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MODELS OF PEANO ARITHMETIC AS MODULES OVER INITIAL SEGMENTS

By

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Abstract. Let M be a countable non-standard model of first order Peano arithmetic (**PA**) and I a weakly definable proper initial segment that is closed under addition, multiplication and factorial. We show that there is another model N of **PA** such that the structure of I-module of M coincides with that of N and the multiplication of M coincides with that of N on I but does not coincide at some $(a,b) \notin I^2$.

1. Introduction and Preliminaries

Let **PA** denote the first order Peano arithmetic formulated in the language $L = \{0, 1, +, \cdot, <\}$. Let L_0 denote the language $\{0, 1, +, <\}$. Some papers, including [3] and [4], dealt with the connection between L_0-reducts of models of **PA** and their multiplicative structures. Along these lines of research, in [1], Tsuboi and Murakami considered the following question asked by M. Yasumoto.

Let M be a countable non-standard model of **PA**. Does there exist a model N of **PA** such that

(1) the structure of ordered additive semigroup of M coincides with that of N (i.e. $M|L_0 = N|L_0$),

(2) the multiplication of M coincides with that of N on some non-standard initial segment I but does not coincide at some $(a,b) \notin I^2$? (i.e. $\cdot^M|I = \cdot^N|I$ and $\cdot^M \neq \cdot^N$)

In [1], they showed the existence of such N and I in a strong way. They proved that for any (not necessarily countable) non-standard model M of **PA**, there exist a model N of **PA** and an initial segment I that is closed under multiplication of M and N such that

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(1) $M|L_0 = N|L_0$,

(2) $\cdot^M|I = \cdot^N|I$,

(3) $a \cdot^M a = a \cdot^N a$ if and only if $a \in I$ for all $a \in M$.

In this paper, we prove a related result also answering the question above (see Theorem 13). Our result differs from that of [1] in the following points. First of all, we consider an arbitrary initial segment I of M satisfying minor conditions. Our model N coincides with M not only as an ordered abelian group, but also as an I-module. (We can treat $M|\{+\}$ as an abelian group by adding negative elements.) These two points strengthen the consequence, but we must weaken the condition 3 as below:

(3′) The multiplication of M coincides with that of N on I but does not coincide at some $(a, b) \notin I^2$.

In the paper [4], it is proved that for any countable model of **PA**, the isomorphism type of the additive semigroup determines the isomorphism type of the multiplicative semigroup. If the structure of additive semigroup of M coincides with that of N, the multiplication of M is isomorphic to that of N. So to prove the statement above, we need to construct an I-module automorphism on M which does not preserve the multiplication of M (see Lemma 12).

Before going further, we need some preparations. Let M be a model of **PA**.

DEFINITION 1. Let $I \subset M$. We say that I is *weakly definable* if there exists an $L(M)$-formula $\phi(x, y)$ such that

$$I = \{x \in M : M \models \phi(x, n) \text{ for some } n \in \omega\}.$$

In the definition above, we can always assume that the sets defined by $\phi(x, n)$ $(n \in \omega)$ are increasing in n, by replacing $\phi(x, y)$ with $\phi'(x, y) = \exists z \leq y \phi(x, z)$ as necessary. We shall thus always assume this hereafter.

For the remainder of this paper, we fix a weakly definable proper initial segment $I \subset M$ that is closed under $* + *$, $* \cdot *$ and $*!$ where $x! = x \cdot (x - 1) \cdots 1$. For example let α be an element larger than 1 in M. We define the function $f(n)$ by $f(1) = \alpha$ and $f(n + 1) = f(n)!$, which is definable in M. Then

$$I = \{x \in M : x \leq f(n) \text{ for some } n \in \omega\}$$

satisfies all the requirements stated above.

We can embed M into the ordered ring $M \cup \{-a : a \in M\}$ which is *eq*-definable in M*. We usually work in this extended structure, which is also de-

* We say that a subset A of M^{eq} is *eq*-definable if it is definable in M^{eq}. Notice that every *eq*-definable subset of M is definable in M, and characteristics of M is preserved in $M \cup \{-a : a \in M\}$.

noted by M if there is no confusion. Similarly, we identify I with the extended structure $I \cup \{-a : a \in I\}$, which can be considered as an ordered subring of the ordered ring $M(= M \cup -M)$.

For some $a \in I$, we will define a new unary relation symbol D_a and a new unary function symbol f_a interpreted as:

- $x \in D_a$ if x can be divided by a;
- $f_a(x) = a \cdot x$.

Now let the language L_0^I of I-modules with ordering denote the set $L_0 \cup \{f_a, D_a : a \in I\}$. We consider M and N (in our theorem) as L_0^I-structures. For simplicity of notation, $f_a(x)$ will be written as ax if there is no confusion. We can naturally consider an L-structure M as an L_0^I-structure if $I \subset M$. In other words, we can consider M as an I-module with total ordering.

We write $a \sim_I b$ if $D_d(a - b)$ holds for any nonzero $d \in I$. We write $a <_I b$ if $a + d < b$ holds for any $d \in I$. For an element $q = d/e$ $(e > 0)$ of the quotient field $Q(I)$ of I, qx denotes the maximum element $y \in M$ with $ey \leq dx$. Let $\bar{q} = \langle q_1, \dots, q_n \rangle$ be a tuple of elements of $Q(I)$. Let $\bar{v} = \langle v_1, \dots, v_n \rangle$ be a tuple of elements of M with the same length as $\bar{q}$. We introduce a notation:

$$\bar{q} * \bar{v} = \sum q_i v_i.$$

For an n-tuple $\bar{a}$ of M, we define

$$\langle \bar{a} \rangle_I = \{\bar{q} * \bar{a} + d : \bar{q} \in Q(I), d \in I\}.$$

The quantifier free type $\mathrm{qftp}(\bar{a}/A)$ of $\bar{a}$ over A is the set of all quantifier free $L(A)$-formulas $\phi(\bar{x})$ satisfied by $\bar{a}$.

2. Main Result

From now on, let M be a countable non-standard model of **PA** and I a weakly definable proper initial segment of M that is closed under $+$, $\cdot$ and $!$.

LEMMA 2. *Let $a, b \in M$ with $a <_I b$. Then for any $c \in M$, there exist infinitely many $d \in M$ such that $c \sim_I d$ and $a < d < b$.*

PROOF. Let $\phi(x, y, z)$ be the L-formula asserting that

$$0 < x \cdot y < z \wedge \forall w \quad (0 < w \leq x \rightarrow y \text{ can be divided by } w).$$

Putting $e = b - a$, we see that $M \models \exists y \phi(s, y, e)$ for all $s \in I$. In fact, since I is closed under $\cdot$ and $!$, we have $M \models \phi(s, s!, e)$. By overspill, there exists $t > I$ such that $M \models \exists y \phi(t, y, e)$. Let $u \in M$ be the solution y of $\phi(t, y, e)$, then it follows that $u \sim_I 0$ and $t \cdot u < e$. So there exist infinitely many $m \in M$ such that $a < c + mu < b$. □

LEMMA 3. *Let $\bar{a}$ be an n-tuple of M and $A = \langle \bar{a} \rangle_I$. Then A includes I, and is closed under +, − and multiplication by $d \in Q(I)$.*

PROOF. First we claim that

$$\langle \bar{a} \rangle_I = \{x \in M : dx = \bar{d} * \bar{a} + e, d \neq 0, \bar{d}, e \in I\}.$$

We may assume that the length of $\bar{a}$ equals 1, and put $\bar{a} = a$. Let $x \in \langle a \rangle_I$. Then there exist $p = s/t \in Q(I)$ and $e \in I$ such that $x = pa + e$. By the definition of pa, there exist $r \in M$ $(0 \leq r < t)$ such that $sa = t(pa) + r$. Since I is an initial segement and $t \in I$, it follows that $r \in I$. So $tx = t(pa + e) = t(pa) + te = sa - r + te$. Recalling that we have identified I with $I \cup -I$, we can assume that I is closed under $-$. Since I is closed under $+$, $-$, and $\cdot$, it follows that $-r + te \in I$.

Conversely suppose that $tx = sa + e$ for some $t > 0$, s and $e \in I$. Then there exists $r \in I$ $(0 \leq r < t)$ such that $sa = t((s/t)a) + r$, so that $tx = t((s/t)a) + r + e$. Since $r + e$ can be divided by t, there exist $u \in M$ such that $r + e = tu$. We ramark that (1) I is an initial segment, (2) $|u| < |tu|$ and (3) $tu \in I$. So we have $u \in I$, and $x = (s/t)a + u$.

We claim that $\langle \bar{a} \rangle_I$ is closed under $+$, $-$. It suffices to show for the case of $+$. Let $x, y \in \langle \bar{a} \rangle_I$. Then

$$dx = \bar{d} * \bar{a} + e,$$

$$d'y = \bar{d}' * \bar{a} + e'$$

for some d, d', $\bar{d}$, $\bar{d}'$, e and $e' \in I$. By multiplying the above by d' and d respectively, we have

$$d'dx = d'\bar{d} * \bar{a} + d'e,$$

$$dd'y = d\bar{d}' * \bar{a} + de'.$$

By adding the both sides of equations above, $d'd(x + y) = (d'\bar{d} + d\bar{d}') * \bar{a} + (d'e + de')$. So $x + y \in \langle \bar{a} \rangle_I$.

We claim that $\langle \bar{a} \rangle_I$ is closed under multiplication by $s/t \in Q(I)$ $(t > 0)$. Let $x \in \langle \bar{a} \rangle_I$. Then $dx = \bar{d} * \bar{a} + e$ for some d, $\bar{d}$ and $e \in I$. By the definition, there exists $r \in I$ $(0 \leq r < t)$ such that $sx = t((s/t)x) + r$. Then we have

$$dt((s/t)x) = s\bar{d} * \bar{a} + se - dr,$$

and so $(s/t)x \in \langle \bar{a} \rangle_I$.

LEMMA 4. *Let $\bar{a}$ be an n-tuple of M. Then $\langle \bar{a} \rangle_I$ coincides with the set of all $\bar{a}$-definable elements of M using quantifier free L_0^I-formulas.*

PROOF. Let x_0 be an element defined by the quantifier free $L_0^I(\bar{a})$-formula $\phi(x)$. We may assume that $\phi(x)$ is of the form $\bigvee_j \bigwedge_i \phi_{i,j}(x)$ where $\phi_{i,j}(x)$ is an atomic formula or a negation of an atomic formula. A negation of atomic L_0^I-formula is of the form $\neg(t(x) < s(x))$, $\neg(t(x) = s(x))$ or $\neg D_d(t(x))$ where $t(x)$, $s(x)$ are terms. If x_0 satisfies a negation of an atomic L_0^I-formula, then x_0 satisfies some formula of the form $t(x) < s(x)$, $t(x) = s(x)$ or $D_d(t(x) - e)$ for some e $(0 < e < d)$. So we may assume that $\phi_{i,j}(x)$ is an atomic formula.

We remark that x_0 is definable by the formula $\bigwedge_i \phi(x)_{i,j}$ for some j. So we may assume that $\phi(x)$ is of the form $\bigwedge_i \phi_i(x)$ where $\phi_i(x)$ is an atomic formula.

First, we suppose that there exists i such that $\phi_i(x)$ is of the form $t(x) = s(x)$. We remark that $L_0^I(\bar{a})$-terms $t(x)$ and $s(x)$ are of the form $cx + \bar{d} * \bar{a} + d$ where c, d and $\bar{d} \in I$. So we have $x_0 \in \langle \bar{a} \rangle_I$.

Next, we suppose that there does not exist i such that $\phi_i(x)$ is of the form $t(x) = s(x)$. We remark that an $L_0^I(\bar{a})$-term is of the form $cx + \bar{d} * \bar{a} + d$ where c, d and $\bar{d} \in I$. So we may assume that x_0 is defined by some conjunction of atomic formulas as follows:

$$\bigwedge_i (s_i < c_i x < t_i) \wedge \bigwedge_i (D_{d_i}(e_i x + u_i))$$

where s_i, t_i, and u_i are elements of $\langle \bar{a} \rangle_I$ and c_i $(c_i > 0)$, d_i and e_i are elements of I. The first conjunction is equivalent to $s < x < t$ where $s = max_i\{(1/c_i)s_i\}$ and $t = min_i\{(1/c_i)(t_i - 1) + 1\}$. By Lemma 3, it follows that $s, t \in \langle \bar{a} \rangle_I$. Suppose that $s <_I t$. By Lemma 2, there exist infinitely many $d \in M$ such that $x_0 \sim_I d$ and $s < d < t$. Therefore, there are infinitely many d for which the formula $\phi(x)$ holds. This is a contradiction. So we can assume that $s \not<_I t$. Since I is an initial segment, there exists $g \in I$ such that $x_0 = s + g$ and we have $x_0 \in \langle \bar{a} \rangle_I$.

Conversely, let $x_0 \in \langle \bar{a} \rangle_I$. We may assume that the length of $\bar{a}$ equals 1, and put $\bar{a} = a$. We put $x_0 = pa + d$, where $p = b/c \in Q(I)$, and $d \in I$. By the definition of pa, there exists $g \in I$ such that $ba = c(x_0 - d) + g$ and $0 \le g < c$. Then the $L_0^I(a)$-formula $ba = c(x - d) + g$ defines x_0. □

LEMMA 5. *Let $\bar{a}$ be an n-tuple of M. Then $\langle \bar{a} \rangle_I$ is weakly definable.*

PROOF. Let I be weakly definable by the formula $\phi(x, n)$. Let $I_n = \{x \in M : M \models \phi(x, n)\}$, so that $I = \bigcup_{n \in \omega} I_n$. Let $\psi(x, n)$ be the formula asserting that

there exist d $(d \neq 0)$, $\bar{d}$ and $e \in I_n$ such that $dx = \bar{d} * \bar{a} + e$.

By the claim in the proof of Lemma 3, $\langle \bar{a} \rangle_I$ is weakly definable by the formula $\psi(x, n)$.

LEMMA 6. *Let $\bar{a}$ be an n-tuple of M, and let $A = \langle \bar{a} \rangle_I$. Then the L_0^I-quantifier free type $p(x) = \mathrm{qftp}(b/A)$ of b over A is determined by the following sets:*

(1) $\{D_d(t(x)) : M \models D_d(t(b)), d \in I, t(x) \text{ a term in } L_0^I(A)\}$,

(2) $\{\neg D_d(t(x)) : M \models \neg D_d(t(b)), d \in I, t(x) \text{ a term in } L_0^I(A)\}$,

(3) $\{c < x : M \models c < b, c \in A\}$,

(4) $\{x < c : M \models b < c, c \in A\}$.

PROOF. Let Γ be the union of the four sets above. If $b \in A$, then $p(x)$ is generated by $x = b$. It is clear that $x = b$ is equivalent to $b - 1 < x$ and $x < b + 1$, both of which belong to Γ. So we can assume $b \notin A$. Let us consider the formula $dx < c$ in $p(x)$ where $d \in I$ $(d > 0)$ and c is an $L_0^I(A)$-term. First, suppose that $D_d(c)$ holds. Then we have $d((1/d)c) = c$, and $dx < c$ is equivalent to $x < (1/d)c$. The last formula belongs to Γ. Then we assume that $a = d((1/d)c) + e$ for some e such that $0 < e < d$. In this case, $dx < c$ is equivalent to $x < (1/d)c + 1$. □

LEMMA 7. *Let $\sigma : \bar{a} \to \bar{b}$ be an L_0^I-isomorphism. Then σ can be extended to the L_0^I-isomorphism $\sigma' : \langle \bar{a} \rangle_I \to \langle \bar{b} \rangle_I$ such that $\bar{p} * \bar{a} + d \mapsto \bar{p} * \bar{b} + d$, where $\bar{p} \in Q(I)$ and $d \in I$.*

PROOF. We may assume that the length of $\bar{a}$ equals 1, and so put $\bar{a} = a$ and $\bar{b} = b$. First we show that σ' preserves addition. Let $x_1 + x_2 = x_3$, with $x_i = p_i a + c_i$, $p_i = d_i/e_i \in Q(I)$, and $c_i \in I$ for $i = 1, 2, 3$. There exist $g_i \in I$ $(i = 1, 2, 3)$ such that $d_i a = e_i(x_i - c_i) + g_i$ and $0 \le g_i < e_i$. Since $x_i = (d_i a - g_i)/e_i + c_i$, we have

$$\left(\frac{d_1 a - g_1}{e_1} + c_1\right) + \left(\frac{d_2 a - g_2}{e_2} + c_2\right) = \frac{d_3 a - g_3}{e_3} + c_3.$$

By multiplying both sides by $e = e_1 e_2 e_3$, this is equivalent to the quantifier free $L_0^I(a)$-formula

$$((e_2 e_3(d_1 a - g_1) + e c_1) + (e_1 e_3(d_2 a - g_2) + e c_2)$$
$$= e_1 e_2(d_3 a - g_3) + e c_3) \wedge \bigwedge_i D_{e_i}(d_i a - g_i).$$

Since a and b have the same quantifier free type, the $L_0^I(b)$-formula obtained by replacing a in the formula above by b also holds. So we have $\sigma(x_1) + \sigma(x_2) = \sigma(x_3)$. Similarly, we can show that σ' is the L_0^I-isomorphism. □

DEFINITION 8. Let A be a subset of M.
(1) We say that a pair (A_-, A_+) is a *cut* of A if $A = A_- \cup A_+$ and $A_- < A_+$.
(2) Let $a \notin A$. We say that a defines the *cut* (A_-, A_+) if $A_- < a < A_+$.

LEMMA 9. *Let $\bar{a}$ and $\bar{b}$ be n-tuples of M. Put $A = \langle \bar{a} \rangle_I$ and $B = \langle \bar{b} \rangle_I$. Let $\sigma : A \to B$ be an L_0^I-isomorphism with $\sigma(\bar{a}) = \bar{b}$. Let a define the cut (A_-, A_+) of A. Then there exists $b \in M$ such that b defines the cut $(\sigma(A_-), \sigma(A_+))$ of B.*

PROOF. Let A be weakly definable by the $L(M)$-formula $\psi(x, n)$. Let $A_n = \{x \in M : M \models \psi(x, n)\}$, so that $A = \bigcup_{n \in \omega} A_n$. Let $c(n) = max\{x \in M : x < a, x \in A_n\}$ and $d(n) = min\{x \in M : x > a, x \in A_n\}$. Then the $L(M)$-formula $c(n) < d(n)$ holds for every $n \in \omega$. There exist definable functions $\bar{q}(n)$, $\bar{r}(n)$, $l(n)$ and $m(n)$ such that
(1) $c(n) = \bar{q}(n) * \bar{a} + l(n)$ and $d(n) = \bar{r}(n) * \bar{a} + m(n)$,
(2) $\bar{q}(n), \bar{r}(n) \in Q(I)$ and $l(n), m(n) \in I$ for every $n \in \omega$.
We put $c'(n) = \bar{q}(n) * \bar{b} + l(n)$ and $d'(n) = \bar{r}(n) * \bar{b} + m(n)$. Since σ preserves the ordering, the $L(M)$-formula $c'(n) < d'(n)$ holds for every $n \in \omega$. By overspill, there exists $e > \omega$ such that $c'(e) < d'(e)$. As stated in the introduction, we may assume that the sets A_n $(n \in \omega)$ are increasing in n. Therefore it follows that $\sigma(A_-) < c'(e)$ and $d'(e) < \sigma(A_+)$. So any element b between $c'(e)$ and $d'(e)$ defines the cut $(\sigma(A_-), \sigma(A_+))$ of B. □

LEMMA 10. *Let $\bar{a}$ be an n-tuple of M and $A = \langle \bar{a} \rangle_I$. Then for all $a \notin A$, there exists $b \in M$ such that a and b define the same cut of A and $a <_I b$.*

PROOF. Let I be weakly definable by the formula $\phi(x, n)$. Let A be weakly definable by the formula $\psi(x, n)$, and let $A_n = \{x \in M : M \models \psi(x, n)\}$. Let $d(n)$ be the maximum element x satisfying $\phi(x, n)$. Let $\theta(n)$ be a formula asserting that

the interval between a and $a + d(n)$ does not intersect the set A_n.

Then $M \models \theta(n)$ for every $n \in \omega$. In fact, if there exist $n \in \omega$ and $x \in A_n$ such that $a < x < a + d(n)$, then $0 < x - a < d(n)$. Since I is an initial segment, $x - a \in I \subset A$. Since $x \in A$, this is contradictory to $a \notin A$. By overspill, there exits $e > \omega$ such that the interval between a and $a + d(e)$ does not intersect the set A. Since $d(e) > I$, we have $b = a + d(e)$. □

LEMMA 11. *Let $A = \langle \bar{a} \rangle_I$ and $B = \langle \bar{b} \rangle_I$. Let σ be an L_0^I-isomorphism from A to B with $\sigma(\bar{a}) = \bar{b}$. Suppose that*

(1) *$a \sim_I b$;*
(2) *a and b define the cut (A_-, A_+) of A and the cut $(\sigma(A_-), \sigma(A_+))$ of B respectively.*

Then σ is extended to an L_0^I-isomorphism $\sigma' : \langle \bar{a}a \rangle_I \to \langle \bar{b}b \rangle_I$ with $\sigma'(a) = b$.

PROOF. By Lemma 7, it suffices to show that $\mathrm{qftp}(a/A) = \mathrm{qftp}(b/B)$. By Lemma 6, we consider the following cases:

Case 1. $D_c(\bar{d} * \bar{a} + ea)$ holds where $\bar{d}$ and $e \in I$. Since $\bar{a}$ and $\bar{b}$ have the same quantifier free type over I, $\bar{d} * \bar{a} \sim_I \bar{d} * \bar{b}$. Since $a \sim_I b$, $\bar{d} * \bar{a} + ea \sim_I \bar{d} * \bar{b} + eb$. So $D_c(\bar{d} * \bar{b} + eb)$ holds.

Case 2. $\bar{p} * \bar{a} + d < a$ holds where $\bar{p} \in Q(I)$ and $d \in I$. By the second condition of the lemma, $\sigma(\bar{p} * \bar{a} + d) = \bar{p} * \bar{b} + d < b$ holds. □

LEMMA 12. *Let M be a countable non-standard model of* **PA**, *and I a weakly definable proper initial segment of M that is closed under $+$, $\cdot$ and $!$. Then there is an L_0^I-automorphism σ of M such that $\sigma(c \cdot d) \neq \sigma(c) \cdot \sigma(d)$ for some $(c, d) \notin I^2$.*

PROOF. We fix $a \in M$ such that $a > I$. Then $a^2 \notin A = \langle a \rangle_I$. In fact, if $a^2 \in A$, then there exists a formula $dx^2 + ex + f = 0$ having the solution a where d $(d \neq 0)$, e and $f \in I$. Since $a > I$, $|da^2 + ea + f| > I$. This is a contradiction.

Let $\sigma_0 : A \to A$ be the identity mapping. By Lemmata 10 and 2, there exists $b \neq a^2$ such that a^2 and b define the same cut of A and $a^2 \sim_I b$. By Lemma 11, σ_0 can be extended to an L_0^I-isomorphism $\sigma_1 : \langle aa^2 \rangle_I \to \langle ab \rangle_I$. Using Lemmata 9, 10, 2 and 11, σ_1 can be extended to an L_0^I-automorphism σ on M by a back and forth argument. This automorphism does not preserve the multiplication of M. In fact, $\sigma(a^2) = b \neq a^2 = \sigma(a)^2$. □

THEOREM 13. *Let M be a countable non-standard model of* **PA**, *and I a weakly definable proper initial segment of M that is closed under $+$, $\cdot$ and $!$. Then there exists a model N of* **PA** *such that*
(1) *$M|L_0^I = N|L_0^I$.*
(2) *$\cdot^M|I = \cdot^N|I$ and $\cdot^M \neq \cdot^N$.*

PROOF. By Lemma 12, we have a model $N = (M|L_0^I, \cdot^N)$ of **PA** such that $x \cdot^N y = \sigma^{-1}(\sigma(x) \cdot^M \sigma(y))$. In fact, if $\sigma(c \cdot^M d) \neq \sigma(c) \cdot^M \sigma(d)$, then $c \cdot^M d \neq c \cdot^N d$. □

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