

A MASCHKE TYPE THEOREM FOR HOPF π -COMODULES

By

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Abstract. Let H be a Hopf π -coalgebra and let A be a right group H -comodule algebra with a total integral ϕ . In this article we will find some sufficient conditions under which an epimorphism of (H, A) -Hopf π -comodule splits if it splits A -linearly. As an application, we obtain a characterization for an (H, A) -Hopf π -comodule to be projective as an A -module, generalizing the one of the Maschke type theorem found in [D2].

1. Introduction

Let H be a Hopf algebra over a commutative ring R and A a right H -comodule algebra. Doi ([D2] under review) gave some sufficient conditions under which an epimorphism of (H, A) -Hopf module splits if it splits A -linearly. As an application in the case when R is a field, he got that an (H, A) -Hopf module is finitely generated projective as an A -module if and only if it is a Hopf module direct summand of $M \otimes A$ for some finite dimensional H -comodule M .

In [Tur] Turaev introduced, for a group π , the notion of a Hopf π -coalgebra, which can induce a π -category, i.e., group-category, and showed that such a category gives rise to a three-dimensional homotopy quantum field theory with target space $K(\pi, 1)$. Virelizier [Vir1] studied some algebraic properties of Hopf π -coalgebras; the results are then applied in [Vir2] to construct Hennings-like (see [KR]) and Kuperberg-like (see [Ku]) invariants of principal π -bundles over link complements and over 3-manifolds.

Now, it is natural to ask whether there exists a Maschke type theorem in the generalized context of Hopf π -coalgebras. This question motivates the present research.

In this paper we will give a positive answer to the above question.

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2. Preliminaries and Basic Definitions

Throughout this paper, R denotes a commutative ring. We will work over R . We always let π be a discrete group with a neutral element 1, and let \otimes denote \otimes_R . If U and V are R -modules, $T_{U,V} : U \otimes V \rightarrow V \otimes U$ will denote the flip map defined by $T_{U,V}(u \otimes v) = v \otimes u$ for any $u \in U$ and $v \in V$.

Similar to [Vir1], A π -coalgebra is a family of R -modules $C = \{C_\alpha\}_{\alpha \in \pi}$ together with a family of R -linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \rightarrow C_\alpha \otimes C_\beta\}_{\alpha,\beta \in \pi}$ (called a comultiplication) and an R -linear map $\varepsilon : C_1 \rightarrow R$ (called a counit), such that Δ is coassociative in the sense that, for any $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}, \quad (2.1)$$

and for all $\alpha, \beta \in \pi$,

$$(id_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha})\Delta_{1,\alpha}. \quad (2.2)$$

REMARK. $(C_1, \Delta_{1,1}, \varepsilon)$ is an ordinary coalgebra in the usual sense of the word (cf. [Sw] or [Mon]).

Following the Sweedler's notation for Hopf π -coalgebras introduced in [Vir1], we have that, for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$,

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)} \in C_\alpha \otimes C_\beta.$$

The coassociativity axiom (2.1) gives that, for any $\alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta\gamma}$,

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)}, \quad (2.3)$$

which is written as $c_{(1,\alpha)} \otimes c_{(2,\beta)} \otimes c_{(3,\gamma)}$. Inductively, we can define $c_{(1,\alpha_1)} \otimes c_{(2,\alpha_2)} \otimes \cdots \otimes c_{(n,\alpha_n)}$ for any $c \in C_{\alpha_1\alpha_2\cdots\alpha_n}$.

A π -grouplike element of a π -coalgebra C is a family of elements $x = (x_\alpha)_{\alpha \in \pi} \in \prod_{\alpha \in \pi} C_\alpha$ such that $\Delta_{\alpha,\beta}(x_{\alpha\beta}) = x_\alpha \otimes x_\beta$ for all $\alpha, \beta \in \pi$ and $\varepsilon(x_1) = 1$ (or equivalently $x_1 \neq 0$). Note that x_1 is then an ordinary grouplike element of the coalgebra C_1 .

Given a π -coalgebra C . A right $\pi - C$ -comodule is a family $M = \{M_\alpha\}_{\alpha \in \pi}$ of R -modules endowed with a family $\rho^M = \{\rho_{\alpha,\beta}^M : M_{\alpha\beta} \rightarrow M_\alpha \otimes C_\beta\}_{\alpha,\beta \in \pi}$ of R -linear maps (the structure maps) such that for any $\alpha, \beta, \gamma \in \pi$,

$$(\rho_{\alpha,\beta}^M \otimes id_{C_\gamma})\rho_{\alpha\beta,\gamma}^M = (id_{M_\alpha} \otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma}^M, \tag{2.4}$$

and for any $\alpha \in \pi$,

$$(id_{M_\alpha} \otimes \varepsilon)\rho_{\alpha,1}^M = id_{M_\alpha}. \tag{2.5}$$

Similarly, we use the Sweedler's notation for coactions, for any $\alpha, \beta \in \pi$ and $m \in M_{\alpha\beta}$,

$$\rho_{\alpha,\beta}^M(m) = m_{(0,\alpha)} \otimes m_{(1,\beta)} \in M_\alpha \otimes C_\beta.$$

Axiom (2.4) gives that, for any $\alpha, \beta, \gamma \in \pi$ and $m \in M_{\alpha\beta\gamma}$,

$$m_{(0,\alpha\beta)(0,\alpha)} \otimes m_{(0,\alpha\beta)(1,\beta)} \otimes m_{(1,\gamma)} = m_{(0,\alpha)} \otimes m_{(1,\beta\gamma)(1,\beta)} \otimes m_{(1,\beta\gamma)(2,\gamma)}, \tag{2.6}$$

which is written as $m_{(0,\alpha)} \otimes m_{(1,\beta)} \otimes m_{(2,\gamma)}$. By iterating the procedure, we define inductively $m_{(0,\alpha_0)} \otimes m_{(1,\alpha_1)} \otimes \dots \otimes m_{(n,\alpha_n)}$ for any $m \in M_{\alpha_0\alpha_1\dots\alpha_n}$.

REMARK. M_1 endowed with the structure map $\rho_{1,1}$ is an ordinary right comodule over the coalgebra C_1 (cf. [Sw] or [Mon]).

A π -comodule map between two right $\pi - C$ -comodules M and N is a family $f = \{f_\alpha : M_\alpha \rightarrow N_\alpha\}_{\alpha \in \pi}$ of right R -module maps such that $\rho_{\alpha,\beta}^N f_{\alpha\beta} = (f_\alpha \otimes id_{C_\beta})\rho_{\alpha,\beta}^M$ for all $\alpha, \beta \in \pi$.

In a similar way, we can define the notion of a left $\pi - C$ -comodule and the concept of a $\pi - (C, C)$ -bicomodule.

The category of right $\pi - C$ -comodules is denoted by $\mathcal{M}^{\pi-C}$, and their morphisms are π -comodule maps. Similarly, we can introduce the categories ${}^{\pi-C}\mathcal{M}$ of left $\pi - C$ -comodules, and the category ${}^{\pi-C}\mathcal{M}^{\pi-C}$ of $\pi - C$ -bicomodules. For $M \in {}^{\pi-C}\mathcal{M}$, we will use the Sweedler's notation, for any $\alpha, \beta \in \pi$ and $m \in M_{\alpha\beta}$,

$${}^M\rho_{\alpha,\beta}(m) = m_{(-1,\alpha)} \otimes m_{(0,\beta)} \in C_\alpha \otimes M_\beta.$$

Similar to [Tur] or [Vir1], a Hopf π -coalgebra is a π -coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)$ together with a family $S = \{S_\alpha : H_\alpha \rightarrow H_{\alpha^{-1}}\}_{\alpha \in \pi}$ of R -linear maps (called an antipode) such that

$$\text{each } H_\alpha \text{ is an } R\text{-algebra with multiplication } m_\alpha \text{ and unit element } 1_\alpha, \tag{2.7}$$

for all $\alpha, \beta \in \pi$, $\Delta_{\alpha, \beta}$ and $\varepsilon : H_1 \rightarrow R$ are algebra maps, (2.8)

for any $\alpha \in \pi$, $m_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1}, \alpha} = \varepsilon 1_\alpha = m_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha, \alpha^{-1}}$. (2.9)

Let $H = (\{H_\alpha, m_\alpha, 1_\alpha\}, \Delta, \varepsilon, S)$ be a Hopf π -coalgebra. Then we have

- (a) $S_{\alpha^{-1}}(hg) = S_{\alpha^{-1}}(g)S_{\alpha^{-1}}(h)$ for any $\alpha \in \pi$ and $h, g \in H_{\alpha^{-1}}$;
- (b) $S_{\alpha^{-1}}(1_{\alpha^{-1}}) = 1_\alpha$ for any $\alpha \in \pi$;
- (c) $\Delta_{\beta, \alpha}S_{\alpha^{-1}\beta^{-1}} = T_{H_\alpha, H_\beta}(S_{\alpha^{-1}} \otimes S_{\beta^{-1}})\Delta_{\alpha^{-1}, \beta^{-1}}$ for any $\alpha, \beta \in \pi$;
- (d) $\varepsilon S_1 = \varepsilon$.

Note that $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is an ordinary Hopf algebra (cf. [Sw] or [Mon]) and that the notion of a Hopf π -coalgebra is not self-dual.

Let H be a Hopf π -coalgebra and $M = \{M_\alpha\}_{\alpha \in \pi}$ a right $\pi - H$ -comodule with structure maps $\rho = \{\rho_{\alpha, \beta}\}_{\alpha, \beta \in \pi}$. The coinvariants of H on M are the elements of the R -module

$$M^{coH} = \left\{ m = (m_\alpha)_{\alpha \in \pi} \in \prod_{\alpha \in \pi} M_\alpha \mid \rho_{\alpha, \beta}(m_{\alpha\beta}) = m_\alpha \otimes 1_\beta \text{ for all } \alpha, \beta \in \pi \right\}.$$

For any $\alpha \in \pi$, let M_α^{coH} be the image of the canonical projection of this set onto M_α . Similar to [Vir1, Example 2.1], $M^{coH} = \{M_\alpha^{coH}\}_{\alpha \in \pi}$ is a right π -subcomodule of M , called a π -subcomodule of coinvariants.

DEFINITION 2.1. Let $H = (\{H_\alpha, m_\alpha, 1_\alpha, \Delta, \varepsilon\})$ be a Hopf π -coalgebra and let $A = \{A_\alpha, m_\alpha, 1_\alpha\}_{\alpha \in \pi}$ be a family of R -algebras. A is called a right $\pi - H$ -comodule algebra if there is a family $\rho^A = \{\rho_{\alpha, \beta}^A : A_{\alpha\beta} \rightarrow A_\alpha \otimes H_\beta\}_{\alpha, \beta \in \pi}$ of R -linear maps such that

$$(A, \rho^A) \text{ is a right } \pi - C\text{-comodule,} \tag{2.10}$$

$$\rho_{\alpha, \beta}^A(ab) = a_{(0, \alpha)}b_{(0, \alpha)} \otimes a_{(1, \beta)}b_{(1, \beta)}, \text{ for all } \alpha, \beta \in \pi \text{ and } a, b \in A_{\alpha\beta}, \tag{2.11}$$

$$\rho_{\alpha\beta}^A(1_{\alpha\beta}) = 1_\alpha \otimes 1_\beta \text{ for any } \alpha, \beta \in \pi. \tag{2.12}$$

In this occasion, we say that (A, ρ^A) is a right $\pi - H$ -comodule algebra. Note that A_1 endowed with the $\rho_{1,1}^{A_1}$ is an ordinary right H_1 -comodule algebra (cf. [D1] and [Mon]).

Similarly, we can define the notions of a left $\pi - H$ -comodule algebra and a $\pi - H$ -bicomodule algebra.

In what follows, let H be a Hopf π -coalgebra. Let A be a right $\pi - H$ -comodule algebra and let

$$C = \left\{ a = (a_\alpha)_{\alpha \in \pi} \in \prod_{\alpha \in \pi} A_\alpha \mid \rho_{\alpha, \beta}(a_{\alpha\beta}) = a_\alpha \otimes 1_\beta \text{ for all } \alpha, \beta \in \pi \right\}.$$

We denote by $\mathcal{M}_A^{\pi-H}$ the category of right (H, A) -Hopf π -comodules; its object is a family $M = \{M_\alpha\}_{\alpha \in \pi}$ of right A_α -modules M_α which is also a right $\pi - H$ -comodule such that

$$\rho_{\alpha, \beta}(ma) = m_{(0, \alpha)}a_{(0, \alpha)} \otimes m_{(1, \beta)}a_{(1, \beta)}, \quad \text{for all } m \in M_{\alpha\beta}, a \in A_{\alpha\beta}. \quad (2.13)$$

Its morphism is a family of A_α -module maps which is also a $\pi - H$ -comodule map.

REMARK. We remark that the category $\mathcal{M}_{A_1}^{H_1}$ is an ordinary relative Hopf module category studied in [D1].

EXAMPLE 2.2. 1) Obviously, A is an object in $\mathcal{M}_A^{\pi-H}$.

2) For every $M \in \mathcal{M}_A^{\pi-H}$, ρ^M induces a family of R -linear maps $\bar{\rho} = \{\bar{\rho}_\alpha : M_\alpha \rightarrow M_1 \otimes H_\alpha\}_\alpha$. Then we can view $M_1 \otimes H = \{(M_1 \otimes H)_\alpha = M_1 \otimes H_\alpha\}_{\alpha \in \pi}$ as a right (H, A) -Hopf module by $(m \otimes x) \cdot a = ma_{(0, 1)} \otimes xa_{(1, \alpha)}$ for any $m \in M_1$, $x \in H_\alpha$, $a \in A_\alpha$ and $\delta_{\alpha, \beta}(m \otimes x) = m \otimes x_{(1, \alpha)} \otimes x_{(2, \beta)}$ for all $m \in M_1$, $x \in H_{\alpha\beta}$, and then $\bar{\rho} : M \rightarrow M_1 \otimes H$ becomes a morphism of $\mathcal{M}_A^{\pi-H}$.

3. A Maschke Type Theorem for $\mathcal{M}_A^{\pi-H}$

In this section we will prove that there exists a Maschke type theorem in the generalized context of Hopf π -comodules.

Now, we have some definitions as follows:

DEFINITION 3.1. A total integral is a family of R -maps $\phi = \{\phi_\alpha\}_{\alpha \in \pi} : H_\alpha \rightarrow A_\alpha$ such that ϕ is a π -comodule map, i.e., $\rho_{\alpha, \beta}^A \phi_{\alpha\beta} = (\phi_\alpha \otimes id_{C_\beta})\Delta_{\alpha, \beta}$ and $\phi_\alpha(1_\alpha) = 1_\alpha$ for any $\alpha, \beta \in \pi$.

DEFINITION 3.2. For $M \in \mathcal{M}_A^{\pi-C}$, a trace map associated with ϕ is a family of maps $tr^M = \{tr_\alpha^M : M_1 \rightarrow M_\alpha\}$ defined by

$$tr_\alpha^M(m) = m_{(0, \alpha)}\phi_\alpha(S_{\alpha^{-1}}(m_{(1, \alpha^{-1})})) \quad (3.1)$$

for any $m \in M_1$.

PROPOSITION 3.3. For any $m \in M_1$, we have $tr^M(m) \in M^{coH}$. Moreover, the condition $\phi(1_\alpha) = 1_\alpha$ for any $\alpha \in \pi$, implies that tr^M is the identity on M_1 .

PROOF. For any $m \in M_1$, we have

$$\begin{aligned}
& \rho_{\alpha, \beta}(tr_{\alpha\beta}^M(m)) \\
& \stackrel{(3.1)}{=} \rho_{\alpha, \beta}(m_{(0, \alpha\beta)}\phi_{\alpha\beta}S_{\beta^{-1}\alpha^{-1}}(m_{(1, \beta^{-1}\alpha^{-1})})) \\
& \stackrel{(2.11)}{=} m_{(0, \alpha\beta)(0, \alpha)}\phi_{\alpha}S_{\alpha^{-1}}(m_{(1, \beta^{-1}\alpha^{-1})(2, \alpha^{-1})}) \otimes m_{(0, \alpha\beta)(1, \beta)}S_{\beta^{-1}}(m_{(1, \beta^{-1}\alpha^{-1})(1, \beta^{-1})}) \\
& = m_{(0, \alpha)}\phi_{\alpha}S_{\alpha^{-1}}(m_{(1, \alpha^{-1})(2, \alpha^{-1})}) \otimes \underline{m_{(1, \alpha^{-1})(1, 1)(1, \beta)}S_{\beta^{-1}}(m_{(1, \alpha^{-1})(1, 1)(2, \beta^{-1})})} \\
& = m_{(0, \alpha)}\phi_{\alpha}S_{\alpha^{-1}}(m_{(1, \alpha^{-1})}) \otimes 1_{\beta} \\
& = tr_{\alpha}^M(m) \otimes 1_{\beta},
\end{aligned}$$

where we have used the properties of the antipode and the π -comodule structure for the second and third equations, respectively. ■

Now, we define $\lambda = \{\lambda_{\alpha}\} : M_1 \otimes H_{\alpha} \rightarrow M_{\alpha}$ by

$$\lambda_{\alpha}(m \otimes x) = m_{(0, \alpha)}\phi_{\alpha}(S_{\alpha^{-1}}(m_{(1, \alpha^{-1})}S_{\alpha}(x)))$$

for any $x \in H_{\alpha}$.

We will denote $\rho_{1, \alpha}^M : M_{\alpha} \rightarrow M_1 \otimes H_{\alpha}$ by ρ_{α} as follows.

LEMMA 3.4. *For any $\alpha \in \pi$, we have $\lambda_{\alpha} \circ \rho_{\alpha} = id_{M_{\alpha}}$.*

PROOF. For any $m \in M_{\alpha}$ and $\alpha \in \pi$, one has

$$\begin{aligned}
\lambda_{\alpha} \circ \rho_{\alpha}(m) &= m_{(0, 1)(0, \alpha)}\phi(S_{\alpha^{-1}}(m_{(0, 1)(1, \alpha^{-1})}S_{\alpha}(m_{(1, \alpha)}))) \\
&= m_{(0, \alpha)}\phi(S_{\alpha^{-1}}(m_{(1, 1)(1, \alpha^{-1})}S_{\alpha}(m_{(1, 1)(2, \alpha)}))) \\
&= m_{(0, \alpha)}\phi(S_{\alpha^{-1}}(\varepsilon(m_{(1, 1)}1_{\alpha^{-1}}))) \\
&= m\phi S_{\alpha^{-1}}(1_{\alpha^{-1}}) = m,
\end{aligned}$$

where one has used the properties of the antipode. ■

Define the center of A as

$$Z(A) = \{Z(A_{\alpha}) = \{x \in A_{\alpha} \mid ax = xa \text{ for all } a \in A_{\alpha}\}\}_{\alpha \in \pi}.$$

We now have the following two lemmas.

LEMMA 3.5. *For any $\alpha \in \pi$, if $\phi_{\alpha}(H_{\alpha}) \subset Z(A_{\alpha})$, then λ_{α} is an A_{α} -module map,*

here $M_1 \otimes A_\alpha$ affords the module structure given by Example 2.2(2). Moreover, for any $\alpha \in \pi$, the following diagram is commutative:

$$\begin{array}{ccc} M_1 \otimes_C A_\alpha & \xrightarrow{\mu_{M_1}} & M_1 \otimes H_\alpha \\ \text{tr}^M \otimes_C \text{id}_{A_\alpha} \downarrow & & \downarrow \lambda_\alpha \\ M^{\text{co}H} \otimes_C A_\alpha & \xrightarrow{\Psi_M} & M_\alpha \end{array}$$

where we define $\mu_{M_1}(m \otimes a) = ma_{(0,1)} \otimes a_{(1,\alpha)}$ and $\Psi_M((m) \otimes a) = m_\alpha a$ for any $m \in M_1$ and $a \in A_\alpha$.

PROOF. For any $\alpha \in \pi$, $a \in A_\alpha$ and $b \in H_\alpha$, we have

$$\begin{aligned} & \lambda_\alpha(ma_{(0,1)} \otimes ba_{(1,\alpha)}) \\ &= (ma_{(0,1)})_{(0,\alpha)} \phi(S_{\alpha^{-1}}((ma_{(0,1)})_{(1,\alpha^{-1})} S_\alpha(ba_{(1,\alpha)}))) \\ &= m_{(0,\alpha)} a_{(0,1)(0,\alpha)} \phi(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})} a_{(0,1)(1,\alpha^{-1})} S_\alpha(ba_{(1,\alpha)}))) \\ &= m_{(0,\alpha)} a_{(0,\alpha)} \phi(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})} a_{(2,1)(1,\alpha^{-1})} S_\alpha(a_{(2,1)(2,\alpha)} S_\alpha(b)))) \\ &= m_{(0,\alpha)} a \phi(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})} S_\alpha(b))) \\ &= m_{(0,\alpha)} \phi(S_{\alpha^{-1}}(m_{(1,\alpha^{-1})} S_\alpha(b))) a \\ &= \lambda_\alpha(m \otimes b) a, \end{aligned}$$

here we used the properties of the antipode in the fourth equation and the condition: $\phi_\alpha(H_\alpha) \subset Z(A_\alpha)$ in the fifth equation. Hence this proves that λ_α is an A_α -module map, i.e., $\lambda_\alpha(ma_{(0,1)} \otimes ba_{(1,\alpha)}) = \lambda_\alpha(m \otimes b) a$.

For the second assertion, we only note that $\Psi_M \circ (\text{tr}^M \otimes_C \text{id}_{A_\alpha})(m \otimes a) = \text{tr}_\alpha^M(m) a$ and $\lambda_\alpha(m \otimes 1_\alpha) = \text{tr}_\alpha^M(m)$ for any $m \in M_1$. So the diagram clearly commutes from this and the formula $\lambda_\alpha(ma_{(0,1)} \otimes a_{(1,\alpha)}) = \lambda_\alpha(m \otimes 1) a$. ■

LEMMA 3.6. For any $\alpha \in \pi$, λ is a morphism in $\mathcal{M}_A^{\pi-H}$ if either of the following two conditions is fulfilled:

- (i) A is faithful as an R -module and $\phi_\alpha(A_\alpha) \subset R$,
- (ii) H is involutory (i.e., $S_{\alpha^{-1}} S_\alpha = \text{id}$ for all $\alpha \in \pi$), $\phi_\alpha(H_\alpha) \subset Z(A_\alpha)$ for all $\alpha \in \pi$ and

$$\phi_\alpha(ab) = \phi_\alpha(ba) \quad \text{for all } a, b \in A_\alpha, \alpha \in \pi.$$

PROOF. By Lemma 3.5, it suffices to verify that λ is a $\pi - H$ -comodule map.

If (i) holds true, then, for all $x \in H_{\alpha\beta}$, $\alpha, \beta \in \pi$ we have $\phi_\alpha(x_{(1,\alpha)})x_{(2,\beta)} = \phi_{\alpha\beta}(x)_{(0,\alpha)}1_{(1,\beta)} := \phi_{\alpha\beta}(x)1_\beta$ since ϕ is a $\pi - H$ -comodule map and $\phi_\alpha(A_\alpha) \subset R \subset A^{coH}$. This implies that $\phi_{\beta^{-1}}(\mathcal{S}_\beta(x_{(2,\beta)}))\mathcal{S}_\alpha(x_{(1,\alpha)}) = \phi_{\beta^{-1}\alpha^{-1}}(\mathcal{S}_{\alpha\beta}(x))_{(0,\beta^{-1})}1_{(1,\alpha^{-1})}$. In what follows, we claim:

$$x_{(1,\alpha)}\phi_{\beta^{-1}}\mathcal{S}_\beta(x_{(2,\beta)})_{(0,\beta^{-1}\alpha^{-1})} = \phi_{\beta^{-1}\alpha^{-1}}(\mathcal{S}_{\alpha\beta}(x))1_\alpha. \quad (3.2)$$

In fact, we have

$$\begin{aligned} & x_{(1,\alpha)}\phi_{\beta^{-1}}\mathcal{S}_\beta(x_{(2,\beta)})_{(0,\beta^{-1}\alpha^{-1})} \\ &= x_{(1,\alpha)}1_{(1,\alpha)}\phi_{\beta^{-1}\alpha^{-1}}\mathcal{S}_{\alpha^{-1}\alpha\beta}(x_{(2,\alpha^{-1}\alpha\beta)})_{(0,\beta^{-1}\alpha^{-1})} \\ &= x_{(1,\alpha)}\phi_{\beta^{-1}\alpha^{-1}}\mathcal{S}_{\alpha\beta}(x_{(2,\alpha^{-1}\alpha\beta)(2,\alpha\beta)})\mathcal{S}_{\alpha^{-1}}(x_{(2,\alpha^{-1}\alpha\beta)(1,\alpha^{-1})}) \\ &= x_{(1,1)(1,\alpha)}\mathcal{S}_{\alpha^{-1}}(x_{(1,1)(2,\alpha^{-1})})\phi_{\beta^{-1}\alpha^{-1}}\mathcal{S}_{\alpha\beta}(x_{(2,\alpha\beta)}) \\ &= \phi_{\beta^{-1}\alpha^{-1}}\mathcal{S}_{\alpha\beta}(x)1_\alpha, \end{aligned}$$

as required.

We will next show that λ is a $\pi - H$ -comodule map. For all $m \in M_1$, $x \in H_{\alpha\beta}$, one has

$$\begin{aligned} (\lambda_\alpha \otimes id)\delta_{\alpha,\beta}(m \otimes x) &= \lambda_\alpha(m \otimes x_{(1,\alpha)}) \otimes x_{(2,\beta)} \\ &= m_{(0,\alpha)}\phi_\alpha\mathcal{S}_{\alpha^{-1}}(m_{(1,\alpha^{-1})}\mathcal{S}_\alpha(x_{(1,\alpha)})) \otimes x_{(2,\beta)} \\ &= m_{(0,\alpha)} \otimes \phi_\alpha\mathcal{S}_{\alpha^{-1}}(m_{(1,\alpha^{-1})}\mathcal{S}_\alpha(x_{(1,\alpha)}))x_{(2,\beta)} \\ &= m_{(0,\alpha)} \otimes \phi_{\alpha\beta\beta^{-1}}\mathcal{S}_{\beta\beta^{-1}\alpha^{-1}}(m_{(1,\beta\beta^{-1}\alpha^{-1})}\mathcal{S}_{\alpha\beta\beta^{-1}}(x_{(1,\alpha\beta\beta^{-1})}))1_\beta x_{(2,\beta)} \\ &\stackrel{(3.2)}{=} m_{(0,\alpha)} \otimes (m_{(1,\beta\beta^{-1}\alpha^{-1})}\mathcal{S}_{\alpha\beta\beta^{-1}}(x_{(1,\alpha\beta\beta^{-1})}))_{(1,\beta)} \\ &\quad \phi_{\alpha\beta\beta^{-1}}\mathcal{S}_{\beta\beta^{-1}\alpha^{-1}}(m_{(1,\beta\beta^{-1}\alpha^{-1})}\mathcal{S}_{\alpha\beta\beta^{-1}}(x_{(1,\alpha\beta\beta^{-1})}))_{(2,\beta^{-1}\alpha^{-1})}x_{(2,\beta)} \\ &= m_{(0,\alpha)} \otimes m_{(1,\alpha^{-1})(1,\beta)}\mathcal{S}_{\beta^{-1}}(x_{(1,\alpha)(2,\beta^{-1})})x_{(2,\beta)} \\ &\quad \phi_{\alpha\beta}\mathcal{S}_{\beta^{-1}\alpha^{-1}}(m_{(1,\alpha^{-1})(2,\beta^{-1}\alpha^{-1})}\mathcal{S}_{\alpha\beta}(x_{(1,\alpha)(1,\alpha\beta)})) \\ &= m_{(0,\alpha)} \otimes m_{(1,\alpha^{-1})(1,\beta)}\mathcal{S}_{\beta^{-1}}(x_{(2,1)(1,\beta^{-1})})x_{(2,1)(2,\beta)} \\ &\quad \phi_{\alpha\beta}\mathcal{S}_{\beta^{-1}\alpha^{-1}}(m_{(1,\alpha^{-1})(2,\beta^{-1}\alpha^{-1})}\mathcal{S}_{\alpha\beta}(x_{(1,\alpha\beta)})) \\ &= m_{(0,\alpha)}\phi_{\alpha\beta}\mathcal{S}_{\beta^{-1}\alpha^{-1}}(m_{(1,\alpha^{-1})(2,\beta^{-1}\alpha^{-1})}\mathcal{S}_{\alpha\beta}(x)) \otimes m_{(1,\alpha^{-1})(1,\beta)} \end{aligned}$$

$$\begin{aligned}
 &= m_{(0, \alpha\beta)(0, \alpha)} \phi_\alpha S_{\alpha^{-1}}(m_{(1, \beta^{-1}\alpha^{-1})} S_{\alpha\beta}(x)) \otimes m_{(0, \alpha\beta)(1, \beta)} \\
 &= \rho_{\alpha, \beta}^M \lambda_{\alpha\beta}(m \otimes x),
 \end{aligned}$$

where we used the properties of Δ_H and S_H in the sixth equation; the association of Δ_H in the seventh equation and the association of ρ^M in the ninth equation.

If (ii) holds, then $\lambda_\alpha(m \otimes x) = m_{(0, \alpha)} \phi_\alpha(x S_{\alpha^{-1}}(m_{(1, \alpha^{-1})}))$, and so

$$\begin{aligned}
 \rho_{\alpha, \beta}^M \lambda_{\alpha\beta}(m \otimes x) &= m_{(0, \alpha\beta)(0, \alpha)} \phi_{\alpha\beta}(x S_{\beta^{-1}\alpha^{-1}}(m_{(1, \beta^{-1}\alpha^{-1})}))_{(0, \alpha)} \\
 &\quad \otimes m_{(0, \alpha\beta)(1, \beta)} \phi_{\alpha\beta}(x S_{\beta^{-1}\alpha^{-1}}(m_{(1, \beta^{-1}\alpha^{-1})}))_{(1, \beta)} \\
 &= m_{(0, \alpha\beta)(0, \alpha)} \phi_\alpha(S_{\alpha^{-1}}(m_{(1, \beta^{-1}\alpha^{-1})(2, \alpha^{-1})} x_{(1, \alpha)})) \\
 &\quad \otimes m_{(0, \alpha\beta)(1, \beta)} S_{\beta^{-1}}(m_{(1, \beta^{-1}\alpha^{-1})(1, \beta^{-1})} x_{(2, \beta)}) \\
 &= m_{(0, \alpha)} \phi_\alpha(S_{\alpha^{-1}}(m_{(1, \alpha^{-1})(3, \alpha^{-1})} x_{(1, \alpha)})) \\
 &\quad \otimes m_{(1, \alpha^{-1})(1, \beta)} S_{\beta^{-1}}(m_{(1, \alpha^{-1})(2, \beta^{-1})} x_{(2, \beta)}) \\
 &= m_{(0, \alpha)} \phi_\alpha(S_{\alpha^{-1}}(m_{(1, \alpha^{-1})} x_{(1, \alpha)})) \otimes x_{(2, \beta)} \\
 &= (\lambda_\alpha \otimes id) \delta_{\alpha, \beta}(m \otimes x).
 \end{aligned}$$

Here we have used the properties of Δ_H, S_H , and the π -comodule map ϕ and the condition: $\phi_\alpha(ab) = \phi_\alpha(ba)$ for all $a, b \in A_\alpha, \alpha \in \pi$ in the second equation; the association of ρ^M in the third equation.

This concludes the proof of the lemma 3.6. ■

Let $M = \{M_\alpha\}_{\alpha \in \pi}$ and $N = \{N_\alpha\}_{\alpha \in \pi}$ be a family of $A = \{A_\alpha\}_{\alpha \in \pi}$ -module respectively. Let $f = \{f_\alpha : M_\alpha \rightarrow N_\alpha\}_{\alpha \in \pi} : M \rightarrow N$ be a family of A -module maps. By definition, an epimorphism $f : M \rightarrow N$ of A -modules consists of A_α -module epimorphisms $f_\alpha : M_\alpha \rightarrow N_\alpha$.

We now show the main result of this paper as follows.

THEOREM 3.7. *Let H be a Hopf π -coalgebra. Let A be a right $\pi - H$ -comodule algebra with a total integral ϕ , and suppose that (i) or (ii) in Lemma 3.6 holds. Then an epimorphism $f : M \rightarrow N$ of (H, A) -Hopf π -comodules splits, if the A_1 -linear epimorphism $f_1 : M_1 \rightarrow N_1$ between the neutral components splits.*

PROOF. Let $j = \{j_\alpha : M_\alpha \rightarrow N_\alpha\}_{\alpha \in \pi}$ be a morphism of $\mathcal{M}_A^{\pi-H}$ such that there is an A_1 -module map $p_1 : N_1 \rightarrow M_1$ with $p_1 j_1 = id_{M_1}$. Define an R -linear map

$\tilde{p} = \{\tilde{p}_\alpha : N_\alpha \rightarrow M_\alpha\}_{\alpha \in \pi}$ by $\tilde{p}_\alpha = \lambda_\alpha(p_1 \otimes id)\rho_{1,\alpha}^N$. Then for any $\alpha \in \pi$, $n \in N_\alpha$ and $x \in H_\alpha$, we have

$$\begin{aligned} \tilde{p}(n \cdot x) &\stackrel{(2.13)}{=} \lambda_\alpha(p_1(n_{(0,1)} \cdot x_{(1,1)}) \otimes n_{(1,\alpha)}x_{(2,\alpha)}) \\ &= \lambda_\alpha(p_1(n_{(0,1)}) \cdot x_{(1,1)} \otimes n_{(1,\alpha)}x_{(2,\alpha)}) \\ &= \lambda_\alpha(p_1(n_{(0,1)}) \otimes n_{(1,\alpha)})x \quad (\text{by Lemma 3.5}) \\ &= \tilde{p}(n)x, \end{aligned}$$

and so \tilde{p} is an A_α -module map.

Next we show that \tilde{p} is a $\pi - H$ -comodule map. Indeed, for all $\alpha, \beta \in \pi$, $n \in N_{\alpha,\beta}$, we have

$$\begin{aligned} (\tilde{p} \otimes id)\rho_{\alpha,\beta}^N(n) &= \lambda_\alpha(p_1(n_{(0,\alpha)(0,1)}) \otimes n_{(0,\alpha)(1,\alpha)}) \otimes n_{(1,\beta)} \\ &= \lambda_\alpha(p_1(n_{(0,1)}) \otimes n_{(1,\alpha\beta)(1,\alpha)}) \otimes n_{(1,\alpha\beta)(2,\beta)} \\ &= (\lambda_\alpha \otimes id)\delta_{\alpha,\beta}(p_1(n_{(0,1)}) \otimes n_{(1,\alpha\beta)}) \quad (\text{by Example 2.2}) \\ &= \rho_{\alpha,\beta}^M \lambda_{\alpha\beta}(p_1(n_{(0,1)}) \otimes n_{(1,\alpha\beta)}) \quad (\text{by Lemma 3.6}) \\ &= \rho_{\alpha,\beta}^M \lambda_{\alpha\beta} \tilde{p}(n). \end{aligned}$$

In final, one has $\tilde{p}_\alpha j_\alpha = \lambda_\alpha(p_1 \otimes id)\rho_{1,\alpha}^N j_\alpha = \lambda_\alpha(p_1 \otimes id)(j_1 \otimes id_{H_\alpha})\rho_{1,\alpha}^M = \lambda_\alpha \rho_\alpha = id_{M_\alpha}$, completing the proof. ■

Similar to [D2, Theorem 2], as an application of Theorem 3.7, we have:

THEOREM 3.8. *Let H be a Hopf π -coalgebra. Let A be a right $\pi - H$ -comodule algebra with a total integral ϕ , and suppose that (i) or (ii) in Lemma 3.6 holds. Then an (H, A) -Hopf π -comodule P is projective as an A -module iff P is an (H, A) -Hopf π -comodule direct summand of $W \otimes A$ for some $\pi - H$ -comodule W , where $W \otimes A$ is regarded as an (H, A) -Hopf π -comodule via $(w \otimes a)b = w \otimes ab$ for any $w \in W_\alpha$, $a, b \in A_\alpha$ and $\rho^{W \otimes A} = \{\rho_{\alpha,\beta}^{W \otimes A} : (W \otimes A)_{\alpha,\beta} \rightarrow (W \otimes A)_\alpha \otimes H_\beta\}_{\alpha,\beta \in \pi}$ is given by $\rho_{\alpha,\beta}^{W \otimes A}(w \otimes a) = w_{(0,\alpha)} \otimes a_{(0,\alpha)} \otimes w_{(1,\beta)}a_{(1,\beta)}$ for any $\alpha, \beta \in \pi$, $w \in W_{\alpha\beta}$ and $a \in A_{\alpha\beta}$.*

PROOF. For an (H, A) -Hopf π -comodule W , we first show that $W \otimes A = \{(W \otimes A)_\alpha = W_\alpha \otimes A_\alpha\}_{\alpha \in \pi}$ endowed with the above structures is also an (H, A) -Hopf π -comodule. It is easy to see that each $W_\alpha \otimes A_\alpha$ is an A_α -module and to

see that $(id_{(W \otimes A)_\alpha} \otimes \varepsilon)\rho_{\alpha,1}^{W \otimes A} = id_{(W \otimes A)_\alpha}$ for any $\alpha \in \pi$. Also, for any $\alpha, \beta, \gamma \in \pi$, $w \in W_{\alpha\beta\gamma}$ and $a \in A_{\alpha\beta\gamma}$, we have

$$\begin{aligned} & (\rho_{\alpha,\beta}^{W \otimes A} \otimes id)\rho_{\alpha\beta,\gamma}^{W \otimes A}(w \otimes a) \\ &= w_{(0,\alpha\beta)(0,\alpha)} \otimes a_{(0,\alpha\beta)(0,\alpha)} \otimes w_{(0,\alpha\beta)(1,\beta)} a_{(0,\alpha\beta)(1,\beta)} \otimes w_{(1,\gamma)} a_{(1,\gamma)} \\ &= w_{(0,\alpha)} \otimes a_{(0,\alpha)} \otimes w_{(1,\beta\gamma)(1,\beta)} a_{(1,\beta\gamma)(1,\beta)} \otimes w_{(1,\beta\gamma)(2,\gamma)} a_{(1,\beta\gamma)(2,\gamma)} \\ &= (id_{(W \otimes A)_\alpha} \otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma}^{W \otimes A}, \end{aligned}$$

and this proves that $W \otimes A$ is a π -comodule. The condition of the $\pi - H$ -comodule algebra A will implies Eq. (2.13). Hence $W \otimes A$ is an object of $\mathcal{M}_A^{\pi-H}$.

Finally, by the argument similar to [D2, Theorem 2], it suffices to prove the *only if* part holds true since it is obvious that the *if* part holds based on $W \otimes A$ being a free A -module, i.e. each $W_\alpha \otimes A_\alpha$ is a free A_α -module for any $\alpha \in \pi$. Now for $\alpha \in \pi$, let $L_\alpha \subset P_\alpha$ be a subset that generates P_α as an A_α -module, and let W be the smallest $\pi - H$ -subcomodule of P satisfying $W_\alpha \supset L_\alpha$, i.e., W_α is an R -submodule of P_α such that $\rho_{\alpha,\beta}^P(W_{\alpha\beta}) \subset W_\alpha \otimes H_\beta$ for any $\alpha, \beta \in \pi$. It is known that if L_α is a finite set then W_α is finite dimensional. Thus we build a $\pi - H$ -comodule $W = \{W_\alpha\}_{\alpha \in \pi}$ and so $W \in \mathcal{M}_A^{\pi-H}$. Consider that the module structure map $\phi = \{\phi_\alpha : W_\alpha \otimes A_\alpha \rightarrow P_\alpha\}_{\alpha \in \pi}$, $w \otimes a \mapsto wa$ for any $w \in W_\alpha$, $a \in A_\alpha$, is a surjective morphism of $\mathcal{M}_A^{\pi-H}$. Now since P is A -projective and by Theorem 3.7, we have that P is an (H, A) -Hopf π -comodule direct summand of $W \otimes A$, i.e., each P_α is an A_α -module direct summand of $W \otimes A$ such that P is an (H, A) -Hopf π -comodule.

This concludes the proof of the theorem. ■

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