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A RIGIDITY THEOREM FOR HYPERSURFACES WITH POSITIVE MÖBIUS RICCI CURVATURE IN S^{n+1}

By

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Abstract. Let M^n $(n \geq 3)$ be an immersed hypersurface without umbilic points in the $(n+1)$-dimensional unit sphere S^{n+1}. Then M^n is associated with a so-called Möbius form Φ and a Möbius metric g which are invariants of M^n under the Möbius transformation group of S^{n+1}. In this paper, we show that if Φ is identically zero and the Ricci curvature Ric_g is pinched: $(n-1)(n-2)/n^2 \leq Ric_g \leq (n^2-2n+5)(n-2)/[n^2(n-1)]$, then it must be the case that $n = 2p$ and M^n is Möbius equivalent to $S^p(1/\sqrt{2}) \times S^p(1/\sqrt{2})$.

§ 1. Introduction

Let $x : M^m \to S^{n+1}$ be an m-dimensional submanifold in the $(n+1)$-dimensional unit sphere S^{n+1} without umbilic point and $\{e_i\}$ be a local orthonormal basis for the first fundamental form $I = dx \cdot dx$ with dual basis $\{\theta_i\}$. Let $II = \sum_{i,j,\alpha} h_{ij}^{\alpha} \theta_i \otimes \theta_j \otimes e_\alpha$ be the second fundamental form and $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha = \frac{1}{m} \sum_{i,\alpha} h_{ii}^{\alpha} e_\alpha$ the mean curvature vector of x, respectively, where $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of x. We define $\rho^2 = m/(m-1) \cdot (\|II\|^2 - m\|\mathbf{H}\|^2)$, where $\|\cdot\|$ is the norm with respect to the induced metric $dx \cdot dx$ on M^n, then $g = \rho^2 dx \cdot dx$ is a Möbius invariant and is called the Möbius metric of $x : M^m \to S^{n+1}$. The normalized scalar curvature of g will be denoted by R and is called the normalized Möbius scalar curvature. A basic Möbius invariant of x, the Möbius form $\Phi = \sum_{i,\alpha} C_i^{\alpha} \theta_i \otimes e_\alpha$ is defined by (cf. [12])

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$$C_i^\alpha = -\rho^{-2}\left(H_{,i}^\alpha + \sum_j (h_{ij}^\alpha - H^\alpha \delta_{ij}) e_j(\log \rho)\right), \tag{1.1}$$

where $\{H_{,i}^\alpha\}$ denotes the covariant derivative of the mean curvature vector field of x in the normal bundle with respect to the basis $\{e_i\}$ and $\{e_\alpha\}$. We see easily from the definition that all minimal submanifolds with constant scalar curvature in S^{n+1} will satisfy $\Phi \equiv 0$, and further for $m = n$, all hypersurfaces with constant mean curvature and constant scalar curvature in S^{n+1} will also satisfy $\Phi \equiv 0$. The Möbius form plays an important role in the Möbius differential geometry. In a series papers by C. P. Wang, H. Li and F. Wu [8, 9, 12], the authors have obtained a completely classification for all surfaces in S^{n+1} with $\Phi \equiv 0$. For the general case $m \geq 3$, there have achieved interesting results recently (cf. [5, 7, 10]), but to author's knowledge, the study for submanifolds with $\Phi \equiv 0$ is far from completed.

In this paper, we will restrict to the hypersurface case, i.e. $m = n$, and prove the following locally rigidity result.

Main Theorem. *Let $x : M^n \to S^{n+1}$ $(n \geq 3)$ be an immersed umbilic free hypersurface with vanishing Möbius form. Suppose (M^n, g) has pinched Ricci curvature with*

$$\frac{(n-1)(n-2)}{n^2} \leq Ric_g \leq \frac{(n^2-2n+5)(n-2)}{n^2(n-1)}, \tag{1.2}$$

then $Ric_g \equiv \frac{(n-1)(n-2)}{n^2}$ with $n = 2p$ an even and M^n is Möbius equivalent to the Einstein hypersurface $S^p(1/\sqrt{2}) \times S^p(1/\sqrt{2})$ of S^{n+1}.

Remark 1.1. In fact, we shall prove a more general result in Theorem 4.1.

This paper is organized as follows: Section 2 is devoted to some notations and preliminaries. In Section 3, we make calculations for a standard example concerned with the Main Theorem whose proof is given in Section 4. The paper ends up with an appendix as Section 5, where we prove Lemma 4.1, which is elementary and is crucial for our proof of the Main Theorem.

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§ 2. Möbius Invariants for Hypersurfaces in S^{n+1}

In this section we define Möbius invariants and recall structure equations for hypersurfaces in S^{n+1}. For more detail we refer to [12].

Let L^{n+3} be the Lorentz space, i.e., $\mathbf{R}^{n+3}$ with inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle x, w\rangle = -x_0w_0 + x_1w_1 + \cdots + x_{n+2}w_{n+2} \tag{2.1}$$

for $x = (x_0, x_1, \ldots, x_{n+2})$, $w = (w_0, w_1, \ldots, w_{n+2}) \in \mathbf{R}^{n+3}$.

Let $x : M^n \to S^{n+1} \hookrightarrow \mathbf{R}^{n+2}$ be an immersed hypersurface of S^{n+1} without umbilics. We define the Möbius position vector $Y : M^n \to L^{n+3}$ of x by

$$Y = \rho(1, x), \quad \rho^2 = \frac{n}{n-1}(\|II\|^2 - nH^2) > 0, \tag{2.2}$$

where and in sequel, for simplicity, we write H^{n+1} as H.

Wang [12, Theorem 1.2] proved that $g = \langle dY, dY\rangle = \rho^2 dx\cdot dx$ is Möbius invariant (cf. also [4, 11, 13]), and hence named g the Möbius metric for x. Combining this fact and a classical theorem for Möbius equivalence of two hypersurfaces, we have the following

Theorem 2.1 ([12]). *Two hypersurfaces x, $\tilde{x} : M^n \to S^{n+1}$ without umbilic points are Möbius equivalent if and only if there exists T in the Lorentz group $O(n+2,1)$ on L^{n+3} such that $Y = \tilde{Y}T$.*

Let us denote by Δ the Laplace operator with respect to g and define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle\Delta Y, \Delta Y\rangle Y, \tag{2.3}$$

then we have (cf. [12])

$$\langle\Delta Y, Y\rangle = -n, \quad \langle\Delta Y, dY\rangle = 0, \quad \langle\Delta Y, \Delta Y\rangle = 1 + n^2R, \tag{2.4}$$

$$\langle Y, Y\rangle = 0, \quad \langle N, Y\rangle = 1, \quad \langle N, N\rangle = 0, \tag{2.5}$$

where R is the normalized scalar curvature of g and is called the normalized Möbius scalar curvature of x.

Let $\{E_1, \ldots, E_n\}$ be a local orthonormal basis for (M^n, g) with dual basis $\{\omega_1, \ldots, \omega_n\}$ and write $Y_i = E_i(Y)$, then from (2.2), (2.4) and (2.5), we have

$$\langle Y_i, Y\rangle = \langle Y_i, N\rangle = 0, \quad \langle Y_i, Y_j\rangle = \delta_{ij}, \quad 1 \le i, j \le n. \tag{2.6}$$

Let V be the orthogonal complement to the subspace $Span\{Y,N,Y_1,\dots Y_n\}$ in L^{n+3}. Along M we have the following orthogonal decomposition:

$$L^{n+3} = Span\{Y,N\} \oplus Span\{Y_1,\dots,Y_n\} \oplus V. \tag{2.7}$$

V is called the Möbius normal bundle of x. A local unit vector basis $E = E_{n+1}$ for V can be written as (cf. [12])

$$E = E_{n+1} = (H, Hx + e_{n+1}). \tag{2.8}$$

Then $\{Y,N,Y_1,\dots,Y_n,E\}$ forms a moving frame in L^{n+3} along M^n.

If not otherwise stated, we will use the following range of indices throughout this paper: $1 \le i,j,k,l,t \le n$.

We can write the structure equations as follows:

$$dY = \sum_i Y_i\omega_i, \tag{2.9}$$

$$dN = \sum_{i,j} A_{ij}\omega_j Y_i + \sum_i C_i\omega_i E, \tag{2.10}$$

$$dY_i = -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij}Y_j + \sum_i B_{ij}\omega_j E, \tag{2.11}$$

$$dE = -\sum_i C_i\omega_i Y - \sum_{i,j} B_{ij}\omega_j Y_i, \tag{2.12}$$

where ω_{ij} are the components of the connection form of the Möbius metric g, $\mathbf{A} = \sum_{i,j} A_{ij}\omega_i \otimes \omega_j$, $\Phi = \sum_i C_i\omega_i$ and $\mathbf{B} = \sum_{i,j} B_{ij}\omega_i \otimes \omega_j$ are called the Blaschke tensor, the Möbius form and the Möbius second fundamental form of x, respectively. The relations between Φ, $\mathbf{B}$, $\mathbf{A}$ and the Euclidean invariants of x are given by (1.1) and (cf. [12])

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}), \tag{2.13}$$

$$A_{ij} = -\rho^{-2}[\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}] \\ -\frac{1}{2}\rho^{-2}(\|\nabla\log\rho\|^2 - 1 + H^2)\delta_{ij}, \tag{2.14}$$

where Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to $dx\cdot dx$.

The covariant derivative of C_i, A_{ij}, B_{ij} are defined by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j\omega_{ji}, \tag{2.15}$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2.16}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \tag{2.17}$$

The integrability conditions for the structure equations (2.9)–(2.12) are given by (cf. [12])

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k, \tag{2.18}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{kj}A_{ki}), \tag{2.19}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.20}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}. \tag{2.21}$$

Then we have the following identities (cf. [12])

$$R_{ij} = \sum_k R_{ikjk} = -\sum_k B_{ik}B_{jk} + \operatorname{tr}\mathbf{A}\delta_{ij} + (n-2)A_{ij}, \tag{2.22}$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j}(B_{ij})^2 = \frac{n-1}{n}, \quad \operatorname{tr}\mathbf{A} = \sum_i A_{ii} = \frac{1}{2n}(1 + n^2R), \tag{2.23}$$

where R_{ijkl} denote the components of the curvature tensor of g. $R = \frac{1}{n(n-1)}\sum_{i,j} R_{ijij}$ is the normalized Möbius scalar curvature of $x : M^n \to S^{n+1}$.

The second covariant derivative of B_{ij} are defined by

$$\sum_l B_{ij,kl}\omega_l = dB_{ij,k} + \sum_l B_{lj,k}\omega_{li} + \sum_l B_{il,k}\omega_{lj} + \sum_l B_{ij,l}\omega_{lk}. \tag{2.24}$$

By exterior differentiation of (2.17), we have the following Ricci identities

$$B_{ij,kl} - B_{ij,lk} = \sum_t B_{tj}R_{tikl} + \sum_t B_{it}R_{tjkl}. \tag{2.25}$$

We get from (2.13) that

$$\mathscr{S} = \rho^{-1}(S - H \cdot id) = \sum_{i,j} B_{ij}\omega_i E_j, \tag{2.26}$$

where S is the Weingarten operator for $x: M^n \to S^{n+1}$ and we call $\mathscr{S}$ the Möbius shape operator of x. For $n \geq 3$, it is easy to find that all coefficients in (2.9)–(2.12) are determined by $\{g, \mathscr{S}\}$ and thus we have

THEOREM 2.2 (see [12], [1]–[3]). *Two umbilic free hypersurfaces $x: M^n \to S^{n+1}$ and $\tilde{x}: \tilde{M}^n \to S^{n+1}$ $(n \geq 3)$ are Möbius equivalent if and only if there exists a diffeomorphism $\sigma: M^n \to \tilde{M}^n$ which preserves the Möbius metric and the Möbius shape operator.*

§3. Calculation of Möbius Invariants for $\tilde{x}: S^p(a) \times S^p(b) \to S^{2p+1}$

For the purpose of establishing the Main Theorem, we will consider in this section the umbilic free hypersurface $\tilde{x}: S^p(a) \times S^p(b) \to S^{2p+1}$ with $a^2 + b^2 = 1$.

We write $\mathbf{R}^{2p+2} = \mathbf{R}^{p+1} \times \mathbf{R}^{p+1}$. Let $\tilde{x}_1 : S^p(a) \to \mathbf{R}^{p+1}$ and $\tilde{x}_2 : S^p(b) \to \mathbf{R}^{p+1}$ be the standard embedding of spheres with radius a and b, respectively. Then $\tilde{x} = \tilde{x}_1 + \tilde{x}_2$ and one of the unit normal vector of $\tilde{x}$ is given by $e_{2p+1} = \frac{b}{a}\tilde{x}_1 - \frac{a}{b}\tilde{x}_2$. The induced Euclidean metric of $\tilde{x}$ is given by $I = d\tilde{x}_1 \cdot d\tilde{x}_1 + d\tilde{x}_2 \cdot d\tilde{x}_2$ and the second fundamental form of $\tilde{x}$ is $II = -d\tilde{x} \cdot de_{2p+1} = -\frac{b}{a} d\tilde{x}_1 \cdot d\tilde{x}_1 + \frac{a}{b} d\tilde{x}_2 \cdot d\tilde{x}_2$. Take an orthonormal frame $\{e_1, e_2, \ldots, e_{2p}\}$ with dual frame $\{\theta_1, \theta_2, \ldots, \theta_{2p}\}$ such that $d\tilde{x}_1 = \sum_{i=1}^{p} \theta_i e_i$ and $d\tilde{x}_2 = \sum_{i=p+1}^{2p} \theta_i e_i$, then we have

$$I = \sum_{i=1}^{2p} (\theta_i)^2, \quad II = -\frac{b}{a}\sum_{i=1}^{p}(\theta_i)^2 + \frac{a}{b}\sum_{i=p+1}^{2p}(\theta_i)^2 = \sum_{i,j=1}^{2p} h_{ij}\theta_i\theta_j \tag{3.1}$$

with

$$h_{ij} = \lambda_i \delta_{ij}, \quad \lambda_1 = \cdots = \lambda_p = -\frac{b}{a}, \quad \lambda_{p+1} = \cdots = \lambda_{2p} = \frac{a}{b}. \tag{3.2}$$

From (3.2) we see that

$$H = \frac{1}{2p}\sum_{i=1}^{2p} h_{ii} = \frac{a^2 - b^2}{2ab}, \quad \|II\|^2 = \sum_{i,j=1}^{2p}(h_{ij})^2 = \frac{p(a^4 + b^4)}{a^2 b^2}. \tag{3.3}$$

Note that $\tilde{x}: S^p(a) \times S^p(b) \to S^{2p+1}$ is of constant mean curvature and constant length of second fundamental form, and its Möbius form is thus identically zero. By definition

$$\rho^2 = \frac{2p}{2p-1}(\|II\|^2 - 2pH^2) = \frac{p^2}{2p-1} \cdot \frac{1}{a^2 b^2}, \tag{3.4}$$

so that the Möbius metric g of $\tilde{x}$ is given by

$$(3.5)\qquad g = \rho^2 d\tilde{x}\cdot d\tilde{x} = \rho^2 d\tilde{x}_1\cdot d\tilde{x}_1 + \rho^2 d\tilde{x}_2\cdot d\tilde{x}_2 = \tilde{g}_1 + \tilde{g}_2 = \sum_{i=1}^{2p}(\omega_i)^2,$$

where $\omega_i = \frac{p}{\sqrt{2p-1}ab}\theta_i$. Let us define

$$(3.6)\qquad E_i = \frac{\sqrt{2p-1}ab}{p}e_i,\quad Y_i = (0, e_i),\quad E_{2p+1} = (H, e_{2p+1} + H\tilde{x}).$$

From (2.13), the Möbius second fundamental form is given by

$$(3.7)\quad B_{ij} = b_i\delta_{ij},\quad b_1 = \cdots = b_p = -\frac{\sqrt{2p-1}}{2p},\quad b_{p+1} = \cdots = b_{2p} = \frac{\sqrt{2p-1}}{2p}.$$

From (2.14) and (3.2)–(3.4), we get

$$(3.8)\qquad A_{ij} = a_i\delta_{ij},\quad a_1 = \cdots = a_p = \frac{2p-1}{4p^2}\left[2b^4 - \frac{1}{2}(a^2-b^2)^2\right],$$

$$a_{p+1} = \cdots = a_{2p} = \frac{2p-1}{4p^2}\left[2a^4 - \frac{1}{2}(a^2-b^2)^2\right].$$

The Ricci curvature and normalized scalar curvature of g can be calculated, by (3.7), (3.8) and (2.22)

$$(3.9)\qquad R_{11} = \cdots = R_{pp} = (2p-1)(p-1)p^{-2}b^2,$$

$$R_{p+1,p+1} = \cdots = R_{2p,2p} = (2p-1)(p-1)p^{-2}a^2,\quad R_{ij} = 0,\ i \neq j,$$

$$R = \frac{1}{2p(2p-1)}\sum_{i=1}^{2p} R_{ii} = \frac{p-1}{2p^2}.$$

It follows that for $n = 2p$ and if $Ric_g \geq (n-1)(n-2)/n^2$, then it must be the case $a^2 = b^2 = \frac{1}{2}$.

Remark 3.1. Our example here is a Möbius isoparametric hypersurface with two distinct principal curvatures. We note that all Möbius isoparametric hypersurface with two distinct principal curvatures have been classified in [7].

§4. Proof of the Main Theorem

Firstly, we state an algebraic lemma, Lemma 4.1, which will play a crucial role in our proof of the Main Theorem. Because this lemma's proof is not much concerned in the theme of this article, we will leave it in section 5 as Appendix.

Lemma 4.1. *For constant* $X \geq (n-1)(n-2)/n$, $n \geq 3$, *let* $x_1, \dots, x_n$; $y_1, \dots, y_n \in \mathbf{R}$ *satisfy*

$$\sum_{i=1}^{n} x_i = X, \quad x_i \geq C_n = \frac{(n-1)(n-2)}{n^2} \quad \text{for all } i, \tag{4.1}$$

$$\sum_{i=1}^{n} y_i = 0, \quad \sum_{i=1}^{n} y_i^2 = \frac{n-1}{n}. \tag{4.2}$$

Then we have

$$\sum_{i=1}^{n} (x_i y_i^2 + y_i^4) \geq -\frac{n-1}{4n}\left[X - nC_n + \frac{2(n-1)}{n}\right]^2 + \frac{n-1}{n}\left[X - nC_n + \frac{2(n-1)}{n}\right] - \frac{2(n-1)^2}{n^3}, \tag{4.3}$$

and the equality sign in (4.3) *holds if and only if one of the following two cases occurs*

(i) $n = 2p$ *is even and* $x_1 = \cdots = x_n = C_n$, $X = nC_n$; $y_1^2 = \cdots = y_n^2 = \frac{n-1}{n^2}$.
(ii) $nC_n < X \leq nC_n + \frac{2}{n}$ *and* $n-1$ *of* $\{x_i\}$ *equal* C_n, *say* $x_1 = \cdots = x_{n-1} = C_n$, $x_n = X - (n-1)C_n$ *and correspondingly* $y_1^2 = \cdots = y_{n-1}^2 = \frac{1}{2n}\left[X - nC_n + \frac{2(n-1)}{n}\right]$.

Now, we shall prove the following more general theorem from which the Main Theorem is proved immediately.

Theorem 4.1. *Let* $x : M^n \to S^{n+1}$ $(n \geq 3)$ *be an immersed umbilic free hypersurface with vanishing Möbius form. Suppose the curvature of* (M^n, g) *satisfy* $Ric_g \geq (n-1)(n-2)/n^2$ *and* $R \leq (n^2 - 2n + 5)(n-2)/[n(n-1)]^2$. *Then* $Ric_g \equiv (n-1)(n-2)/n^2$, $n = 2p$ *is even and* M^n *is Möbius equivalent to the Einstein hypersurface* $S^p(1/\sqrt{2}) \times S^p(1/\sqrt{2})$ *of* S^{n+1}.

Proof. To achieve the result expected, we start with calculating the Laplacian of the norm square of the Möbius second fundamental form.

Let $x : M^n \to S^{n+1}$ be an umbilic free hypersurface with vanishing Möbius form $\Phi \equiv 0$. Since our consideration is of local nature, without loss of generality, we may assume that M^n is simply connected. From (2.19) and (2.20), we see that

$\{B_{ij}\}$ is a Codazzi tensor (i.e. $B_{ij,k} = B_{ik,j}$) and that $\{A_{ij}\}$ and $\{B_{ij}\}$ can be diagonalized simultaneously. We choose $\{E_1, \dots, E_n\}$ such that

$$A_{ij} = a_i\delta_{ij}, \quad B_{ij} = b_i\delta_{ij}, \quad 1 \le i, j \le n. \tag{4.4}$$

Now we have, by using (2.23), (2.25), (4.4) and the fact $B_{ij,k} = B_{ik,j}$,

$$\begin{aligned}
0 &= \frac{1}{2}\Delta \sum_{i,j}(B_{ij})^2 = \sum_{i,j,k}(B_{ij,k})^2 + \sum_{i,j,k} B_{ij}B_{ij,kk} \\
&= \sum_{i,j,k}(B_{ij,k})^2 + \sum_{i,j,k} B_{ij}\left(B_{kk,ij} + \sum_m B_{mk}R_{mijk} + \sum_m B_{im}R_{mkjk}\right) \\
&= \sum_{i,j,k}(B_{ij,k})^2 + \sum_{i,j} b_ib_jR_{ijji} + \sum_{i,j} b_i^2 R_{ijij} \\
&= \sum_{i,j,k}(B_{ij,k})^2 + \frac{1}{2}\sum_{i,j}(b_i - b_j)^2 R_{ijij}.
\end{aligned} \tag{4.5}$$

From (2.21)–(2.23) and (4.4), we have

$$\begin{aligned}
\frac{1}{2}\sum_{i,j}(b_i - b_j)^2 R_{ijij} &= \frac{1}{2}\sum_{i,j}(b_i - b_j)^2(b_ib_j + a_i + a_j) \\
&= -\frac{(n-1)^2}{n^2} + \frac{n-1}{n}\sum_i a_i + n\sum_i a_ib_i^2 \\
&= -\frac{(n-1)^2}{n^2} - \frac{2(n-1)}{n(n-2)}\operatorname{tr} \mathbf{A} + \frac{n}{n-2}\sum_i (R_{ii}b_i^2 + b_i^4).
\end{aligned} \tag{4.6}$$

From (2.23) and the assumption of the theorem, we have

$$\sum_i b_i = 0, \quad \sum_i b_i^2 = \frac{n-1}{n}; \tag{4.7}$$

$$\sum_i R_{ii} = n(n-1)R \le \frac{(n^2 - 2n + 5)(n-2)}{n(n-1)}; \quad R_{ii} \ge C_n = \frac{(n-1)(n-2)}{n^2}. \tag{4.8}$$

Now we can apply Lemma 4.1 to obtain

$$\begin{aligned}
\sum_{i=1}^n (R_{ii}b_i^2 + b_i^4) \ge &-\frac{n-1}{4n}\left[n(n-1)R - nC_n + \frac{2(n-1)}{n}\right]^2 \\
&+ \frac{n-1}{n}\left[n(n-1)R - nC_n + \frac{2(n-1)}{n}\right] - \frac{2(n-1)^2}{n^3}.
\end{aligned} \tag{4.9}$$

From (4.6), (4.9) and (2.23), we get

$$\begin{aligned}(4.10)\quad & \frac{1}{2}\sum_{i,j}(b_i - b_j)^2 R_{ijij}\\ & \geq \frac{n}{n-2}\left\{-\frac{n-1}{4n}\left[n(n-1)R - nC_n + \frac{2(n-1)}{n}\right]^2\right.\\ & \quad \left. - \frac{(n-1)^2}{n^2} - \frac{n-1}{n^3}(1+n^2R) + \frac{n-1}{n}\left[n(n-1)R - nC_n + \frac{2(n-1)}{n}\right]\right\}\\ & = -\frac{n-1}{4(n-2)}[n(n-1)R - nC_n]\left[n(n-1)R - nC_n - \frac{4(n-2)}{n(n-1)}\right]\\ & \geq 0,\end{aligned}$$

where the last inequality is implied by (4.8).

From (4.5) and (4.10), we have

$$(4.11)\qquad B_{ij,k} \equiv 0, \quad \text{for all } i, j, k,$$

and

$$(4.12)\qquad n(n-1)R \equiv nC_n \quad \text{or} \quad n(n-1)R \equiv nC_n + \frac{4(n-2)}{n(n-1)}.$$

Case (I) $n(n-1)R \equiv nC_n$.

Since (4.9) and (4.10) now become equality, from the proof of Lemma 4.1 in section 5, we see that $n = 2p$ is even and $R_{11} = \cdots = R_{nn} = C_n$, $b_1^2 = \cdots = b_n^2 = (n-1)/n^2$. Without loss of generality, we assume that

$$(4.13)\qquad b_1 = \cdots = b_p = -\frac{\sqrt{n-1}}{n}, \quad b_{p+1} = \cdots = b_{2p} = \frac{\sqrt{n-1}}{n}.$$

Then we have the following decomposition: $TM = V_1 \oplus V_2$, where V_1 and V_2 are the eigenspaces of the Möbius shape operator $\mathcal{S}$ corresponding to eigenvalue $-\sqrt{n-1}/n$ and $\sqrt{n-1}/n$, respectively.

From (4.11), (4.13), (4.4) and (2.17), we obtain

$$\begin{aligned}(4.14)\quad 0 &= \sum_{k=1}^{n} B_{i\alpha,k}\omega_k = dB_{i\alpha} + \sum_{k=1}^{n} B_{ik}\omega_{k\alpha} + \sum_{k=1}^{n} B_{k\alpha}\omega_{ki}\\ &= b_i\omega_{i\alpha} + b_\alpha\omega_{k\alpha} = (b_i - b_\alpha)\omega_{i\alpha}, \quad 1 \leq i \leq p,\ p+1 \leq \alpha \leq 2p,\end{aligned}$$

which gives that

$$\omega_{i\alpha} = 0, \quad 1 \le i \le p,\ p+1 \le \alpha \le 2p, \tag{4.15}$$

and thus

$$d\omega_i = \sum_{j=1}^{p} \omega_{ij} \wedge \omega_j, \quad d\omega_\alpha = \sum_{\beta=p+1}^{2p} \omega_{\alpha\beta} \wedge \omega_\beta, \quad 1 \le i \le p,\ p+1 \le \alpha \le 2p. \tag{4.16}$$

Therefore, V_1 and V_2 are integrable and we can write $M = M_1 \times M_2$ for some simply connected manifolds M_1 and M_2 with $\dim M_1 = \dim M_2 = p$. Moreover, if we define

$$g_1 = \sum_{i=1}^{p} \omega_i^2, \quad g_2 = \sum_{\alpha=p+1}^{2p} \omega_\alpha^2,$$

then we have

$$(M, g) = (M_1, g_1) \times (M_2, g_2). \tag{4.17}$$

From (4.4), (4.13) and (2.22), we see that

$$R_{11} = \cdots = R_{nn} = C_n, \quad \text{or equivalently,} \quad a_1 = \cdots = a_n = \frac{n-1}{2n^2}. \tag{4.18}$$

It follows from (2.21), (4.4), (4.13) and (4.18) that

$$R_{ijkl} = \frac{2(n-1)}{n^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad 1 \le i, j, k, l \le p, \tag{4.19}$$

$$R_{\alpha\beta\gamma\sigma} = \frac{2(n-1)}{n^2}(\delta_{\alpha\gamma}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\gamma}), \quad p+1 \le \alpha, \beta, \gamma, \sigma \le 2p, \tag{4.20}$$

that is, (M_1, g_1) and (M_2, g_2) are space forms with the same constant sectional curvature $2(n-1)/n^2$.

Let $\tilde{x} : S^p(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \to S^{n+1}$ be the hypersurface defined as in Section 3 with $a = b = 1/\sqrt{2}$. Then by (3.5) the Möbius metric $\tilde{g}$ of $\tilde{x}$ is given by $\tilde{g} = \tilde{g}_1 + \tilde{g}_2$, where

$$\tilde{g}_1 = \frac{n^2}{n-1} d\tilde{x}_1 \cdot d\tilde{x}_1, \quad \tilde{g}_2 = \frac{n^2}{n-1} d\tilde{x}_2 \cdot d\tilde{x}_2. \tag{4.21}$$

Note that $d\tilde{x}_i \cdot d\tilde{x}_i$ $(i = 1, 2)$ have constant sectional curvature 2, from (4.19)–(4.21) we know that $(S^p(1/\sqrt{2}), \tilde{g}_i)$ and (M_i, g_i) $(i = 1, 2)$ are simply connected spaces with the same constant curvature $2(n-1)/n^2$. Hence, we can find local isometries

$$\varphi_i : (M_i, g_i) \to (S^p(1/\sqrt{2}), \tilde{g}_i), \quad i = 1, 2.$$

Then we obtain a diffeomorphism $\varphi = (\varphi_1, \varphi_2) : M^n \to S^p(1/\sqrt{2}) \times S^p(1/\sqrt{2})$ which preserves the Möbius metric and the Möbius shape operator (cf. (3.7) and (4.13)).

From Theorem 2.2 we know that $x : M^n \to S^{n+1}$ is Möbius equivalent to $\tilde{x} : S^p(1/\sqrt{2}) \times S^p(1/\sqrt{2}) \to S^{n+1}$ which has been considered in Section 3.

Case (II) $n(n-1)R \equiv nC_n + \frac{4(n-2)}{n(n-1)}$.

Since (4.9) and (4.10) now become equality, from the proof of Lemma 4.1 in section 5, we should have

$$n(n-1)R \le nC_n + \frac{2}{n},$$

which implies $4(n-2)/[n(n-1)] \le 2/n$ and thus we get $n = 3$. Apply Lemma 4.1 again we deduce (up to re-arranging the order of the lower index)

$$b_1 = -\frac{1}{\sqrt{3}}, \quad b_2 = \frac{1}{\sqrt{3}}, \quad b_3 = 0; \tag{4.22}$$

$$R_{11} = R_{22} = \frac{2}{9}, \quad R_{33} = \frac{8}{9}. \tag{4.23}$$

On the other hand, just as deriving (4.15), from (4.11) and (4.22) we can show that $\omega_{ij} = 0$, $1 \le i, j \le 3$, which imply that (M, g) has constant sectional curvature zero. This is a contradiction to (4.23). Therefore case (II) can not occur.

We have completed the proof of Theorem 4.1.

REMARK 4.1. We expect that Theorem 4.1 should be true without the upper bound restriction for R. But our method depends heavily on Lemma 4.1, which is already the best possible.

§5. Appendix: Proof of Lemma 4.1

If $X = nC_n$, then $x_1 = \cdots = x_n = C_n$, and by (4.2)

$$\sum_{i=1}^n (x_i y_i^2 + y_i^4) = C_n \sum_{i=1}^n y_i^2 + \sum_{i=1}^n y_i^4 = \frac{(n-1)^2(n-2)}{n^3} + \sum_{i=1}^n y_i^4 \tag{5.1}$$

$$\ge \frac{(n-1)^2(n-2)}{n^3} + \frac{1}{n}\left(\sum_{i=1}^n y_i^2\right)^2 = \frac{(n-1)^3}{n^3},$$

which shows that (4.3) is correct, and it is also easy to see that the equality sign in (4.3) holds if and only if (i) occurs.

Hereafter we assume $X > nC_n$. Define a bounded domain Ω in $\mathbf{R}^{2n}$ by

$$\Omega = \{(x, y) \in \mathbf{R}^{2n} \mid x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \text{ satisfy (4.1) and (4.2)}\},$$

with boundary $\partial\Omega = \bigcup_{i=1}^{n-1} \Omega_i$, where

$$\Omega_i = \{(x, y) \in \Omega \mid \text{exactly } i \text{ of } \{x_1, \dots, x_n\} \text{ equal } C_n\}.$$

Let us consider the function

$$f(x, y) = \sum_{i=1}^{n} (x_i y_i^2 + y_i^4)$$

defined on Ω. Since Ω is bounded and closed, $f(x, y)$ will attain its minimum at somewhere on Ω. We apply the method of Lagrange's multiplier for seeking this minimum.

Consider the following auxiliary function

$$F(x, y, \lambda, \mu, \gamma) = \sum_{i=1}^{n} (x_i y_i^2 + y_i^4) + \lambda \sum_{i=1}^{n} y_i + \mu\left(\sum_{i=1}^{n} y_i^2 - \frac{n-1}{n}\right) + \gamma\left(\sum_{i=1}^{n} x_i - X\right)$$

defined on $\mathbf{R}^{2n+3}$.

If $(\bar{x}, \bar{y})$ is a critical point of $f(x, y)$ in the interior of Ω, then it must satisfy the equations

$$\frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = \frac{\partial F}{\partial y_1} = \cdots = \frac{\partial F}{\partial y_n} = 0, \tag{5.2}$$

$$\frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial \mu} = \frac{\partial F}{\partial \gamma} = 0. \tag{5.3}$$

From (5.2) and $\frac{\partial F}{\partial x_i} = y_i^2 + \gamma$, we get for each i, $\bar{y}_i^2 + \gamma = 0$. Furthermore, by making summation over i and using $\frac{\partial F}{\partial \mu} = 0$, we obtain at the critical point

$$\gamma = -\frac{n-1}{n^2}; \quad \bar{y}_1^2 = \cdots = \bar{y}_n^2 = \frac{n-1}{n^2}. \tag{5.4}$$

Combining (4.7) with $\sum_{i=1}^{n} \bar{y}_i = 0$ we see that n must be even, say $n = 2p$, and up to re-arranging the lower index we have

$$\bar{y}_1 = \cdots = \bar{y}_p = -\frac{\sqrt{n-1}}{n}, \quad \bar{y}_{p+1} = \cdots = \bar{y}_n = \frac{\sqrt{n-1}}{n}. \tag{5.5}$$

From $\frac{\partial F}{\partial y_i} = 2x_i y_i + 4y_i^3 + \lambda + 2\mu y_i$ and (5.2), (5.5) we have

$$(5.6)\qquad \frac{2\sqrt{n-1}}{n}\bar{x}_i + \frac{4(n-1)\sqrt{n-1}}{n^3} - \lambda + \frac{2\sqrt{n-1}}{n}\mu = 0, \quad 1 \le i \le p,$$

$$(5.7)\qquad \frac{2\sqrt{n-1}}{n}\bar{x}_i + \frac{4(n-1)\sqrt{n-1}}{n^3} + \lambda + \frac{2\sqrt{n-1}}{n}\mu = 0, \quad p+1 \le i \le n.$$

Summing up (5.6) and (5.7), and using $\frac{\partial F}{\partial y} = 0$, we get

(5.8)

$$\frac{2\sqrt{n-1}}{n}\sum_{i=1}^{n}\bar{x}_i + \frac{4(n-1)\sqrt{n-1}}{n^2} + 2\sqrt{n-1}\mu = 0, \quad \text{thus } \mu = -\frac{X}{n} - \frac{2(n-1)}{n^2}.$$

Combining (5.6)–(5.8), we obtain

$$(5.9)\qquad \begin{cases} \bar{x}_1 = \cdots = \bar{x}_p = \frac{X}{n} + \frac{n\lambda}{2\sqrt{n-1}}, \\ \bar{x}_{p+1} = \cdots = \bar{x}_n = \frac{X}{n} - \frac{n\lambda}{2\sqrt{n-1}}. \end{cases}$$

where $\bar{x}_i > C_n$ implies that the parameter λ satisfies

$$(5.10)\qquad -\frac{2\sqrt{n-1}(X-nC_n)}{n^2} < \lambda < \frac{2\sqrt{n-1}(X-nC_n)}{n^2}.$$

We have proved the following

Claim 1. *$f(x,y)$ has critical points in the interior of Ω if and only if $n = 2p$ is even. In the case $n = 2p$, the critical points $(\bar{x}, \bar{y})$, which depends on λ, is given by* (5.5) *and* (5.9) *(up to re-arranging the lower index). It is easy to see that*

$$f(\bar{x}, \bar{y}) = \frac{n-1}{n^2}X + \frac{(n-1)^2}{n^3},$$

and it is in fact a local minimum of $f(x,y)$ on Ω.

Our next purpose is to find out the minimum of $f(x,y)$ on $\partial\Omega$. We first prove the following

Claim 2. *On Ω_{n-1}, it holds*

$$(5.11)\quad f(x,y) \ge -\frac{n-1}{4n}\left[X - nC_n + \frac{2(n-1)}{n}\right]^2 + \frac{n-1}{n}\left[X - (n-1)C_n + \frac{n-1}{n}\right],$$

with the equality sign attainable if and only if $X \le nC_n + \frac{2}{n}$. If it is the case, then Lemma 4.1(ii) *occurs.*

PROOF OF CLAIM 2. Without loss of generality, we assume that on Ω_{n-1}

(5.12) $$x_1 = \cdots = x_{n-1} = C_n, \quad x_n = X - (n-1)C_n > C_n.$$

By use of (4.2), we have

(5.13) $$\begin{aligned} f(x,y) &= C_n \sum_{i=1}^{n-1} y_i^2 + [X - (n-1)C_n] y_n^2 + \sum_{i=1}^{n-1} y_i^4 + y_n^4 \\ &= \sum_{i=1}^{n-1} y_i^4 + \left(\sum_{i=1}^{n-1} y_i^2\right)^2 - \left[X - nC_n + \frac{2(n-1)}{n}\right] \sum_{i=1}^{n-1} y_i^2 \\ &\quad + \frac{n-1}{n}[X - (n-1)C_n] + \frac{(n-1)^2}{n^2} \\ &\geq \frac{n}{n-1}\left(\sum_{i=1}^{n-1} y_i^2\right)^2 - \left[X - nC_n + \frac{2(n-1)}{n}\right] \sum_{i=1}^{n-1} y_i^2 \\ &\quad + \frac{n-1}{n}\left[X - (n-1)C_n + \frac{n-1}{n}\right], \end{aligned}$$

where in the last inequality, we have used

(5.14) $$\sum_{i=1}^{n-1} y_i^4 \geq \frac{1}{n-1}\left(\sum_{i=1}^{n-1} y_i^2\right)^2.$$

From (5.13), we find that

(5.15) $$\begin{aligned} f(x,y) &\geq \frac{n}{n-1}\left[\sum_{i=1}^{n-1} y_i^2 - \frac{n-1}{2n}\left(X - nC_n + \frac{2(n-1)}{n}\right)\right]^2 \\ &\quad - \frac{n-1}{4n}\left[X - nC_n + \frac{2(n-1)}{n}\right]^2 + \frac{n-1}{n}\left[X - (n-1)C_n + \frac{n-1}{n}\right] \\ &\geq -\frac{n-1}{4n}\left[X - nC_n + \frac{2(n-1)}{n}\right]^2 + \frac{n-1}{n}\left[X - (n-1)C_n + \frac{n-1}{n}\right]. \end{aligned}$$

This proves (5.11). The equality sign in (5.11) holds if and only if

(5.16) $$\sum_{i=1}^{n-1} y_i^4 = \frac{1}{n-1}\left(\sum_{i=1}^{n-1} y_i^2\right)^2 \quad \text{and} \quad \sum_{i=1}^{n-1} y_i^2 = \frac{n-1}{2n}\left[X - nC_n + \frac{2(n-1)}{n}\right]$$

are satisfied. Moreover, from (5.15) one see that (5.16) is possible if and only if

$$\frac{n-1}{2n}\left[X - nC_n + \frac{2(n-1)}{n}\right] \leq \frac{n-1}{n}, \quad \text{i.e. } X \leq nC_n + \frac{2}{n}.$$

If it is the case, then the equality sign in (5.11) holds if

$$y_1^2 = \cdots = y_{n-1}^2 = \frac{1}{2n}\left[X - nC_n + \frac{2(n-1)}{n}\right].$$

This proves Claim 2.

CLAIM 3. *For each* $q \in \{2, 3, \dots, n-1\}$, *it holds on* Ω_{n-q}

$$(5.17)\quad f(x,y) > -\frac{n-1}{4n}\left[X - nC_n + \frac{2(n-1)}{n}\right]^2 + \frac{n-1}{n}\left[X - (n-1)C_n + \frac{n-1}{n}\right].$$

PROOF OF CLAIM 3. For given $q \in \{2, 3, \dots, n-1\}$, we consider the function $f(x,y)$ being defined on Ω_{n-q}. Without loss of generality, we assume

$$(5.18)\qquad x_{q+1} = \cdots = x_n = C_n; \quad x_i > C_n,\ 1 \le i \le q.$$

By use of (4.2), we have

$$\begin{aligned}
(5.19)\qquad f(x,y) &= \sum_{i=1}^{q} x_i y_i^2 + C_n \sum_{i=q+1}^{n} y_i^2 + \sum_{i=1}^{n} y_i^4 \\
&= \sum_{i=1}^{q} x_i y_i^2 + \frac{n-1}{n} C_n - C_n \sum_{i=1}^{q} y_i^2 + \sum_{i=1}^{q} y_i^4 + \sum_{i=q+1}^{n} y_i^4 \\
&\ge \sum_{i=1}^{q} x_i y_i^2 + \frac{n-1}{n} C_n - C_n \sum_{i=1}^{q} y_i^2 \\
&\quad + \frac{1}{q}\left(\sum_{i=1}^{q} y_i^2\right)^2 + \frac{1}{n-q}\left(\sum_{i=q+1}^{n} y_i^2\right)^2 \\
&= \sum_{i=1}^{q} x_i y_i^2 + \frac{n-1}{n} C_n - C_n \sum_{i=1}^{q} y_i^2 + \frac{1}{q}\left(\sum_{i=1}^{q} y_i^2\right)^2 \\
&\quad + \frac{1}{n-q}\left[\frac{(n-1)^2}{n^2} - \frac{2(n-1)}{n}\sum_{i=1}^{q} y_i^2 + \left(\sum_{i=1}^{q} y_i^2\right)^2\right] \\
&= \sum_{i=1}^{q} x_i y_i^2 + \frac{n-1}{n} C_n + \frac{(n-1)^2}{n^2(n-q)} \\
&\quad - \left[C_n + \frac{2(n-1)}{n(n-q)}\right]\sum_{i=1}^{q} y_i^2 + \frac{n}{q(n-q)}\left(\sum_{i=1}^{q} y_i^2\right)^2,
\end{aligned}$$

with equality sign holds if and only if $y_1^2 = \cdots = y_q^2$ and $y_{q+1}^2 = \cdots = y_n^2$.

Denote $Q = \sum_{i=1}^{q} y_i^2$, then $0 \leq Q \leq \frac{n-1}{n}$. If $Q = 0$, the following discussion is trivially hold, so we will assume $Q > 0$. We shall find the minimum of the function $g(x, y) = \sum_{i=1}^{q} x_i y_i^2$ for $x = (x_1, \ldots, x_q)$ and $y = (y_1, \ldots, y_q)$ subjecting to the constraint

$$\sum_{i=1}^{q} x_i = X - (n-q)C_n, \quad \sum_{i=1}^{q} y_i^2 = Q; \quad x_i > C_n,\ 1 \leq i \leq q. \tag{5.20}$$

Define

$$G(x, y, \lambda, \mu) = \sum_{i=1}^{q} x_i y_i^2 + \lambda \sum_{i=1}^{q} x_i + \mu \sum_{i=1}^{q} y_i^2. \tag{5.21}$$

Let $(\bar{x}, \bar{y})$ be a critical point of $g(x, y)$ under condition (5.20), then we have at $(\bar{x}, \bar{y})$

$$\frac{\partial G}{\partial x_i} = \bar{y}_i^2 + \lambda = 0, \quad 1 \leq i \leq q, \tag{5.22}$$

$$\frac{\partial G}{\partial y_i} = 2\bar{x}_i \bar{y}_i + 2\mu \bar{y}_i = 0, \quad 1 \leq i \leq q. \tag{5.23}$$

From (5.22) and (5.23), we obtain $\bar{y}_1^2 = \cdots = \bar{y}_q^2 = Q/q$ and $\bar{x}_1 = \cdots = \bar{x}_q = [X - (n-q)C_n]/q$. This gives

$$g(\bar{x}, \bar{y}) = \frac{Q}{q} \sum_{i=1}^{q} \bar{x}_i = \frac{1}{q}[X - (n-q)C_n]Q.$$

It is easy to check that $g(\bar{x}, \bar{y})$ is in fact the absolute minimum of $g(x, y)$ under the condition (5.20). Therefore, from (5.19) and the above fact we obtain that

$$\begin{aligned} f(x, y) &\geq \frac{n-1}{n} C_n + \frac{(n-1)^2}{n^2(n-q)} + \left[\frac{X}{q} - \frac{n}{q} C_n - \frac{2(n-1)}{n(n-q)}\right] Q + \frac{n}{q(n-q)} Q^2 \\ &= \frac{n}{q(n-q)} \left[Q - \frac{n-q}{2n} \left(\frac{2(n-1)q}{n(n-q)} + nC_n - X\right)\right]^2 \\ &\quad + \frac{n-1}{n} C_n + \frac{(n-1)^2}{n^2(n-q)} - \frac{n-q}{4nq} \left[X - nC_n - \frac{2(n-1)q}{n(n-q)}\right]^2 \\ &\geq \frac{n-1}{n} C_n + \frac{(n-1)^2}{n^2(n-q)} - \frac{n-q}{4nq} \left[X - nC_n - \frac{2(n-1)q}{n(n-q)}\right]^2, \end{aligned} \tag{5.24}$$

with equality sign attainable if and only if

$$0 \leq \frac{n-q}{2n}\left[\frac{2(n-1)q}{n(n-q)} + nC_n - X\right] \leq \frac{n-1}{n},$$

or equivalently

$$nC_n - \frac{2(n-1)}{n} \leq X \leq nC_n + \frac{2(n-1)q}{n(n-q)}.$$

A direct calculation shows that

$$\frac{n-1}{n}C_n + \frac{(n-1)^2}{n^2(n-q)} - \frac{n-q}{4nq}\left[X - nC_n - \frac{2(n-1)q}{n(n-q)}\right]^2 \tag{5.25}$$

$$\geq -\frac{n-1}{4n}\left[X - nC_n + \frac{2(n-1)}{n}\right]^2 + \frac{n-1}{n}\left[X - (n-1)C_n + \frac{n-1}{n}\right],$$

with equality sign holds if and only if $q = 1$.

Now, Claim 3 follows from (5.24) and (5.25).

Finally, a direct calculation will verify the following, which shows that $f(x, y)$ on Ω will not achieve its minimum in the interior of Ω in case $X > nC_n$.

CLAIM 4.

$$\frac{n-1}{n^2}X + \frac{(n-1)^2}{n^3} \tag{5.26}$$

$$\geq -\frac{n-1}{4n}\left[X - nC_n + \frac{2(n-1)}{n}\right]^2 + \frac{n-1}{n}\left[X - (n-1)C_n + \frac{n-1}{n}\right],$$

with equality sign holds if and only if $X = nC_n$.

We have completed the proof of Lemma 4.1 by Claim 1~Claim 4.

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