

RESTRICTED ENERGY INEQUALITIES AND NUMERICAL APPROXIMATIONS

By

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Introduction

Let $\{A, B_j\}$ be linear partial differential operators. Let $\Omega(\subset \mathbf{R}^n)$ be a bounded domain with smooth boundary Γ . Our boundary value problem is to find $u \in L^2(\Omega)$ satisfying

$$(P) \quad \begin{cases} Au = f & \text{in } \Omega, \\ B_j u = f_j & \text{on } \Gamma \ (j \in J) \end{cases}$$

for given data $\{f, f_j\}$. We are particularly interested in a method of numerical approximation of solutions of (P).

The problem (P) is closely connected with its adjoint problem (P*). The adjoint problem is to find $v \in L^2(\Omega)$ satisfying

$$(P^*) \quad \begin{cases} A^* v = g & \text{in } \Omega \\ \mathcal{B}_j^* v = g_j & \text{on } \Gamma \ (j \in J^*) \end{cases}$$

for given data $\{g, g_j\}$.

Recently, it has become clear that a solution $u \in L^2(\Omega)$ of (P) can be constructed numerically, assuming an energy inequality

$$(E^*) \quad \|v\| \leq C \left(\|A^* v\| + \sum_{j \in J^*} \langle \mathcal{B}_j^* v |_{\Gamma} \rangle_{\mu_j} \right) \quad (v \in H^q(\Omega))$$

([1]).

Here we have two questions:

- (1) In case when L^2 -solutions of (P) are not unique, how can we characterize the solution, obtained in [1]?
- (2) In case when L^2 -solutions of (P*) are not unique, (E*) can not be satisfied. Is there any numerical method to approach to one of solutions of (P)?

In this paper, in stead of (E^*) , we assume a restricted energy inequality:

$$(\mathcal{E}_0^*) \quad \|v\| \leq C \|A^*v\| \quad (v \in M^*, \mathcal{B}_j^*v|_\Gamma = 0 \ (j \in J^*)),$$

where M^* is a subspace in $H^q(\Omega)$ defined in §1. Then we will see that the method in [1] is applicable. Moreover, we will see that the solution obtained by our method is unique in a subspace τ in $L^2(\Omega)$.

§1. Restricted Energy Inequalities

Let

$$A = \sum_{|\nu| \leq m} a_\nu(x) \partial_x^\nu$$

be a differential operator with smooth coefficients defined in a neighborhood of $\bar{\Omega}$. Let

$$B_j = \sum_{|\nu| \leq j} b_{j\nu}(x) \partial_x^\nu \quad (j \in J, J \subset \{0, 1, \dots, m-1\})$$

be differential operators with smooth coefficients defined in a neighborhood of Γ . We assume that Γ is non-characteristic for $\{A, B_j \ (j \in J)\}$. Namely,

$$\sum_{|\nu|=m} a_\nu(x) \mathbf{n}(x)^\nu \neq 0 \quad \text{on } \Gamma,$$

$$\sum_{|\nu|=j} b_{j\nu}(x) \mathbf{n}(x)^\nu \neq 0 \quad \text{on } \Gamma,$$

where $\mathbf{n}(x)$ is a unit inner normal at $x(\in \Gamma)$.

Set

$$J^c \cup J = \{0, 1, \dots, m-1\}, \quad J^c \cap J = \emptyset, \quad J^* = \{j \mid m-1-j \in J^c\},$$

$$B_j = (d/d\mathbf{n})^j \quad (j \in J^c).$$

Then we can define

$$\mathcal{B}_j = \sum_{|\nu| \leq j} \beta_{j\nu}(x) \partial_x^\nu \quad (j \in \{0, 1, \dots, m-1\}),$$

for which Γ is non-characteristic, such that the following Green's Theorem holds.

LEMMA 1.1 (Green's Theorem). *Suppose that $u, Au \in L^2(\Omega)$, then it holds that*

$$\langle (d/dn)^k u|_\Gamma \rangle_{-k-m+1/2} \leq C(\|u\| + \|Au\|) \quad (k = 0, 1, \dots, m-1)$$

and

$$\begin{aligned} (Au, v) - (u, A^*v) &= - \sum_{j \in J} \langle B_j u|_\Gamma, \mathcal{B}_{m-1-j}^* v|_\Gamma \rangle \\ &\quad - \sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, \mathcal{B}_j^* v|_\Gamma \rangle \quad (v \in H^{2m-1}(\Omega)), \end{aligned}$$

where

$$A^* = \sum_{|v| \leq m} (-\partial_x)^v \overline{a_v(x)}, \quad \mathcal{B}_j^* = \sum_{|v| \leq j} (-\partial_x)^v \overline{\beta_{jv}(x)}.$$

NOTATIONS.

- (1) $(u, v) = (u, v)_{L^2(\Omega)}$, $\|u\| = \|u\|_{L^2(\Omega)}$,
- (2) $\Lambda = (1 - \Delta)^{1/2}$, where Δ is the Laplace-Beltrami operator on Γ ,
- (3) $\langle u, v \rangle_\sigma = (u, v)_{H^\sigma(\Gamma)} = (\Lambda^\sigma u, \Lambda^\sigma v)_{L^2(\Gamma)}$, $\langle u \rangle_\sigma = \|u\|_{H^\sigma(\Gamma)}$ for $u, v \in H^\sigma(\Gamma)$ (σ : real),
- (4) $u \in H^{-\sigma}(\Gamma)$: $H^\sigma(\Gamma) \ni v \mapsto \langle u, v \rangle \in \mathbf{C}$ ($\sigma > 0$).

Lemma 1.1 is well known for $u \in H^m(\Omega)$. See Appendix of [3] in case when $u, Au \in L^2(\Omega)$.

REMARK. Set

$$\begin{aligned} Pu &= \{Au, B_j u|_\Gamma \ (j \in J)\}, \quad Qu = \{u, -B_{m-1-j} u|_\Gamma \ (j \in J^*)\}, \\ P^*v &= \{A^*v, \mathcal{B}_j^* v|_\Gamma \ (j \in J^*)\}, \quad Q^*v = \{v, \mathcal{B}_{m-1-j}^* v|_\Gamma \ (j \in J)\}, \end{aligned}$$

then the problem (P) denotes $Pu = \{f, f_j \ (j \in J)\}$ and the problem (P^*) denotes $P^*v = \{g, g_j \ (j \in J^*)\}$, and Green's Theorem is stated as follows.

GREEN'S THEOREM. *Suppose that $u, Au \in L^2(\Omega)$, then it holds that*

$$\langle (d/dn)^k u|_\Gamma \rangle_{-k-m+1/2} \leq C(\|u\| + \|Au\|) \quad (k = 0, 1, \dots, m-1)$$

and

$$[Pu, Q^*v] = [Qu, P^*v]_* \quad (v \in H^{2m-1}(\Omega)),$$

where

$$[F, G] = (f, g) + \sum_{j \in J} \langle f_j, g_j \rangle \quad \text{for } F = \{f, f_j (j \in J)\} \text{ and } G = \{g, g_j (j \in J)\},$$

$$[F, G]_* = (f, g) + \sum_{j \in J^*} \langle f_j, g_j \rangle \quad \text{for } F = \{f, f_j (j \in J^*)\} \text{ and } G = \{g, g_j (j \in J^*)\}.$$

NULL SPACES. Set

$$K = \{\phi \in L^2(\Omega) \mid P\phi = 0\}, \quad K^* = \{\phi \in L^2(\Omega) \mid P^*\phi = 0\}.$$

Owing to Green's Theorem, we have

$$K = \{\phi \in L^2(\Omega) \mid [Q\phi, P^*v]_* = 0 \ (\forall v \in H^{2m-1}(\Omega))\},$$

$$K^* = \{\phi \in L^2(\Omega) \mid [Pu, Q^*\phi] = 0 \ (\forall u \in H^{2m-1}(\Omega))\}.$$

Therefore, K and K^* are closed subspaces in $L^2(\Omega)$. Set

$$K^\perp = \{f \in L^2(\Omega) \mid (f, \phi) = 0 \ (\forall \phi \in K)\},$$

$$K^{*\perp} = \{f \in L^2(\Omega) \mid (f, \phi) = 0 \ (\forall \phi \in K^*)\}.$$

We assume

(A-I) there exists an integer $p (\geq 2m - 1)$ such that

$$K, K^* \subset H^p(\Omega)$$

throughout this paper.

We define

$$M^* = K^{*\perp} \cap H^q(\Omega) = \{f \in H^q(\Omega) \mid (f, \phi) = 0 \ (\forall \phi \in K^*)\}$$

for an integer q ($m \leq q \leq p$). Then M^* is a closed subspace in $H^q(\Omega)$. Let $u \in H^q(\Omega)$, then there exist $\phi \in K^*$ and $\xi \in M^*$ such that

$$u = \phi + \xi \quad \text{and} \quad \|u\|^2 = \|\phi\|^2 + \|\xi\|^2.$$

We say that *restricted energy inequality* (\mathcal{E}^*) holds, if it holds

$$(\mathcal{E}^*) \quad \|v\| \leq C \left(\|A^*v\| + \sum_{j \in J^*} \langle \mathcal{B}_j^* v|_\Gamma \rangle_{\mu_j} \right) \quad (v \in M^*),$$

where $\mu_j = q - 1/2 - j$.

We say that *restricted energy inequality* (\mathcal{E}_0^*) holds, if it holds

$$(\mathcal{E}_0^*) \quad \|v\| \leq C \|A^* v\| \quad (v \in M^*, \mathcal{B}_j^* v|_\Gamma = 0 \quad (j \in J^*)).$$

Since $\{B_j \ (j = 0, 1, \dots, m-1)\}$ and $\{\mathcal{B}_j^* \ (j = 0, 1, \dots, m-1)\}$ are Dirichlet sets, we have ([4])

LEMMA 1.2. *Let $s \geq m$.*

i) *Let $f_j \in H^{s-1/2-j}(\Gamma)$ ($j = 0, 1, \dots, m-1$), then there exists $U \in H^s(\Omega)$ such that*

$$B_j U|_\Gamma = f_j \quad (j = 0, 1, \dots, m-1), \quad \|U\|_s \leq C \sum_{j \in \{0, 1, \dots, m-1\}} \langle f_j \rangle_{s-1/2-j}.$$

ii) *Let $g_j \in H^{s-1/2-j}(\Gamma)$ ($j = 0, 1, \dots, m-1$), then there exists $V \in H^s(\Omega)$ such that*

$$\mathcal{B}_j^* V|_\Gamma = g_j \quad (j = 0, 1, \dots, m-1), \quad \|V\|_s \leq C \sum_{j \in \{0, 1, \dots, m-1\}} \langle g_j \rangle_{s-1/2-j}.$$

LEMMA 1.3. *(\mathcal{E}^*) holds iff (\mathcal{E}_0^*) holds.*

PROOF. Suppose that (\mathcal{E}_0^*) holds. Let $v \in M^*(\subset H^q(\Omega))$.

(1) Set

$$g = A^* v \in H^{q-m}(\Omega), \quad g_j = \mathcal{B}_j^* v|_\Gamma \in H^{q-1/2-j}(\Gamma) \quad (j \in J^*).$$

Then there exists $V \in H^q(\Omega)$ such that

$$\mathcal{B}_j^* V|_\Gamma = g_j \quad (j \in J^*), \quad \|V\|_q \leq C \sum_{j \in J^*} \langle g_j \rangle_{q-1/2-j}$$

from (ii) of Lemma 1.2 ($s = q$).

(2) Set $w = v - V$, then $w \in H^q(\Omega)$ satisfies

$$\begin{cases} A^* w = g - A^* V, \\ \mathcal{B}_j^* w|_\Gamma = 0 \quad (j \in J^*). \end{cases}$$

Since $w \in H^q(\Omega)$, there exist $\phi \in K^*$ and $\xi \in M^*$ such that

$$w = \phi + \xi, \quad \|w\|^2 = \|\phi\|^2 + \|\xi\|^2.$$

Therefore $\xi \in M^*$ satisfies

$$\begin{cases} A^*\xi = g - A^*V, \\ \mathcal{B}_j^*\xi|_\Gamma = 0 \quad (j \in J^*). \end{cases}$$

Since (\mathcal{E}_0^*) holds, we have

$$\|\xi\| \leq C\|A^*\xi\| = C\|g - A^*V\| \leq C' \left(\|g\| + \sum_{j \in J^*} \langle g_j \rangle_{q-1/2-j} \right).$$

(3) In the same way, since $V \in H^q(\Omega)$, there exist $\psi \in K^*$ and $\eta \in M^*$ such that

$$V = \psi + \eta, \quad \|V\|^2 = \|\psi\|^2 + \|\eta\|^2.$$

Hence we have

$$v = w + V = (\phi + \psi) + (\xi + \eta), \quad \phi + \psi \in K^*, \quad \xi + \eta \in M^*.$$

Since $v \in M^*$, we have

$$v = \xi + \eta.$$

Hence we have

$$\|v\| \leq \|\xi\| + \|\eta\| \leq \|\xi\| + \|V\| \leq C \left(\|g\| + \sum_{j \in J^*} \langle g_j \rangle_{q-1/2-j} \right). \quad \square$$

We assume

(A-II) (\mathcal{E}_0^*)

throughout this paper. Then we can define a Hilbert space \mathcal{H} as the closure of M^* by the norm []:

$$[v]^2 = \|A^*v\|^2 + \sum_{j \in J^*} \langle \mathcal{B}_j^*v|_\Gamma \rangle_{\mu_j}^2.$$

Inner product of \mathcal{H} is defined by

$$[w, v] = (A^*w, A^*v) + \sum_{j \in J^*} \langle \mathcal{B}_j^*w|_\Gamma, \mathcal{B}_j^*v|_\Gamma \rangle_{\mu_j}.$$

For a fixed $f \in L^2(\Omega)$, define

$$f : \mathcal{H} \ni v \mapsto (v, f) \in \mathbf{C}$$

then f is a continuous linear functional on \mathcal{H} . In fact, it holds

$$|(v, f)| \leq \|v\| \|f\| \leq C[v]\|f\| \quad (v \in \mathcal{H})$$

from Lemma 1.3. Therefore, owing to Riesz' Theorem in \mathcal{H} , there exists $w \in \mathcal{H}$ such that

$$(f, v) = [w, v] \quad (v \in \mathcal{H}),$$

where we say that $w \in \mathcal{H}$ is a *Riesz function* of f ($\in L^2(\Omega)$).

§2. Existence and Uniqueness

THEOREM 2.1. *Assume (A-I) and (A-II). Suppose that $f \in K^{\perp}$. Let $w \in \mathcal{H}$ be a Riesz function of f . Set $u = A^*w \in L^2(\Omega)$, then u satisfies*

$$(P_0) \quad \begin{cases} Au = f & \text{in } \Omega, \\ B_j u|_{\Gamma} = 0 & (j \in J), \end{cases}$$

and

$$B_{m-1-j}u|_{\Gamma} = -\Lambda^{2\mu_j} \mathcal{B}_j^* w|_{\Gamma} \quad (j \in J^*).$$

PROOF. (1) Since $w \in \mathcal{H}$ satisfy

$$(f, v) = (A^*w, A^*v) + \sum_{j \in J^*} \langle \mathcal{B}_j^* w|_{\Gamma}, \mathcal{B}_j^* v|_{\Gamma} \rangle_{\mu_j} \quad (v \in \mathcal{H}),$$

$u = A^*w$ satisfies

$$(f, v) - (u, A^*v) = \sum_{j \in J^*} \langle \mathcal{B}_j^* w|_{\Gamma}, \mathcal{B}_j^* v|_{\Gamma} \rangle_{\mu_j} \quad (v \in \mathcal{H}) \dots\dots \textcircled{1}.$$

(2) Moreover, we have

$$(f, v) - (u, A^*v) = \sum_{j \in J^*} \langle \mathcal{B}_j^* w|_{\Gamma}, \mathcal{B}_j^* v|_{\Gamma} \rangle_{\mu_j} \quad (v \in H^q(\Omega)) \dots\dots \textcircled{1}'.$$

In fact, let $v \in H^q(\Omega)$. Then there exist $\phi \in K^*$ and $\xi \in M^*$ such that $v = \phi + \xi$. Since $\xi \in M^* \subset \mathcal{H}$, we have from $\textcircled{1}$

$$(f, \xi) - (u, A^*\xi) = \sum_{j \in J^*} \langle \mathcal{B}_j^* w|_{\Gamma}, \mathcal{B}_j^* \xi|_{\Gamma} \rangle_{\mu_j}.$$

We remark that ϕ satisfies

$$A^*\phi = 0, \quad \mathcal{B}_j^*\phi|_{\Gamma} = 0 \quad (j \in J^*),$$

and $(f, \phi) = 0$. Hence we have

$$(f, v) - (u, A^*v) = \sum_{j \in J^*} \langle \mathcal{B}_j^* w|_\Gamma, \mathcal{B}_j^* v|_\Gamma \rangle_{\mu_j}.$$

(3) From ①', we have

$$(f, v) - (u, A^*v) = 0 \quad (v \in \mathcal{D}'(\Omega)).$$

which means

$$Au = f \quad \text{in } \mathcal{D}'(\Omega).$$

Therefore we have

$$(Au, v) - (u, A^*v) = \sum_{j \in J^*} \langle \mathcal{B}_j^* w|_\Gamma, \mathcal{B}_j^* v|_\Gamma \rangle_{\mu_j} \quad (v \in H^q(\Omega)) \dots\dots ①''.$$

(4) Owing to Green's Theorem, we have

$$\begin{aligned} (Au, v) - (u, A^*v) &= - \sum_{j \in J} \langle B_j u|_\Gamma, \mathcal{B}_{m-1-j}^* v|_\Gamma \rangle \\ &\quad - \sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, \mathcal{B}_j^* v|_\Gamma \rangle \quad (v \in H^{2m-1}(\Omega)) \dots\dots ②. \end{aligned}$$

Hence, from ①'' and ②, we have

$$\begin{aligned} \sum_{j \in J^*} \langle \mathcal{B}_j^* w|_\Gamma, \mathcal{B}_j^* v|_\Gamma \rangle_{\mu_j} &= - \sum_{j \in J} \langle B_j u|_\Gamma, \mathcal{B}_{m-1-j}^* v|_\Gamma \rangle \\ &\quad - \sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, \mathcal{B}_j^* v|_\Gamma \rangle \quad (v \in H^{q'}(\Omega), \quad q' = \max(q, 2m - 1)), \end{aligned}$$

which means

$$B_j u|_\Gamma = 0 \quad (j \in J), \quad B_{m-1-j} u|_\Gamma = -\Lambda^{2\mu_j} \mathcal{B}_j^* w|_\Gamma \quad (j \in J^*). \quad \square$$

COROLLARY 2.1. Assume (A-I) and (A-II). Suppose that $\{f \in L^2(\Omega), f_j \in H^{m-1/2-j}(\Gamma) \ (j \in J)\}$ satisfy

$$(\mathcal{R}) \quad (f, \phi) + \sum_{j \in J} \langle f_j, \mathcal{B}_{m-1-j}^* \phi|_\Gamma \rangle = 0 \quad (\phi \in K^*),$$

that is,

$$(\mathcal{R}) \quad [\{f, f_j\}, Q^* \phi] = 0 \quad (\phi \in K^*).$$

Let $U \in H^m(\Omega)$ satisfy $\{B_j U|_\Gamma = f_j \ (j \in J)\}$. Set $u = A^*w + U$, where w is a Riesz function of $f - AU$. Then $u \in L^2(\Omega)$ satisfies (P).

PROOF. Since $K^* \subset H^{2m-1}(\Omega)$, we have

$$[PU, Q^*\phi] = [QU, P^*\phi]_* = 0 \quad (\phi \in K^*),$$

owing to Green's Theorem. Namely, we have

$$[\{AU, B_j U|_\Gamma \ (j \in J)\}, \{\phi, \mathcal{B}_{m-1-j}^*\phi|_\Gamma \ (j \in J)\}] = 0 \quad (\phi \in K^*),$$

which means

$$(AU, \phi) + \sum_{j \in J} \langle f_j, \mathcal{B}_{m-1-j}^*\phi|_\Gamma \rangle = 0 \quad (\phi \in K^*).$$

Set $F = f - AU$, then we have

$$(F, \phi) = (f - AU, \phi) = (f, \phi) + \sum_{j \in J} \langle f_j, \mathcal{B}_{m-1-j}^*\phi|_\Gamma \rangle \quad (\phi \in K^*).$$

Therefore, we have $F \in K^{\perp}$, iff $\{f, f_j\}$ satisfies (R).

Now we apply Theorem 2.1, then there exists $v = A^*w \in L^2(\Omega)$ satisfying

$$Av = F, \quad B_j v|_\Gamma = 0 \quad (j \in J),$$

where w is a Riesz function of F . Hence $u = v + U \in L^2(\Omega)$ is a solution of (P). □

Now we define a subspace τ in $L^2(\Omega)$:

$$\tau = \left\{ u \in L^2(\Omega) \mid Au \in L^2(\Omega), (u, \phi) + \sum_{j \in J^*} \langle B_{m-1-j}u|_\Gamma, B_{m-1-j}\phi|_\Gamma \rangle_{-\mu_j} = 0 \ (\forall \phi \in K) \right\}.$$

We remark that $K \cap \tau = \{0\}$, because $u \in K \cap \tau$ satisfies

$$(u, u) + \sum_{j \in J^*} \langle B_{m-1-j}u|_\Gamma, B_{m-1-j}u|_\Gamma \rangle_{-\mu_j} = 0.$$

THEOREM 2.2. Assume (A-I) and (A-II). Suppose that $f \in K^{\perp}$. Let w be a Riesz function of f . Then $u = A^*w \in \tau$ and u is a solution of (P_0) . Moreover, a solution u of (P_0) is unique in τ .

PROOF. (1) From Theorem 2.1, we have

$$\begin{aligned} Qu &= \{u, -B_{m-1-j}u|_{\Gamma} \ (j \in J^*)\} \\ &= \{A^*w, \Lambda^{2\mu_j} \mathcal{B}_j^* w|_{\Gamma} \ (j \in J^*)\}, \end{aligned}$$

that is,

$$\begin{aligned} P^*w &= \{A^*w, \mathcal{B}_j^* w|_{\Gamma} \ (j \in J^*)\} \\ &= \{u, -\Lambda^{-2\mu_j} B_{m-1-j}u|_{\Gamma} \ (j \in J^*)\}. \end{aligned}$$

(2) Since $w, A^*w \in L^2(\Omega)$, we have, owing to Green's Theorem,

$$[P\phi, Q^*w] = [Q\phi, P^*w]_* \quad (\phi \in H^{2m-1}(\Omega)).$$

Since $K \subset H^{2m-1}(\Omega)$, we have

$$[Q\phi, P^*w]_* = 0 \quad (\phi \in K),$$

that is,

$$(\phi, u) + \sum_{j \in J^*} \langle B_{m-1-j}\phi|_{\Gamma}, \Lambda^{-2\mu_j} B_{m-1-j}u|_{\Gamma} \rangle = 0 \quad (\phi \in K),$$

which means $u \in \tau$.

(3) (Uniqueness) Let u_1 and u_2 be solutions of (P_0) , belonging to τ . Then $u = u_1 - u_2 \in K \cap \tau$. Since $K \cap \tau = \{0\}$, we have $u = 0$. \square

Finally, we consider a method to construct a function belonging to $K - \{0\}$.

LEMMA 2.1. *Let $U \in H^m(\Omega)$ satisfy $\{B_j U|_{\Gamma} = 0 \ (j \in J)\}$ and $U \notin \tau$. Set $\phi = A^*w + U$, where w is a Riesz function of $-AU$. Then $\phi \in K - \{0\}$.*

PROOF. From Theorem 2.1, we have $\phi = A^*w + U \in K$. On the other hand, from Theorem 2.2, we have $A^*w \in \tau$. Then we have $\phi \notin \tau$. In fact, if we suppose $\phi \in \tau$, then $U = \phi - A^*w \in \tau$, which contradicts to $U \notin \tau$. The fact $\phi \neq 0$ follows from $\phi \notin \tau$. \square

LEMMA 2.2. *Assume that there exists $\phi_0 \in K - \{0\}$ such that $\phi_0(x) > 0$ ($x \in \Omega$). Let U be a non-negative function satisfying*

$$U \in H^m(\Omega) - \{0\} \quad \text{and} \quad \text{supp}[U] \Subset \Omega,$$

then $U \notin \tau$.

PROOF. We have $U \notin \tau$, because

$$(U, \phi_0) + \sum_{j \in J^*} \langle B_{m-1-j} U|_{\Gamma}, \Lambda^{-2\mu_j} B_{m-1-j} \phi_0|_{\Gamma} \rangle = (U, \phi_0) > 0. \quad \square$$

§3. Numerical Approximation

Let us say that $\{v_k (k = 1, 2, \dots)\}$ is a *basis* of \mathcal{H} , if any finite subset of $\{v_k (k = 1, 2, \dots)\}$ is linearly independent and the space spanned by $\{v_k (k = 1, 2, \dots)\}$ is dense in \mathcal{H} .

The solution u , obtained in §2, can be approximated by the method proposed in [1], that is,

THEOREM 3.1. *Assume (A-I) and (A-II). Let $u = A^*w$, where w is a Riesz function of given $f \in K^{*\perp}$. Let $\{v_k (k = 1, 2, \dots)\}$ be a basis of \mathcal{H} . Set*

$$u_N = ((f, v_1), \dots, (f, v_N)) \Gamma_N^{-1} \begin{pmatrix} A^*v_1 \\ \vdots \\ A^*v_N \end{pmatrix},$$

where

$$\Gamma_N = \begin{pmatrix} [v_1, v_1] & \cdots & [v_1, v_N] \\ \vdots & & \vdots \\ [v_N, v_1] & \cdots & [v_N, v_N] \end{pmatrix}.$$

Then

$$u_N \rightarrow u \quad (N \rightarrow \infty) \quad \text{in } L^2(\Omega).$$

PROOF. (1) (Theory of Fourier Series in \mathcal{H}) Let $\{v_1^\wedge, v_2^\wedge, \dots\}$ be Schmidt's orthonormalization of $\{v_1, v_2, \dots\}$ in \mathcal{H} . For $w \in \mathcal{H}$, we have

$$w_N = \sum_{1 \leq k \leq N} [w, v_k^\wedge] v_k^\wedge \rightarrow w \quad \text{in } \mathcal{H},$$

that is,

$$w_N = ([w, v_1], \dots, [w, v_N]) \Gamma_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \rightarrow w \quad \text{in } \mathcal{H},$$

where

$$\Gamma_N = ((v_j, v_k))_{j,k=1,\dots,N}.$$

Moreover, since $w_N \rightarrow w$ in \mathcal{H} , we have

$$u_N = A^*w_N \rightarrow A^*w = u \quad \text{in } L^2(\Omega).$$

(2) Especially, since $w \in \mathcal{H}$ is a Riesz function of f , that is,

$$[w, v] = (f, v) \quad (v \in \mathcal{H}),$$

we have

$$[w, v_k] = (f, v_k) \quad (k = 1, 2, \dots).$$

Hence we have

$$w_N = ((f, v_1), \dots, (f, v_N)) \Gamma_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix},$$

$$u_N = A^*w_N = ((f, v_1), \dots, (f, v_N)) \Gamma_N^{-1} \begin{pmatrix} A^*v_1 \\ \vdots \\ A^*v_N \end{pmatrix}. \quad \square$$

Since the boundary of Ω is smooth, we have

LEMMA 3.1. *Let $\text{diam}(\Omega) < a\pi$ ($a > 0$). Then*

$$\{\exp(ia^{-1}\alpha \cdot x) \mid \alpha \in \mathbf{Z}^n\}$$

is a basis of $H^q(\Omega)$.

As is shown easily, we have

LEMMA 3.2. *Let $\{v_k$ ($k = 1, 2, \dots$)* be a basis of $H^q(\Omega)$. Set

$$v_k = \phi_k + \xi_k \quad (\phi_k \in K^*, \xi_k \in M^*),$$

then the space spanned by $\{\xi_k$ ($k = 1, 2, \dots$) is dense in \mathcal{H} . Therefore, we can obtain a subset $\{\xi'_j$ ($j = 1, 2, \dots$) = $\{\xi_{k_j}$ ($j = 1, 2, \dots$)} such that $\{\xi'_j$ ($j = 1, 2, \dots$)} is a basis of \mathcal{H} .

Let $\{v'_j$ ($j = 1, 2, \dots$) = $\{v_{k_j}$ ($j = 1, 2, \dots$)} be a subset of $\{v_k$ ($k = 1, 2, \dots$)},

corresponding to $\{\xi'_j \ (j = 1, 2, \dots)\} = \{\xi_{k_j} \ (j = 1, 2, \dots)\}$ in Lemma 3.2. Remark that it holds

$$\begin{aligned} ([\xi_k, \xi_s])_{k,s=1,2,\dots,N} &= ([v_k, v_s])_{k,s=1,2,\dots,N}, \\ (A^* \xi_k)_{k=1,2,\dots,N} &= (A^* v_k)_{k=1,2,\dots,N}, \end{aligned}$$

and

$$((f, \xi_k))_{k=1,2,\dots,N} = ((f, v_k))_{k=1,2,\dots,N} \quad (f \in K^{*\perp}).$$

Hence we have

COROLLARY 3.1. *Assume (A-I) and (A-II). Let $u = A^*w$, where w is a Riesz function of $f \in K^{*\perp}$. Let $\{v_k \ (k = 1, 2, \dots)\}$ be a basis of $H^q(\Omega)$. Let $\{v'_k \ (k = 1, 2, \dots)\}$ be a subset of $\{v_k \ (k = 1, 2, \dots)\}$ chosen in the above way. Set*

$$u_N = ((f, v'_1), \dots, (f, v'_N)) \Gamma_N^{-1} \begin{pmatrix} A^*v'_1 \\ \vdots \\ A^*v'_N \end{pmatrix},$$

where

$$\Gamma_N = ([v'_k, v'_s])_{k,s=1,2,\dots,N}.$$

Then

$$u_N \rightarrow u \quad (N \rightarrow \infty) \quad \text{in } L^2(\Omega).$$

Finally, we consider the approximation of $\phi \in K - \{0\}$ in Theorem 2.3.

THEOREM 3.2. *Assume (A-I) and (A-II). Assume that there exists $\phi_0 \in K - \{0\}$ such that $\phi_0(x) > 0 \ (x \in \Omega)$. Let U be a non-negative function satisfying*

$$U \in H^m(\Omega) - \{0\} \quad \text{and} \quad \text{supp}[U] \subseteq \Omega.$$

Let $\{v_k \ (k = 1, 2, \dots)\}$ be a basis of \mathcal{H} . Set

$$\phi_N = U - ((AU, v_1), \dots, (AU, v_N)) \Gamma_N^{-1} \begin{pmatrix} A^*v_1 \\ \vdots \\ A^*v_N \end{pmatrix},$$

where

$$\Gamma_N = ([v_k, v_s])_{k,s=1,2,\dots,N}.$$

Then

$$\phi_N \rightarrow \phi \quad (N \rightarrow \infty) \quad \text{in } L^2(\Omega),$$

and $\phi \in K - \{0\}$.

§4. Examples

EXAMPLE 1. Consider Neumann problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ (d/d\mathbf{n})u = f_1 & \text{on } \Gamma, \end{cases}$$

where $\Omega \subset (-\pi, \pi)^n$. Then $K(=K^*)$ is a space spanned by 1.

LEMMA 4.1. *It holds*

$$(\mathcal{E}_0) \quad \|u\| \leq C\|\Delta u\| \quad (u \in M, (d/d\mathbf{n})u|_{\Gamma} = 0),$$

where

$$M = \{u \in H^2(\Omega) \mid (u, 1) = 0\}.$$

PROOF. Let $\{\phi_k \ (k = 0, 1, \dots)\}$ be a complete set of eigen-functions, corresponding to eigen-values $\{\lambda_k \ (k = 0, 1, \dots)\}$ such that

$$-\Delta\phi_k = \lambda_k\phi_k \quad \text{in } \Omega, \quad (d/d\mathbf{n})\phi_k = 0 \quad \text{on } \Gamma,$$

and $(\phi_j, \phi_k) = \delta_{jk}$, where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$.

Let $u \in M$ satisfy

$$\begin{cases} -\Delta u = f \in L^2(\Omega) & \text{in } \Omega, \\ (d/d\mathbf{n})u = 0 & \text{on } \Gamma. \end{cases}$$

Then, owing to Green's Theorem, we have

$$(f, \phi_k) = \lambda_k(u, \phi_k) \quad (k = 0, 1, \dots).$$

Therefore, we have $f \in K^\perp$,

$$u = \sum_{k \neq 0} (1/\lambda_k)(f, \phi_k)\phi_k,$$

and

$$\|u\|^2 = \sum_{k \neq 0} |\lambda_k|^{-2} |(f, \phi_k)|^2 \leq c^{-2} \|f\|^2,$$

where $c = |\lambda_1|$. □

Let \mathcal{H} be a Hilbert space, defined by the completion of M by the norm []:

$$[v]^2 = \|\Delta v\|^2 + \langle (d/dn)v|_{\Gamma} \rangle_{1/2}^2.$$

Since $\{e^{ik \cdot x} \ (k \in \mathbf{Z}^n)\}$ is a basis of $H^2(\Omega)$,

$$\{e^{ik \cdot x} - |\Omega|^{-1}(e^{ik \cdot x}, 1) \ (k \in \mathbf{Z}^n - \{0\})\}$$

is a basis of \mathcal{H} . From Theorem 2.1, Theorem 2.2 and Corollary 3.1, we have

PROPOSITION 4.1. *Suppose that $f \in L^2(\Omega)$ and $(f, 1) = 0$. Let $w \in \mathcal{H}$ be a Riesz function of f in \mathcal{H} . Set $u = -\Delta w$, then $u \in L^2(\Omega)$ satisfies*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ (d/dn)u = 0 & \text{on } \Gamma \end{cases}$$

and $(u, 1) + \langle u|_{\Gamma}, 1 \rangle_{-1/2} = 0$. Moreover, set

$$f_N = ((f, e^{ik \cdot x}))_{0 < |k| \leq N},$$

$$\begin{aligned} \Gamma_N &= ((e^{ik \cdot x}, e^{is \cdot x}))_{0 < |k| \leq N, 0 < |s| \leq N} \\ &= (|k|^2 |s|^2 (e^{ik \cdot x}, e^{is \cdot x}) + (k \cdot n)(s \cdot n) \langle e^{ik \cdot x}|_{\Gamma}, e^{is \cdot x}|_{\Gamma} \rangle_{1/2})_{0 < |k| \leq N, 0 < |s| \leq N}, \end{aligned}$$

$$V_N = (-\Delta e^{ik \cdot x})_{0 < |k| \leq N} = (|k|^2 e^{ik \cdot x})_{0 < |k| \leq N},$$

and

$$u_N = {}^t f_N \Gamma_N^{-1} V_N,$$

then it holds

$$u_N \rightarrow u \quad (N \rightarrow \infty) \quad \text{in } L^2(\Omega).$$

EXAMPLE 2. Consider Dirichlet problem:

$$\begin{cases} (-\Delta - \lambda_0)u = f & \text{in } \Omega, \\ u = f_0 & \text{on } \Gamma, \end{cases}$$

where $\Omega \subseteq (-\pi, \pi)^n$ and λ_0 is the least eigen-value for the eigen-value problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

Then, $K(= K^*)$ is a space spanned by ϕ_0 , where ϕ_0 is an eigen-function corresponding to the eigen-value λ_0 .

LEMMA 4.2. *It holds*

$$(\mathcal{E}_0) \quad \|u\| \leq C \|(-\Delta - \lambda_0)u\| \quad (u \in M, u|_\Gamma = 0),$$

where $M = \{u \in H^2(\Omega) \mid (u, \phi_0) = 0\}$.

PROOF. Let $\{\phi_k \ (k = 0, 1, \dots)\}$ be a complete set of eigen-functions, corresponding to eigen-values $\{\lambda_k \ (k = 0, 1, \dots)\}$ such that

$$-\Delta \phi_k = \lambda_k \phi_k \quad \text{in } \Omega, \quad \phi_k = 0 \quad \text{on } \Gamma,$$

and $(\phi_j, \phi_k) = \delta_{jk}$, where $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$.

Let $u \in M$ satisfy

$$\begin{cases} (-\Delta - \lambda_0)u = f \in L^2(\Omega) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

Then, owing to Green's Theorem, we have

$$(f, \phi_k) = (\lambda_k - \lambda_0)(u, \phi_k) \quad (k = 0, 1, \dots).$$

Therefore, we have $f \in K^\perp$,

$$u = \sum_{k \neq 0} (\lambda_k - \lambda_0)^{-1} (f, \phi_k) \phi_k,$$

and

$$\|u\|^2 = \sum_{k \neq 0} (\lambda_k - \lambda_0)^{-2} |(f, \phi_k)|^2 \leq c^{-2} \|f\|^2,$$

where $c = |\lambda_1 - \lambda_0|$. □

Let \mathcal{H} be a Hilbert space, defined by the completion of M by the norm []:

$$[v]^2 = \|(-\Delta - \lambda_0)v\|^2 + \langle v|_\Gamma \rangle_{1+1/2}^2.$$

Since $\{e^{ik \cdot x} \ (k \in \mathbf{Z}^n)\}$ is a basis of $H^2(\Omega)$,

$$\{\xi_k(x) = e^{ik \cdot x} - (e^{ik \cdot x}, \phi_0)\phi_0 \ (k \in \mathbf{Z}^n)\}.$$

is a basis of \mathcal{H} , we have from Theorem 2.1, Theorem 2.2 and Corollary 3.1, we have

PROPOSITION 4.2. *Suppose that $f \in L^2(\Omega)$ and $(f, \phi_0) = 0$. Let $w \in \mathcal{H}$ be a Riesz function of f in \mathcal{H} . Set $u = (-\Delta - \lambda_0)w$, then $u \in L^2(\Omega)$ satisfies*

$$\begin{cases} (-\Delta - \lambda_0)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma \end{cases}$$

and $(u, \phi_0) + \langle (d/dn)u|_{\Gamma}, (d/dn)\phi_0|_{\Gamma} \rangle_{-1-1/2} = 0$. Moreover, set

$$f_N = ((f, e^{ik \cdot x}))_{|k| \leq N},$$

$$\begin{aligned} \Gamma_N &= ((e^{ik \cdot x}, e^{is \cdot x}))_{|k| \leq N, |s| \leq N} \\ &= ((|k|^2 - \lambda_0)(|s|^2 - \lambda_0)(e^{ik \cdot x}, e^{is \cdot x}) + \langle e^{ik \cdot x}, e^{is \cdot x} \rangle_{1+1/2})_{|k| \leq N, |s| \leq N}, \end{aligned}$$

$$V_n = ((-\Delta - \lambda_0)e^{ik \cdot x})_{|k| \leq N} = ((|k|^2 - \lambda_0)e^{ik \cdot x})_{|k| \leq N},$$

and

$$u_N = f_N \Gamma_N^{-1} V_N,$$

then it holds

$$u_N \rightarrow u \quad (N \rightarrow \infty) \quad \text{in } L^2(\Omega).$$

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