since $\sum_{k,h} P_{kh}\omega_k^s \wedge \omega_h^t \xrightarrow{d} -\sum_{k,h,i,j}(P_{jh}c_{ik}^j + P_{kj}c_{ih}^j)\xi_i \wedge \omega_k^s \wedge \omega_h^t$, we see that $\omega_{\mathbf{I}}$, $\omega_{\mathbf{J}}$ and $\omega_{\mathbf{K}}$ are closed. □

REMARK. Let $(M, g, \mathbf{I}, \mathbf{J}, \mathbf{K})$ be a pseudo-hyperkähler manifold. Then the complex 2-form $\omega_{\mathbf{J}} + \sqrt{-1}\omega_{\mathbf{K}}$ is a holomorphic symplectic structure on $(M, \mathbf{I})$. In the above case, we have the following holomorphic symplectic structure on $(M, \mathbf{J})$:

$$\Omega = -\omega_{\mathbf{I}} + \sqrt{-1}\omega_{\mathbf{K}} = 2\sum_{k=1}^{p}\mu_k^{1,3} \wedge \mu_k^{2,4} + \sum_{k,h} P_{kh}(\lambda_k^{1,3} \wedge \lambda_h^{1,3} + \lambda_k^{2,4} \wedge \lambda_h^{2,4}),$$

where $\mu_k^{i,j} = \xi_k^i + \sqrt{-1}\xi_k^j$, $\lambda_k^{i,j} = \omega_k^i + \sqrt{-1}\omega_k^j$. Note that $(M, \mathbf{J})$ has other holomorphic symplectic structures the cohomology classes of which are different from the cohomology class of Ω. For example, by the proof of Theorem 6.4,

$$\Omega' = 2\sum_{k=1}^{p}\mu_k^{1,3} \wedge \mu_k^{2,4} + \sum_{k,h} P_{kh}\lambda_k^{1,3} \wedge \lambda_k^{2,4}$$

is also a holomorphic symplectic structure on $(M, \mathbf{J})$.

Let $\varphi(\mathbf{t}, \mathbf{x})$ $(\mathbf{t} \in \mathbf{R}^l, \mathbf{x} \in \mathbf{R}^n)$ be an automorphism of $\mathbf{R}^{2m}$ constructed in Proposition 1. Consider a solvable Lie group $\tilde{G} = \mathbf{R}^{2n+2l} \ltimes_{\tilde{\varphi}} \mathbf{R}^{4m}$, where $\tilde{\varphi}(\mathbf{t}, \mathbf{x}) = \varphi(\mathbf{t}, \mathbf{x}) \oplus \varphi(\mathbf{t}, \mathbf{x})$, that is, the group structure of $\tilde{G}$ is defined by

$$
\begin{aligned}
&(\mathbf{t}_1, \mathbf{x}_1, \mathbf{s}_1, \mathbf{r}_1, \mathbf{y}_1, \mathbf{z}_1) * (\mathbf{t}_2, \mathbf{x}_2, \mathbf{s}_2, \mathbf{r}_2, \mathbf{y}_2, \mathbf{z}_2) \\
&\qquad = (\mathbf{t}_1 + \mathbf{t}_2, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{s}_1 + \mathbf{s}_2, \mathbf{r}_1 + \mathbf{r}_2, \mathbf{y}_1 + \varphi(\mathbf{t}_1, \mathbf{x}_1)\mathbf{y}_2, \mathbf{z}_1 + \varphi(\mathbf{t}_1, \mathbf{x}_1)\mathbf{z}_2)
\end{aligned}
$$

for $\mathbf{s}_i, \mathbf{t}_i \in \mathbf{R}^l$, $\mathbf{x}_i, \mathbf{r}_i \in \mathbf{R}^n$ and $\mathbf{y}_i, \mathbf{z}_i \in \mathbf{R}^{2m}$. The Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ is

$$\tilde{\mathfrak{g}} = \{A_i, B_j, U_i, V_j, Y_k, Z_k\}_{i=1,\dots,l, j=1,\dots,n, k=1,\dots,2m},$$

where the bracket products are

$$
\begin{aligned}
&[A_i, Y_{2k-1}] = a_i^k Y_{2h-1}, && [A_i, Z_{2k-1}] = a_i^k Z_{2k-1}, \\
&[A_i, Y_{2k}] = -a_i^k Y_{2h}, && [A_i, Z_{2k}] = -a_i^k Z_{2k}, \\
&[B_j, Y_{2h-1}] = \sum_{k<h} b_j^{kh} Y_{2k-1}, && [B_j, Z_{2h-1}] = \sum_{k<h} b_j^{kh} Z_{2k-1}, \\
&[B_j, Y_{2h}] = \sum_{k<h} b_j^{kh} Y_{2k}, && [B_j, Z_{2h}] = \sum_{k<h} b_j^{kh} Z_{2k},
\end{aligned}
$$

for $i = 1, \dots, l$, $j = 1, \dots, n$ and $1 \le k < h \le m$ and the other brackets are zero.
We denote by $\{\alpha_i^1, \beta_j^1, \alpha_i^2, \beta_j^2, \omega_k^1, \omega_k^2\}$ the dual basis of $\{A_i, B_j, U_i, V_j, Y_k, Z_k\}$.
Let us consider the following Lie algebra and its decomposition:

DECOMPOSITION. 1:

$$
\begin{aligned}
\mathfrak{g} &= \operatorname{span}\{A_i, B_j, Y_k\}, \\
\mathfrak{a} &= \operatorname{span}\{A_i, B_j\}, \\
\mathfrak{b} &= \operatorname{span}\{Y_1, \dots, Y_{2m}\}.
\end{aligned}
$$

Then $\tilde{\mathfrak{g}} = \Psi_{\mathbf{I}}(\mathfrak{g})$. Thus we have Proposition 4, 5 and 6 by Proposition 6.3 and Theorem 6.4.

Indeed, we define an almost complex structure $\mathbf{I}$ by

$$
\begin{cases}
\mathbf{I}A_i = U_i & i = 1, \dots, l \\
\mathbf{I}B_j = V_j & j = 1, \dots, n \\
\mathbf{I}Y_k = Z_k & k = 1, \dots, 2m
\end{cases}
$$

By Proposition 6.3, we see that the Nijenhuis tensor $N_{\mathbf{I}}(X, Y)$ vanishes and a basis for $(1,0)$-type forms is given by

$$\begin{cases} \mu_i = \alpha_i^1 + \sqrt{-1}\alpha_i^2 & i = 1, \dots, l \\ \nu_j = \beta_j^1 + \sqrt{-1}\beta_j^2 & j = 1, \dots, n \\ \lambda_k = \omega_k^1 + \sqrt{-1}\omega_k^2 & k = 1, \dots, 2m \end{cases}$$

In particular, if $\mathfrak{g}$ is not A-type, then $M = \Psi_{\mathbf{I}}(G)/\Gamma_{\Psi_{\mathbf{I}}(G)}$ has a pseudo-Kähler structure with respect to which M does not have the Hard Lefschetz property.

REMARK. If G is A-type, then the Frölicher spectral sequence $\{E_r(\tilde{\mathfrak{g}})\}$ satisfies $E_1(\tilde{\mathfrak{g}}) \simeq E_\infty(\tilde{\mathfrak{g}})$. In particular, $\dim H^r(\tilde{\mathfrak{g}}) = \sum_{p+q=r} \dim_{\mathbf{C}} H_{\bar{\partial}}^{p,q}(\tilde{\mathfrak{g}}^{\mathbf{C}})$.

Indeed, by a straightforward computation, we see

$$\bar{\partial}(\lambda_{K_1} \wedge \bar{\lambda}_{K_2}) = \sum a_i^{K_1K_2} \bar{\mu}_i \wedge \lambda_{K_1} \wedge \bar{\lambda}_{K_2}$$

$$\partial(\lambda_{K_1} \wedge \bar{\lambda}_{K_2}) = \sum a_i^{K_1K_2} \mu_i \wedge \lambda_{K_1} \wedge \bar{\lambda}_{K_2},$$

which implies that if $\lambda_{K_1} \wedge \bar{\lambda}_{K_2}$ is $\bar{\partial}$-closed, then ∂-closed. Put $\mu_{I\bar{J}} = \mu_I \wedge \bar{\mu}_J$. Let $\gamma = \sum_I c_{IJK_1K_2} \mu_{I\bar{J}} \wedge \lambda_{K_1} \wedge \bar{\lambda}_{K_2}$ be a $\bar{\partial}$-closed form such that for each K_1, K_2, there exists an i such that $a_i^{K_1K_2} \neq 0$. Similarly to Lemma 2.1, we can check that γ is $\bar{\partial}$-exact. Thus for each $\bar{\partial}$-cohomology class, we can choose a representation $\sum c_{IJK_1K_2} \mu_{I\bar{J}} \wedge \lambda_{K_1} \wedge \bar{\lambda}_{K_2}$ such that $\bar{\partial}(\lambda_{K_1} \wedge \bar{\lambda}_{K_2}) = 0$. Then we have $H_{\bar{\partial}}^{p,q}(\tilde{\mathfrak{g}}) = \overline{H_{\bar{\partial}}^{q,p}(\tilde{\mathfrak{g}}^{\mathbf{C}})}$ and $\partial : H_{\bar{\partial}}^{p,q}(\tilde{\mathfrak{g}}^{\mathbf{C}}) \to H_{\bar{\partial}}^{p+1,q}(\tilde{\mathfrak{g}}^{\mathbf{C}})$ is the zero-mapping by the above argument. Hence $E_1(\tilde{\mathfrak{g}}) \simeq E_\infty(\tilde{\mathfrak{g}})$ (See [1]).

Let $\mathfrak{g} = \mathrm{span}\{A_i, B_j, Y_k\}$ be a solvable Lie algebra constructed in Proposition 1 and consider the following decomposition:

DECOMPOSITION. 2:

$$\mathfrak{a} = \mathrm{span}\{A_1, \dots A_l\},$$

$$\mathfrak{b} = \mathrm{span}\{B_1, \dots, B_n, Y_1, \dots, Y_{2m}\}.$$

Since $\mathfrak{a}$, $\mathfrak{b}$ satisfy the condition in Proposition 6.3, we can construct a solvable Lie algebra $\Psi_{\mathbf{I}}(\mathfrak{g})$. Since $\mathrm{span}\{Y_1, Y_2, \mathbf{I}Y_1, \mathbf{I}Y_2, \dots, \mathbf{I}Y_{2k-1}, \mathbf{I}Y_{2k}\}$ is an ideal of $\Psi_{\mathbf{I}}(\mathfrak{g})$, $\Psi_{\mathbf{I}}(G)$ also has a lattice. We show that $\Psi_{\mathbf{I}}(G)$ has no left $\Psi_{\mathbf{I}}(G)$-invariant pseudo-Kähler structures with respect to $\mathbf{I}$ except the case of A-type. For simplicity we use the following notation:

$$d\omega_k = -\sum_i A_k^i \alpha_i \wedge \omega_k - \sum_{j,h} B_k^{jh} \beta_j \wedge \omega_h.$$

Hence, $A^i_{2k-1} = -A^i_{2k} = a^k_i$, $B^{j2h-1}_{2k-1} = B^{j2h}_{2k} = b^{kh}_j$. By a straightforward computation, we see

$$d\lambda_k = d(\omega^1_k + \sqrt{-1}\omega^2_k) = -\frac{1}{2}\sum_{k,i} A^i_k(\mu_i + \bar{\mu}_i) \wedge \lambda_k - \sum_{j,h} B^{jh}_k \nu_j \wedge \lambda_h.$$

PROPOSITION 6.7. *If a compact solvmanifold $\Psi_{\mathbf{I}}(G)/\Gamma_{\Psi_{\mathbf{I}}(G)}$ constructed from the decomposition 2 has a left $\Psi_{\mathbf{I}}(G)$-invariant pseudo-Kähler structure, then G is A-type.*

PROOF. By Stokes' theorem and the assumption of the coefficients A^i_k, if there exists a left $\Psi_{\mathbf{I}}(G)$-invariant pseudo-Kähler structure, then there exists a $\bar{\partial}$-closed 2-form $\sum Q^{kh}\lambda_k \wedge \bar{\lambda}_h$ of maximal rank; i.e. the matrix $Q = (Q^{kh})$ is non-degenerate. Thus

$$\begin{aligned} 0 &= \bar{\partial}\sum Q^{kh}\lambda_k \wedge \bar{\lambda}_h \\ &= -\frac{1}{2}\sum_{k,h,i}(A^i_k + A^i_h)\bar{\mu}_i \wedge \lambda_k \wedge \bar{\lambda}_h - \sum_{k,h,j,i} Q^{kh}B^{ji}_h \lambda_k \wedge \bar{\nu}_j \wedge \bar{\lambda}_i. \end{aligned}$$

Hence, $\sum_h Q^{kh}B^{ji}_h = 0$. By the non-degeneracy of $Q = (Q^{kh})$ it implies that $B^{ji}_h = 0$ for each i, j, h. □

By this proposition, we can construct a compact holomorphic symplectic solvmanifold $\Psi_{\mathbf{I}}(G)/\Gamma_{\Psi_{\mathbf{I}}(G)}$ with no left $\Psi_{\mathbf{I}}(G)$-invariant pseudo-Kähler structures with respect to $\mathbf{I}$.

REMARK. Let $(N/\Gamma, \omega)$ be a non-toral compact symplectic nilmanifold. Then a compact complex nilmanifold $(\tilde{N}/\tilde{\Gamma}, \mathbf{I})$, where $\tilde{N}$ is the simply-connected nilpotent Lie group corresponding to a complex nilpotent Lie algebra $({}_{\mathbf{R}}(\mathfrak{n}^{\mathbf{C}}), \mathbf{I})$, has a holomorphic symplectic structure. However, $(\tilde{N}/\tilde{\Gamma}, \mathbf{I})$ has no pseudo-Kähler structures with respect to $\mathbf{I}$ (See [9] and [18, Theorem 1]).

7. Examples of Compact Holomorphic Symplectic Solvmanifolds

EXAMPLE 7.1 ([1]). We consider the following automorphism of $\mathbf{R}^2$:

$$\varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Note that

$$G_3 = \left\{ \begin{pmatrix} e^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & y_2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| y_1, y_2, t \in \mathbf{R} \right\}$$

has a $\bigwedge^{0,2}$-type form $\omega_1 \wedge \omega_2$ with rank 2. As in Proposition 4, we have the following solvable Lie group which has a lattice:

$$\tilde{G} = \mathbf{R}^2 \ltimes_{\tilde{\varphi}} \mathbf{R}^4 = \left\{ \begin{pmatrix} e^t & 0 & 0 & 0 & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & 0 & 0 & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & 0 & z_1 \\ 0 & 0 & 0 & e^{-t} & 0 & 0 & z_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| s, t, y_1, y_2, z_1, z_2 \in \mathbf{R} \right\}.$$

By Theorem 4.1 and Theorem 6.4, $\tilde{G}/\tilde{\Gamma}$ is a compact pseudo-Kähler manifold which has the Hard Lefschetz property. By Theorem 6.4, $\tilde{G}/\tilde{\Gamma} \times \mathbf{C}/\Gamma$ has a holomorphic symplectic structure.

EXAMPLE 7.2. Let $\mathfrak{g}$ be the following Lie algebra:

$$\mathfrak{g} = \mathrm{span}\{A, B, Y_1, Y_2, Y_3, Y_4\},$$

where the bracket products are

$$[A, Y_1] = Y_1, \quad [A, Y_2] = -Y_2,$$
$$[A, Y_3] = Y_3, \quad [A, Y_4] = -Y_4,$$
$$[B, Y_3] = Y_1, \quad [B, Y_4] = Y_2.$$

Consider the following decomposition:

$$\mathfrak{a} = \mathrm{span}\{A, B\},$$
$$\mathfrak{b} = \mathrm{span}\{Y_1, Y_2, Y_3, Y_4\}.$$

By Theorem 6.4, $\Psi_{\mathbf{I}}(G)$ has a holomorphic symplectic structure and a lattice. Moreover, by Theorem 6.6, $M^{24} = \Psi_{\mathbf{J}}(\Psi_{\mathbf{I}}(G))/\Gamma$ has a pseudo-hyperkähler structure.

Next consider

$$\mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g} = \operatorname{span}\{A, B, Y_1, \dots, Y_4, A', B', Y_1', \dots, Y_4'\}.$$

Let $\{\alpha, \beta, \omega_1, \dots, \omega_4, \alpha', \beta', \omega_1', \dots, \omega_4'\}$ be the dual basis corresponding to $\{A, B, Y_1, \dots, Y_4, A', B', Y_1', \dots, Y_4'\}$. Consider the following decomposition:

$$\mathfrak{a} = \operatorname{span}\{A, A'\},$$

$$\mathfrak{b} = \operatorname{span}\{B, B', Y_1, \dots, Y_4, Y_1', \dots, Y_4'\}.$$

By a straightforward computation, we see that $\mathfrak{b}$ has the following non-degenerate closed 2-form:

$$\omega_{\mathfrak{b}} = \beta \wedge \beta' + \sum_{k=0}^{1} (-1)^k (\omega_{2k+1} \wedge \omega_{4-2k} + \omega_{2k+1}' \wedge \omega_{4-2k}').$$

By Theorem 6.4 and Proposition 6.7, $\Psi_{\mathbf{I}}(K)/\Gamma_{\Psi_{\mathbf{I}}(K)}$ is a compact symplectic solvmanifold with no left $\Psi_{\mathbf{I}}(K)$-invariant pseudo-Kähler structures with respect to $\mathbf{I}$.

Remark. It is easy to check that $\Psi_{\mathbf{I}}(K)/\Gamma_{\Psi_{\mathbf{I}}(K)}$ is a total space which has non-toral symplectic solvmanifolds as fiber and base space. Moreover, $\Psi_{\mathbf{I}}(K)/\Gamma_{\Psi_{\mathbf{I}}(K)}$ has a compatible symplectic structure. Indeed, consider the following Lie subalgebras:

$$\mathfrak{n}_1 = \operatorname{span}\{B, \mathbf{I}B, Y_1, \dots, Y_4, \mathbf{I}Y_1, \dots, \mathbf{I}Y_4\},$$

$$\mathfrak{n}_2 = \operatorname{span}\{B', \mathbf{I}B', Y_1', \dots, Y_4', \mathbf{I}Y_1', \dots, \mathbf{I}Y_4'\},$$

$$\mathfrak{t} = \operatorname{span}\{A, \mathbf{I}A, A', \mathbf{I}A'\}.$$

$\mathfrak{n}_1$ and $\mathfrak{t} \ltimes \mathfrak{n}_2$ have non-degenerate 2-forms which are closed on $\Psi_{\mathbf{I}}(K)/\Gamma_{\Psi_{\mathbf{I}}(K)}$. Consider a symplectic fiber bundle $\pi_1 : (T \ltimes N_1)/(\Gamma_T \ltimes \Gamma_{N_1}) \to T/\Gamma_T$ and a mapping $\pi_2 : (T \ltimes N_2)/(\Gamma_T \ltimes \Gamma_{N_2}) \to T/\Gamma_T$, where T, N_1, N_2 are simply-connected Lie groups corresponding to $\mathfrak{t}$, $\mathfrak{n}_1$, $\mathfrak{n}_2$ and Γ_T, Γ_{N_1}, Γ_{N_2} are its lattices. Then the induced fiber bundle $\pi_2^{-1}((T \ltimes N_1)/(\Gamma_T \ltimes \Gamma_{N_1})) = \Psi_{\mathbf{I}}(K)/\Gamma_{\Psi_{\mathbf{I}}(K)}$ is desired.

8. A Construction of Solvable Lie Group with a Parameterized Lattice

In this section we consider some complexification of a solvable Lie group $G = \mathbf{R}^{n+l} \ltimes_{\varphi} \mathbf{R}^{2m}$ constructed in Proposition 1, each of which has a parameterized lattice.

Let $\tilde{G}_{3,3}$ be the simply-connected solvable Lie group defined by

$$\tilde{G}_{3,3} = \left\{ \begin{pmatrix} e^z & 0 & 0 & w_1 \\ 0 & e^{-z} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| w_1, w_2, z \in \mathbf{C} \right\}.$$

Note that $\tilde{G}_{3,3}$ may be described as the semi-direct product $\mathbf{C}^1 \ltimes_{\varphi_3} \mathbf{C}^2$, where $\varphi_3(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}$.

Let $B \in SL(2, \mathbf{Z})$ be a unimodular matrix with distinct real eigenvalues, say, λ, $1/\lambda$ (it's not necessary that λ is positive). Take $t_0 = Log\ \lambda$, i.e., $e^{t_0} = \lambda$. Then there exists a matrix $P \in GL(2, \mathbf{R})$ such that

$$PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Let

$$\tilde{L}_1 = \mathbf{Z}[t_0, \sqrt{-1}\pi] = \{t_0 k + \sqrt{-1}\pi \cdot h \mid k, h \in \mathbf{Z}\},$$

$$\tilde{L}_2 = \left\{ P\begin{pmatrix} \mu \\ \nu \end{pmatrix} \middle| \mu, \nu \in \mathbf{Z}[\sqrt{-1}] \right\},$$

and put $\tilde{\Gamma}_3 = \tilde{L}_1 \ltimes_{\varphi_3} \tilde{L}_2$. Since

$$\begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \cdot \begin{pmatrix} e^{\sqrt{-1}y} & 0 \\ 0 & e^{-\sqrt{-1}y} \end{pmatrix},$$

where $z = x + \sqrt{-1}y$, $\tilde{\Gamma}_3$ is a lattice of $\tilde{G}_{3,3}$. Similarly, the following solvable Lie groups have lattices:

$$\tilde{G}_{3,4} = \left\{ \begin{pmatrix} e^{\bar{z}} & 0 & 0 & w_1 \\ 0 & e^{-\bar{z}} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| w_1, w_2, z \in \mathbf{C} \right\},$$

$$\tilde{G}_{3,5} = \left\{ \begin{pmatrix} e^{z} & 0 & 0 & w_1 \\ 0 & e^{-\bar{z}} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| w_1, w_2, z \in \mathbf{C} \right\}.$$

We define mappings $\varphi_{i,*} : \mathbf{C}^{n+l} \to \mathrm{End}(\mathbf{C}^{2m})$ by the following:

$$\varphi_{1,*}(\mathbf{z},\mathbf{x}) = \sum_{i=1}^{l}\frac{1}{2}(z_i+\bar{z}_i)A_i + \sum_{j=1}^{n}\frac{1}{2}(x_j+\bar{x}_j)B_j,$$

$$\varphi_{2,*}(\mathbf{z},\mathbf{x}) = \sum_{i=1}^{l}\frac{1}{2}(z_i+\bar{z}_i)A_i + \sum_{j=1}^{n}x_jB_j,$$

$$\varphi_{3,*}(\mathbf{z},\mathbf{x}) = \sum_{i=1}^{l}z_iA_i + \sum_{j=1}^{n}x_jB_j,$$

$$\varphi_{4,*}(\mathbf{z},\mathbf{x}) = \sum_{i=1}^{l}\bar{z}_iA_i + \sum_{j=1}^{n}x_jB_j,$$

$$\varphi_{5,*}(\mathbf{z},\mathbf{x}) = \sum_{i=1}^{l}(z_iA_i^{\mathrm{odd}} - \bar{z}_iA_i^{\mathrm{even}}) + \sum_{j=1}^{n}\frac{1}{2}(x_j+\bar{x}_j)B_j,$$

$$\varphi_{6,*}(\mathbf{z},\mathbf{x}) = \sum_{i=1}^{l}(z_iA_i^{\mathrm{odd}} - \bar{z}_iA_i^{\mathrm{even}}) + \sum_{j=1}^{n}x_jB_j,$$

where

$$A_i^{\mathrm{odd}} = \sum_{k=1}^{m}a_i^kE_{2k-1,2k-1}, \quad A_i^{\mathrm{even}} = \sum_{k=1}^{m}a_i^kE_{2k,2k} \quad i=1,\dots,l.$$

Let $\varphi_i(\mathbf{z},\mathbf{x}) = \exp(\varphi_{i,*}(\mathbf{z},\mathbf{x}))$ and we define group structures on $\mathbf{C}^{n+l}\times\mathbf{C}^{2m}$ by

$$(\mathbf{z}_1,\mathbf{x}_1,\mathbf{w}_1) *_i (\mathbf{z}_2,\mathbf{x}_2,\mathbf{w}_2) = (\mathbf{z}_1+\mathbf{z}_2,\mathbf{x}_1+\mathbf{x}_2,\mathbf{w}_1+\varphi_i(\mathbf{z}_1,\mathbf{x}_1)\mathbf{w}_2)$$

for $\mathbf{z}_i\in\mathbf{C}^l$, $\mathbf{x}_i\in\mathbf{C}^n$ and $\mathbf{w}_i\in\mathbf{C}^{2m}$.

We denote the Lie group $(\mathbf{C}^{n+l}\times\mathbf{C}^{2m}, *_i)$ by $\tilde{G}_i = \mathbf{C}^{n+l}\ltimes_{\varphi_i}\mathbf{C}^{2m}$. We call that $\tilde{G}_i$ is the complexification of $G=(\mathbf{R}^{n+l}\ltimes_{\varphi}\mathbf{R}^{2m}, *)$ of type i.

We denote by α_i, β_j, ω_k the left G-invariant 1-forms on $G=\mathbf{R}^{n+l}\ltimes_{\varphi}\mathbf{R}^{2m}$ such that

$$(\alpha_i)_e = (dt_i)_e, \quad (\beta_j)_e = (dx_j)_e, \quad (\omega_k)_e = (dy_k)_e.$$

We denote $\bigwedge^i\{\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_n\}\wedge\bigwedge^j\{\omega_1,\dots,\omega_{2m}\}$ by $\bigwedge^{i,j}$. Moreover, for each $\eta=1,\dots,6$, we denote by $\tilde{\alpha}_{i,\eta}$, $\tilde{\beta}_{j,\eta}$, $\tilde{\omega}_{k,\eta}$ the left $\tilde{G}_\eta$-invariant $(1,0)$-forms on $\tilde{G}_\eta$ such that

$$(\tilde{\alpha}_{i,\eta})_e = (dz_i)_e, \quad (\tilde{\beta}_{j,\eta})_e = (dx_j)_e, \quad (\tilde{\omega}_{k,\eta})_e = (dw_k)_e.$$

If there exists no possibility of confusion, we write $\tilde{\alpha}_i$, $\tilde{\beta}_j$, $\tilde{\omega}_k$ for $\tilde{\alpha}_{i,\eta}$, $\tilde{\beta}_{j,\eta}$, $\tilde{\omega}_{k,\eta}$ respectively. For simplicity, we put $\{\lambda_1, \dots, \lambda_{n+l}\} = \{\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n\}$ and $\{\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n+l}\} = \{\tilde{\alpha}_1, \dots, \tilde{\beta}_n\}$.

Proposition 8.1. *For each i, the solvable Lie group $\tilde{G}_i = \mathbf{C}^{n+l} \ltimes_{\varphi_i} \mathbf{C}^{2m}$ has a parameterized lattice.*

Proof. We construct a co-compact lattice of $\tilde{G}_1$. Let τ be a complex number such that $\operatorname{Im}\tau > 0$ and p_i, q_j $(i = 1, \dots, l, j = 1, \dots, n)$ non-zero purely imaginary numbers. Let $\mathbf{Z}[\tau] = \{k + \tau h \mid k, h \in \mathbf{Z}\}$. We put

$$\tilde{L}_{1,A}(\mathbf{p}) = at_0\mathbf{Z}[p_1] \times \cdots \times at_0\mathbf{Z}[p_l],$$

$$\tilde{L}_{1,B}(\mathbf{q}) = a^{m-1}(m-1)!\mathbf{Z}[q_1] \times \cdots \times a^{m-1}(m-1)!\mathbf{Z}[q_n],$$

$$\tilde{L}_2(\tau) = \left\{P\begin{pmatrix}\mu_1\\ \nu_1\end{pmatrix} \middle| \mu_1, \nu_1 \in \mathbf{Z}[\tau]\right\} \times \cdots \times \left\{P\begin{pmatrix}\mu_m\\ \nu_m\end{pmatrix} \middle| \mu_m, \nu_m \in \mathbf{Z}[\tau]\right\},$$

where a is the least common multiple for denominators of a_i^k, b_j^{kh}. Then $\tilde{\Gamma}_1 = (\tilde{L}_{1,A}(\mathbf{p}) \times \tilde{L}_{1,B}) \ltimes_{\varphi_1} \tilde{L}_2(\tau)$ is a lattice of $\tilde{G}_1$ which has some parameters. Similarly, $\tilde{G}_i$ has a lattice which has some parameters. □

Remarks.

(i) The cases 1 and 2 correspond to the decompositions 1 and 2 respectively.

(ii) More generally, if $\mathfrak{b}_0 = \operatorname{span}_{\mathbf{Q}}\{B_1, \dots, B_n\}$ is a nilpotent Lie algebra over $\mathbf{Q}$, then we have that $\mathbf{C}^{n+l} \ltimes_{\varphi_i} \mathbf{C}^{2m}$ admits a lattice (cf. Raghunathan [17]; Theorem 2.12 of Chapter II).

(iii) We can apply the complexification of type 1 to other solvable Lie groups. For example,

$$\tilde{G} = \left\{ \begin{pmatrix} \cos(z+\bar{z}) & \cos(z+\bar{z}) & 0 & w_1 \\ -\sin(z+\bar{z}) & \sin(z+\bar{z}) & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| w_1, w_2, z \in \mathbf{C} \right\}$$

has a lattice.

(iv) The author thinks it's not trivial that the existence of a lattice of complexificated solvable Lie group (See Guan [9, Example of Section 4]).

(v) If we assume that $z, x, w \in \mathbf{H}$, then we have a complex solvmanifold which admits an almost hypercomplex structure some of which are integrable.

9. Holomorphic Symplectic Structure on $\tilde{G}_i$

In this section we consider holomorphic symplectic structures and pseudo-Kähler structures on $\tilde{G}_i$. Let $G = \mathbf{R}^{n+l} \ltimes_\varphi \mathbf{R}^{2m}$ be a completely solvable Lie group constructed in Proposition 1. In this section we always assume that for each k there exists an i such that $a_i^k \neq 0$ and there exists a j such that $B_j \neq 0$. Moreover, when we consider $\varphi_{5,*}$, $\varphi_{6,*}$, we always assume that for each i the signature of a_i^k is constant. For simplicity, we always assume that $n+l$ are even.

By Lemma 2.1 and 2.2, we have the following:

Lemma 9.1. *Let G/Γ be a compact symplectic solvmanifold constructed in Proposition 1. If G/Γ has a symplectic structure, then there exists a left G-invariant symplectic structure ω which is an element of $\bigwedge^{2,0} + \bigwedge^{0,2}$.*

Proposition 9.2. *If G/Γ has a symplectic structure, then $\tilde{G}_i/\tilde{\Gamma}_i$ $(i = 1,2,3,4)$ has a holomorphic symplectic structure.*

Proof. By Lemma 9.1, there exists the following symplectic structure:

$$\omega = \sum P_{kh}\lambda_k \wedge \lambda_h + \sum Q_{kh}\omega_k \wedge \omega_h,$$

where $P_{kh}, Q_{kh} \in \mathbf{R}$. Then

$$\Omega = \sum P_{kh}\tilde{\lambda}_k \wedge \tilde{\lambda}_h + \sum Q_{kh}\tilde{\omega}_k \wedge \tilde{\omega}_h$$

is a holomorphic symplectic structure on $\tilde{G}_i/\tilde{\Gamma}_i$ for $i = 1,2,3,4$. In the case of $\varphi_{1,*}$, since

$$d\omega_{2k-1} = -\sum_i a_i^k \alpha_i \wedge \omega_{2k-1} - \sum_{k<h}\sum_{j=1}^{n} b_j^{kh}\beta_j \wedge \omega_{2h-1},$$

$$d\omega_{2k} = \sum_i a_i^k \alpha_i \wedge \omega_{2k} - \sum_{k<h}\sum_{j=1}^{n} b_j^{kh}\beta_j \wedge \omega_{2h},$$

we have

$$d\tilde{\omega}_{2k-1} = -\frac{1}{2}\sum_i a_i^k(\tilde{\alpha}_i + \bar{\tilde{\alpha}}_i) \wedge \tilde{\omega}_{2k-1} - \frac{1}{2}\sum_{k<h}\sum_{j=1}^{n} b_j^{kh}(\tilde{\beta}_j + \bar{\tilde{\beta}}_j) \wedge \tilde{\omega}_{2h-1},$$

$$d\tilde{\omega}_{2k} = \frac{1}{2}\sum_i a_i^k(\tilde{\alpha}_i + \bar{\tilde{\alpha}}_i) \wedge \tilde{\omega}_{2k} - \frac{1}{2}\sum_{k<h}\sum_{j=1}^{n} b_j^{kh}(\tilde{\beta}_j + \bar{\tilde{\beta}}_j) \wedge \tilde{\omega}_{2h}.$$

By considering $\tilde{\alpha}_i + \bar{\tilde{\alpha}}_i$ and $\tilde{\beta}_j + \bar{\tilde{\beta}}_j$ as single terms, we see that $\sum Q_{kh}\tilde{\omega}_k \wedge \bar{\tilde{\omega}}_h$ is closed. The other cases are similar and hence omitted. □

PROPOSITION 9.3. *If G/Γ has a symplectic structure, then $\tilde{G}_i/\tilde{\Gamma}_i$ $(i = 1, 5)$ has a pseudo-Kähler structure.*

PROOF. Consider the case of $\varphi_{5,*}$. By our assumption and Lemma 9.1, there exists the following symplectic structure on G:

$$\omega = \sum P_{kh}\lambda_k \wedge \lambda_h + \sum Q_{kh}\omega_{2k-1} \wedge \omega_{2h},$$

where $P_{kh}, Q_{kh} \in \mathbf{R}$. Since

$$d\omega_{2k-1} = -\sum_i a_i^k \alpha_i \wedge \omega_{2k-1} - \sum_{k<h}\sum_{j=1}^n b_j^{kh}\beta_j \wedge \omega_{2h-1},$$

$$d\omega_{2k} = \sum_i a_i^k \alpha_i \wedge \omega_{2k} - \sum_{k<h}\sum_{j=1}^n b_j^{kh}\beta_j \wedge \omega_{2h},$$

we have

$$d\tilde{\omega}_{2k-1} = -\sum_i a_i^k \tilde{\alpha}_i \wedge \tilde{\omega}_{2k-1} - \frac{1}{2}\sum_{k<h}\sum_{j=1}^n b_j^{kh}(\tilde{\beta}_j + \bar{\tilde{\beta}}_j) \wedge \tilde{\omega}_{2h-1},$$

$$d\bar{\tilde{\omega}}_{2k} = \sum_i a_i^k \bar{\tilde{\alpha}}_i \wedge \bar{\tilde{\omega}}_{2k} - \frac{1}{2}\sum_{k<h}\sum_{j=1}^n b_j^{kh}(\tilde{\beta}_j + \bar{\tilde{\beta}}_j) \wedge \bar{\tilde{\omega}}_{2h}.$$

Similarly to the proof of Proposition 9.2, $\sum Q_{kh}\tilde{\omega}_{2k-1} \wedge \bar{\tilde{\omega}}_{2h}$ is a closed $(1,1)$-form. □

By the same argument in the proof of Proposition 6.7 we see that $\tilde{G}_i/\tilde{\Gamma}_i$ $(i = 3, 4, 6)$ has no left $\tilde{G}_i$-invariant pseudo-Kähler structures. Moreover, by a straightforward computation, we see that $\tilde{G}_i/\tilde{\Gamma}_i$ $(i = 5, 6)$ has no left $\tilde{G}_i$-invariant holomorphic symplectic structures.

Table 9.1. Left $\tilde{G}_i$-invariant structure on $\tilde{G}_i/\tilde{\Gamma}_i$

type	holomorphic symplectic	pseudo-Kähler
1	yes	yes
2	yes	no
3	yes	no
4	yes	no
5	no	yes
6	no	no

10. An Application: A Simple Deformation of Holomorphic Symplectic Manifolds

In this section we consider a simple deformation of compact holomorphic symplectic solvmanifolds constructed in section 9. Consider the following solvable Lie group:

$$\tilde{G} = \tilde{G}_{3,1} \times T^1_{\mathbf{C}} = \left\{ \left(z, P\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) \middle| z, w_1, w_2 \in \mathbf{C} \right\} \times T^1_{\mathbf{C}},$$

where

$$\tilde{G}_{3,1} = \left\{ \begin{pmatrix} e^{(1/2)(z+\bar{z})} & 0 & 0 & w_1 \\ 0 & e^{-(1/2)(z+\bar{z})} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| w_1, w_2, z \in \mathbf{C} \right\},$$

$$T^1_{\mathbf{C}} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbf{C} \right\} \Big/ \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \middle| \mu \in \mathbf{Z}[\sqrt{-1}] \right\}.$$

$\tilde{G}$ has holomorphic symplectic structures, for example, $\Omega = \tilde{\alpha} \wedge \tilde{\beta} + \tilde{\omega}_1 \wedge \tilde{\omega}_2$. Put $B = \{\tau \in \mathbf{C} \mid \operatorname{Im} \tau > 0\}$. Let $\varpi : \tilde{G} \times B \to B$ be the natural projection. Consider the group of automorphisms of $\tilde{G} \times B$ defined as follows.

$$K_p = \left\{ \begin{array}{l} g_{khm_1n_1m_2n_2} : \left(z, P\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, x, \tau\right) \\ \qquad \to \left(z + pk + h, PB^h\begin{pmatrix} w_1 + m_1\tau + n_1 \\ w_2 + m_2\tau + n_2 \end{pmatrix}, x, \tau\right) \end{array} \right\}.$$

K_p acts properly discontinuously without fix points. Therefore $\mathscr{M}_p = \tilde{G} \times B/K_p$ is a complex manifold. Since the projection $\varpi : \tilde{G} \times B \to B$ commutes with $g_{khm_1n_1m_2n_2}$, it induces a holomorphic map ϖ of $\mathscr{M}$ on B. By a straightforward computation, we see $\varpi^{-1}(\tau) = \tilde{G}_{3,1}/\tilde{\Gamma}(p,\tau) \times T^1_{\mathbf{C}}$. Thus $\tilde{G}_{3,1}/\tilde{\Gamma}(p,\tau) \times T^1_{\mathbf{C}}$ and $\tilde{G}_{3,1}/\tilde{\Gamma}(p,\tau') \times T^1_{\mathbf{C}}$ are diffeomorphic. Consider the natural projection $\pi_{\tilde{G}} : \tilde{G} \times B \to \tilde{G}$ and a left $\tilde{G}$-invariant holomorphic symplectic structure Ω. Since $\pi^*_{\tilde{G}}\Omega$ is K_p-invariant, i.e., $g^*_{khm_1n_1m_2n_2}\pi^*_{\tilde{G}}\Omega = \pi^*_{\tilde{G}}\Omega$, $\pi^*_{\tilde{G}}\Omega$ induces a form on $\mathscr{M}_p = \tilde{G} \times B/K_p$.

REMARK. $\bigcup_{p,\tau} \tilde{G}_{3,1}/\tilde{\Gamma}(p,\tau) \times T^1_{\mathbf{C}}$ is a differentiable family.

Let (G^{2m}, I) be a solvable Lie group with left G-invariant complex structure I and Ω a left G-invariant holomorphic structure on (G, I). Let Γ_τ be a lattice of G

which has parameter τ. Consider compact holomorphic symplectic solvmanifolds $(M_\tau = G/\Gamma_\tau, I_\tau, \Omega_\tau)$, where I_τ, Ω_τ are complex structures and holomorphic structures induced from I and Ω respectively. We define a volume form $dVol_\tau$ by $dVol_\tau = \Omega_\tau \wedge \cdots \wedge \Omega_\tau \wedge \bar{\Omega}_\tau \wedge \cdots \wedge \bar{\Omega}_\tau$. Moreover we define $Vol_\tau(M_\tau) = \int_{M_\tau} dVol_\tau$. Note that $Vol_\tau(M_\tau)$ can be considered as the volume of fundamental region on G. Then we have the following:

LEMMA 10.1. *If there exists a diffeomorphism* $f_{\tau\tau'} : M_\tau \to M_{\tau'}$ *such that* $f_{\tau\tau'}^* \Omega_{\tau'} = \Omega_\tau$, *then*

$$Vol_\tau(M_\tau) = Vol_{\tau'}(M_{\tau'}) = Vol_\tau(M_{\tau'}).$$

PROOF. By our assumption, we have

$$Vol_\tau(M_\tau) = \int_{M_\tau} dVol_\tau = \int_{M_\tau} f_{\tau\tau'}^* \, dVol_{\tau'} = \int_{M_{\tau'}} dVol_{\tau'}$$
$$= \int_{M_{\tau'}} dVol_\tau = Vol_\tau(M_{\tau'}). \qquad \square$$

In the above case, since we have $\alpha \wedge \omega_1 \wedge \omega_2 = dt \wedge dy_1 \wedge dy_2$, we consider $Vol_\tau(M_\tau)$ as the volume of a fundamental region on $\mathbf{R}^6$. Hence if $\operatorname{Im}\tau \neq \operatorname{Im}\tau'$, then there exists no diffeomorphisms $f_{\tau\tau'} : (\tilde{G}_{3,1}/\tilde{\Gamma}(p,\tau) \times T^1_{\mathbf{C}}, \Omega_\tau) \to (\tilde{G}_{3,1}/\tilde{\Gamma}(p,\tau') \times T^1_{\mathbf{C}}, \Omega_{\tau'})$ such that $f_{\tau\tau'}^* \Omega_{\tau'} = \Omega_\tau$.

By applying the above argument to the complexification of type i of a symplectic solvable Lie group in Proposition 1, we get families of compact holomorphic symplectic non-Kähler solvmanifolds.

REMARK. We can apply the above argument to the case of pseudo-Kähler structures.

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