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HOLOMORPHIC VECTOR BUNDLES ON QUADRIC HYPERSURFACES OF INFINITE-DIMENSIONAL PROJECTIVE SPACES

By

E. BALLICO

Abstract. Here we prove the following result and a few related statements. Let V be a Banach space with countable unconditional basis and the localizing property, $Q \subset \boldsymbol{P}(V)$ a quadric hypersurface with finite-dimensional singular locus and E a holomorphic vector bundle of finite rank on Q. Then $E \cong \bigoplus_{1 \le i \le r} \boldsymbol{O}_Q(a_i)$ for some integers a_i and $h^1(Q, E(t)) = 0$ for every integer t.

1. Introduction

In [L1], Th. 8.5 and Th. 8.1, there is a complete classification of all holomorphic vector bundles of finite rank on $\boldsymbol{P}(V)$ when V is a "good" infinite-dimensional Banach space (e.g. a separable Hilbert space). In this paper we consider holomorphic vector bundles on quadric hypersurfaces of $\boldsymbol{P}(V)$ and prove the following result.

THEOREM 1.1. *Let V be a Banach space with countable unconditional basis and the localizing property and $Q \subset \boldsymbol{P}(V)$ a quadric hypersurface. Assume either Q smooth or that its singular locus is finite-dimensional. Let E be a rank r holomorphic vector bundle on Q. Then there are uniquely determined integers $a_1 \ge \cdots \ge a_r$ such that $E \cong \boldsymbol{O}_Q(a_1) \oplus \cdots \oplus \boldsymbol{O}_Q(a_r)$. Furthermore, $h^1(Q, E(t)) = 0$ for every integer t.*

To avoid any misunderstanding we stress that in this paper every holomorphic vector bundle is assumed to be locally holomorphically trivial in the sense of [L1],

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p. 490. Hence in general our assumptions on holomorphic vector bundles are stronger than the ones in [L1]. The last assertion of Theorem 1.1 was proved (for an arbitrary hypersurface of $\boldsymbol{P}(V)$) in [B2]. The proof of Theorem 1.1 given in section 4 use [L1] and [L2] in an essential way; even paper [B2] which will use several times in the proof of the splitting of E given in section 4 depends heavily from [L1] and [L2].

In section 5 we will also prove in a completely different way the following more elementary result.

Theorem 1.2. *Fix an integer $r \geq 1$. Let V be a Banach space with countable unconditional basis and the localizing property and $X \subset \boldsymbol{P}(V)$ a reducible but reduced quadric hypersurface. Let E be a holomorphic rank r vector bundle on X. Then there is a uniquely determined non-increasing sequence of r integers $a_1 \geq \cdots \geq a_r$ such that $E \cong \bigoplus_{1 \leq i \leq r} \boldsymbol{O}_X(a_i)$. Furthermore, $h^1(X, E(t)) = 0$ for every integer t.*

In section 3 we will consider quadric hypersurfaces of $\boldsymbol{P}(\boldsymbol{C}^{(N)})$ (see Remarks 3.1, 3.2 and 3.3). In section two we collect several results easily obtained from [T] and [S] and which are related to $\boldsymbol{P}(\boldsymbol{C}^{(N)})$.

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2. Finite-Dimensional Results

We will use the following result ([T], Th. 1 at p. 1199, or [S], Main Theorem).

Lemma 2.1. *Fix an integer $r \geq 1$. Let $\boldsymbol{P}^1 \subset \boldsymbol{P}^2 \subset \cdots \subset \boldsymbol{P}^n \subset \boldsymbol{P}^{n+1} \subset \cdots$ be an infinite tower of projective spaces, i.e. for any $n \geq 1$ see $\boldsymbol{P}^n$ as a hyperplane H_n on $\boldsymbol{P}^{n+1}$. Let E_n, $n \geq 1$, be a rank r vector bundle on $\boldsymbol{P}^n$ such that $E_{n+1}|H_n \cong E_n$ for all n. Then there are integers $a_1, \ldots, a_r$ such that $E_n \cong \bigoplus_{1 \leq i \leq r} \boldsymbol{O}_{\boldsymbol{P}^n}(a_i)$ for every n.*

Propositiona 2.2. *Fix an integer $r \geq 1$. Let $\boldsymbol{P}^1 \subset \boldsymbol{P}^2 \subset \cdots \subset \boldsymbol{P}^n \subset \boldsymbol{P}^{n+1} \subset \cdots$ be an infinite tower of projective spaces, i.e. for any $n \geq 1$ see $\boldsymbol{P}^n$ as a hyperplane H_n on $\boldsymbol{P}^{n+1}$. Let $Q_n \subset \boldsymbol{P}^{n+1}$, $n \geq 2$, be a smooth quadric hypersurface such that $Q_n|H_n = Q_{n-1}$ for all $n \geq 3$. Let E_n, $n \geq 1$, be a rank r vector bundle on Q_n such that $E_{n+1}|H_n \cong E_n$ for all n. Then there are integers $a_1, \ldots, a_r$ such that $E_n \cong \bigoplus_{1 \leq i \leq r} \boldsymbol{O}_{Q_n}(a_i)$ for every n.*

Proof. By [T], Lemma 3.2 at p. 1201, the tower of vector bundles E_n,

$n \geq 1$, is level, i.e. for every $n \geq 3$ and any two lines D, R contained in Q_n, the vector bundles $E_n|D$ and $E_n|R$ have the same splitting type; with the terminology of [OSS], §3, and of [B1] each vector bundle E_n is uniform. If $n \geq 2r+3$, then every uniform rank r vector bundle on Q_n is isomorphic to a direct sum of r lines bundles, say $E_n \cong \bigoplus_{1 \leq i \leq r} \boldsymbol{O}_{Q_n}(a_i)$ with $a_1 \geq \cdots \geq a_r$ ([B1], Th. 1). Since $Q_n|H_n = Q_{n-1}$ for all $n \geq 3$, the non-increasing sequence of integer $a_1 \geq \cdots \geq a_r$ is the same for all n.

LEMMA 2.3. *Fix integers r and a such that $r \geq 1$. Let $S \subset \boldsymbol{P}^3$ be an irreducible quadric cone and call P its vertex. Let E be a rank r vector bundle on S such that for every line D with $D \subset S$ we have $E|D \cong \boldsymbol{O}_D(a)^{\oplus r}$. Then $E \cong \boldsymbol{O}_S(a)^{\oplus r}$.*

PROOF. Let $u : A \to S$ be the blowing-up of S at P. Set $h := u^{-1}(P)$ and let f be the strict transform in A of any line $D \subset S$. Thus A is smooth rational surface isomorphic to the Hirzebruch surface F_2 and h is smooth and rational. There is a ruling $\pi : A \to \boldsymbol{P}^1$ and we may take as f any fiber of the ruling π. We have $\mathrm{Pic}(A) \cong \boldsymbol{Z}^{\oplus 2}$ and we may take h and f as a basis of $\mathrm{Pic}(A)$. We have $h^2 = -2$, $h \cdot f = 1$ and $f^2 = 0$. We have $\boldsymbol{O}_A(h+2f) = u^*(\boldsymbol{O}_S(1))$. Set $F := u^*(E)$. The condition $E|D \cong \boldsymbol{O}_D(a)^{\oplus r}$ is equivalent to say that for every fiber T of π the vector bundle $F|T$ is the direct sum of r line bundles of degree a. In particular the splitting type of the restriction of F is the same for all fibers of π, i.e. F is a π-uniform bundle of a π-uniform bundle in the sense of Ishimura ([I]). Since $F = u^*(E)$ and E is locally trivial around P, there is an open neighborhood U of h in A such that $F|U$ is trivial. Since $F|U$ is trivial and $u_*(\boldsymbol{O}_A) = \boldsymbol{O}_S$, we have $E \cong u_*(F)$. If $r = 1$ the triviality of $F|h$ implies the existence of an integer b such that $F \cong \boldsymbol{O}_A(bh + 2bf)$. Thus $E = u_*(F) \cong \boldsymbol{O}_S(b)$. Since $E|D$ has degree a, we have $b = a$, proving the case $r = 1$. Fix a smooth curve $C \in |h + 2f|$. Thus $C \cong \boldsymbol{P}^1$, $C \cap h = \varnothing$, C is a section of π, u is an isomorphism in a neighborhood of C and $u(C)$ is a smooth conic contained is S. Now assumer $r \geq 2$. For every fiber T of π the vector bundle $F(-aC)\,|\,T$ is trivial. Thus $h^0(T, F(-aC)\,|\,T) = r$ and $h^1(T, F(-ah)\,|\,T) = 0$. Thus $\pi_*(F(-aC))$ is a rank r vector bundle on $\boldsymbol{P}^1$ and the natural map $\pi^*(\pi_*(F(-aC))) \to F(-aC)$ is an isomorphism ([OSS], Base-change theorem at p. 11). Since $\pi^*(B)\,|\,h \cong B$ for any vector bundle B on $\boldsymbol{P}^1$ and $F|h$ is trivial, we obtain $F \cong \boldsymbol{O}_A(ah + 2af)^{\oplus r}$. Thus $E \cong u_*(F) \cong \boldsymbol{O}_S(a)^{\oplus r}$, proving the lemma.

PROPOSITION 2.4. *Fix integers r, b, n with $r \geq 1$, $b \geq -1$ and $n \geq 2r + 5 + b$. Let $S \subset \boldsymbol{P}^n$ be an irreducible quadric hypersurface of rank $n - b$, i.e. such that $\mathrm{Sing}(Q)$ has dimension $b - 1$. Let E be a rank r vector bundle on S. Assume the* *existence of integers* $a_1, \dots, a_r$ *such that* $E|D \cong \bigoplus_{1 \le i \le r} \boldsymbol{O}_D(a_i)$ *for every line* $D \subset S$. *Then* $E \cong \bigoplus_{1 \le i \le r} \boldsymbol{O}_S(a_i)$.

PROOF. We stress that the proofs in [B1] are just adaptations to the quadric case and to some more general subvarieties of $\boldsymbol{P}^n$ of the results proved in [EF] for $\boldsymbol{P}^n$. The case $b = -1$, i.e. the case S smooth, is [B1], Th. 1. Thus we may assume $b \ge 0$, i.e. we may assume that S is a quadric cone and assume that the result true for the integer $b' := b - 1$. Let W be the vertex of S. Thus W is a b-dimensional linear space. Consider the exact sequence

$$0 \to \boldsymbol{I}_S(t) \to \boldsymbol{O}_{\boldsymbol{P}^n}(t) \to \boldsymbol{O}_S(t) \to 0 \tag{1}$$

Since the ideal sheaf $\boldsymbol{I}_S$ of S in $\boldsymbol{P}^n$ is isomorphic to $\boldsymbol{O}_{\boldsymbol{P}^n}(-2)$, from (1) and the known cohomology of line bundles on $\boldsymbol{P}^n$ we obtain $h^1(\boldsymbol{P}^n, \boldsymbol{I}_S(t)) = h^i(S, \boldsymbol{O}_S(t)) = 0$ for all integers i, t with $1 \le i \le n-1$. This is equivalent to say that S is an arithmetically Cohen-Macaulay subvariety of $\boldsymbol{P}^n$. Let H be a general hyperplane of $\boldsymbol{P}^n$. Set $Y := S \cap H$. Hence $Y \subset H$ is an irreducible quadric cone and $W \cap H$ is the vertex of Y. By the inductive assumption on b $E|Y \cong \bigoplus_{1 \le i \le r} \boldsymbol{O}_Y(a_i)$. We order the integers $a_1, \dots, a_r$ so that $a_1 \ge \dots \ge a_r$. If $a_r = a_1$, set $k = r$. If $a_r < a_1$, let k be the first integer such that $1 \le k < r$ and $a_k > a_{k+1}$. Since $\dim(Y) \ge 2$, the first part of the proof gives $h^1(Y, (E|Y)(t)) = 0$ for all integers t. Since the conormal bundle of Y in S is isomorphic to $\boldsymbol{O}_Y(-1)$, from [B1], Prop. 1, we obtain $h^1(S, E(t)) = 0$ for every $t \in \boldsymbol{Z}$. From the cohomology exact sequence associated to the exact sequence

$$0 \to E(t-1) \to E(t) \to E(t) \mid Y \to 0 \tag{2}$$

we obtain $h^0(S, E(-a_1)) = k$ and that the restriction map $H^0(S, E(-a_1)) \to H^0(Y, E(-a_1) \mid Y)$ is bijective. Notice that $H^0(Y, E(-a_1) \mid Y)$ spans a trivial rank k factor of $E(-a_1) \mid Y$. Moving H between all hyperplanes not containing W we obtain that $H^0(S, E(-a_1))$ spans a trivial rank subbundle F of $E(-a_1)$. If $k = r$, this implies $E \cong F(a_1)$, i.e. $E \cong \bigoplus_{1 \le i \le r} \boldsymbol{O}_S(a_i)$, proving the result in this case. Now assume $k < r$. For every line $D \subset S$ we have $E/F(a_1) \mid D \cong \bigoplus_{k+1 \le i \le r} \boldsymbol{O}_D(a_i)$. Hence by induction on the rank r we may assume $E/F(a_1) \cong \bigoplus_{k+1 \le i \le r} \boldsymbol{O}_S(a_i)$. Since $h^1(S, \boldsymbol{O}_S(t)) = 0$ for every t, every extension of $E/F(a_1)$ by $F(a_1)$ splits. Thus $E \cong \bigoplus_{1 \le i \le r} \boldsymbol{O}_S(a_i)$.

3. Countable Vector Space

In this section we consider $\boldsymbol{C}^{(N)}$ equipped with the finite-dimensional topology.

REMARK 3.1. For every integer $n \geq 1$ consider the embedding of $\boldsymbol{C}^n$ into $\boldsymbol{C}^{(N)}$ made sending $(z_1, \ldots, z_n)$ into $(z_1, \ldots, z_n, 0, \ldots)$. In this way we obtain an infinite countable tower of projective spaces $\boldsymbol{P}^0 \subset \boldsymbol{P}^1 \subset \boldsymbol{P}^2 \subset \cdots \subset \boldsymbol{P}^n \subset \boldsymbol{P}^{n+1} \subset \cdots \subset \boldsymbol{P}(\boldsymbol{C}^{(N)})$. Fix an integer $r > 0$. Let E be a rank r holomorphic vector bundle on $\boldsymbol{P}(\boldsymbol{C}^{(N)})$. Set $E_n := E|\boldsymbol{P}^n$. Each E_n is a holomorphic vector bundle on $\boldsymbol{P}^n$ and $E_{n+1}|\boldsymbol{P}_n \cong E_n$ for all n. Conversely, the topology of $\boldsymbol{C}^{(N)}$ is such that given any tower E_n, $n \geq 1$, of holomorphic rank r vector bundles with the condition $E_{n+1}|\boldsymbol{P}_n \cong E_n$ for all n there is a unique (up to isomorphisms) rank r vector bundle E on $\boldsymbol{P}(\boldsymbol{C}^{(N)})$ such that $E_n \cong E|\boldsymbol{P}^n$ for all n. By Lemma 2.1 we have $E \cong \bigoplus_{1 \leq i \leq r} \boldsymbol{O}_{\boldsymbol{P}(\boldsymbol{C}^{(N)})}(a_i)$ for some integers $a_1, \ldots, a_r$.

From now on in this section we fix the tower $\boldsymbol{P}^0 \subset \boldsymbol{P}^1 \subset \boldsymbol{P}^2 \subset \cdots \subset \boldsymbol{P}^n \subset \boldsymbol{P}^{n+1} \subset \cdots \subset \boldsymbol{P}(\boldsymbol{C}^{(N)})$ introduced in Remark 3.1.

REMARK 3.2. Every homogeneous polynomial on $\boldsymbol{C}^{(N)}$ is continuous ([D], Ex. 1.63). Hence to give a degree d hypersurface (even not reduced or not irreducible) X of $\boldsymbol{C}^{(N)}$ is equivalent to give for all integers $n \geq 1$ a degree d hypersurface X_{n-1} of $\boldsymbol{P}^n$. Let E be a rank r holomorphic vector bundle on X. For all $n \geq 0$ set $E_n := E|X_n$. Each E_n is a holomorphic vector bundle on X_n and $E_{n+1}|X_n \cong E_n$ for all n. Conversely, the topology of $\boldsymbol{C}^{(N)}$ is such that given any tower E_n, $n \geq 0$, of holomorphic rank r vector bundles with the condition $E_{n+1}|X_n \cong E_n$ for all n there is a unique (up to isomorphisms) rank r vector bundle E on X such that $E_n \cong E|X_n$ for all n.

REMARK 3.3. By [G], Th. 1 at p. 63, every quadratic form on $\boldsymbol{C}^{(N)}$ may be diagonalized. Hence any quadratic form Q on $\boldsymbol{C}^{(N)}$ is uniquely determined, up to the action of $\mathrm{GL}(\boldsymbol{C}^{(N)})$, by a pair (α, β), where α is either a non-negative integer or the simbol ∞ and β is either a non-negative integer or the symbol ∞ and if $a \neq \infty$, then $b = \infty$: α is the rank of Q and β is its corank, i.e. β is the dimension of the maximal linear subspace A of $\boldsymbol{C}^{(N)}$ such that $Q(x, y) = 0$ for every $x \in A$ and every $y \in \boldsymbol{C}^{(N)}$. Conversely, any such pair (α, β) is associated to a quadratic form on $\boldsymbol{C}^{(N)}$; if α is finite, take a diagonal form $Q = \sum_{1 \leq i \leq \alpha} z_i^2$; if β is finite take $Q = \sum_{i \geq \beta+1} z_i^2$; if $(\alpha, \beta) = (\infty, \infty)$ take $Q = \sum_{i \geq 1} z_{2i}^2$. The quadratic form Q associated to the pair (α, β) is non-degenerate if and only if $\beta = 0$.

LEMMA 3.4. *Let V be an infinite dimensional complex vector space and Q an irreducible quadric hypersurface of $\boldsymbol{P}(V)$. Fix lines A, B on Q. Then there are two chains of projective spaces $A_1 \subset A_2 \subset \cdots \subset Q$ and $B_1 \subset B_2 \subset \cdots Q$ such that*

$dim(A_n) = dim(B_n) = n$ for all n, $A = A_1$, $B = B_1$ and $A_i \cap B_i$ containing a line for $i \gg 0$.

PROOF. Taking instead of V any countable infinite vector subspace of V containing the vector subspace of dimension at most 4 associated to the linear span of $A \cup B$, we reduce to the case $V = \boldsymbol{C}^{(N)}$. Since Q is diagonalizable, it is easy to check that both A and B are contained in an infinite increasing tower of projective spaces. If Q is singular and its singular set $\mathrm{Sing}(Q)$ is at least a line, then taking a join of any two such towers of projective spaces with a line, then we obtain two towers $A_1 \subset A_2 \subset \cdots \subset Q$ and $B_1 \subset B_2 \subset \cdots Q$ such that $\dim(A_n) = \dim(B_n) = n$ for all n, $A = A_1$, $B = B_1$ and $A_i \cap B_i$ containing a line for $i \gg 0$. Call $(\alpha(Q), \beta(Q))$ the invariants associated to Q in Remark 3.2. We have solved all cases except the ones with $0 \leq \beta(Q) \leq 1$. We will do the case $\beta(Q) = 0$ (i.e. Q smooth) leaving the very similar case $\beta(Q) = 1$ to the reader. Up to a linear transformation we may assume that $Q = \{\sum_{i \geq 1} z_i^2\}$. Let n be any positive integer such that both A and B are contained in the projectivization of the linear subspace $W = \boldsymbol{C}^n$ of $\boldsymbol{C}^{(N)}$ given by $z_i = 0$ for all $i > n$. We may change the coordinates of $\boldsymbol{C}^{(N)}$ keeping fixed the ones of W in such a way in the new homogeneous coordinates $z_1, \ldots, z_n, x_i, y_i$, $i \geq 1$, $Q = \{\sum_{1 \leq i \leq n} z_i^2 + \sum_{i \geq 1} x_i y_i = 0\}$. Take as $A_1 \subset A_2 \subset \cdots \subset A_k \subset Q$ (resp. $B_1 \subset B_2 \subset \cdots \subset B_k \subset Q$) any tower obtained from A (resp. B) taking the cone with vertex $x_i = 0$ for $1 \leq i \leq k-1$, $y_j = 0$.

4. Proof of Theorem 1.1

LEMMA 4.1. *Let V be a Banach space with countable unconditional basis and the localizing property and $Q \subset \boldsymbol{P}(V)$ a quadric hypersurface whose singular locus is one point. Let E be a rank r holomorphic vector bundle on Q. Assume the existence of a line $D \subset Q$ such that $E|D$ is trivial. Then E is trivial and $h^1(Q, E(t)) = 0$ for every integer t.*

PROOF. Let P be the singular point of Q. Take a closed hyperplane H of $\boldsymbol{P}(V)$ such that $Q \cap H$ is a smooth quadric hypersurface of H. Let $u : Z \to Q$ be the blowing-up of Q at P, i.e. the closure of $v^{-1}(Q \backslash \{P\})$ in $\mathrm{Bl}_P(\boldsymbol{P}(V))$, where $v : \mathrm{Bl}_P(\boldsymbol{P}(V)) \to \boldsymbol{P}(V)$ is the blowing-up of $\boldsymbol{P}(V)$ at P considered in [L1], §7. Z is a smooth manifold and there is a holomorphic map $\pi : Z \to Q \cap H$ such that $Z \cong \boldsymbol{P}(\boldsymbol{O}_{Q \cap H} \oplus \boldsymbol{O}_{Q \cap H}(-1))$ and π is the associated $\boldsymbol{P}^1$-bundle. The closed set $u^{-1}(P)$ is a smooth manifold isomorphic to $Q \cap H$ and $\pi | u^{-1}(P)$ induces an isomorphism between $u^{-1}(P)$ and $Q \cap H$. Furthermore, $u^{-1}(P)$ is a Cartier divisor of Z. Set $F := u^*(E)$. Thus F is a rank r holomorphic vector bundle on F. Since E is locally trivial, there is an open neighborhood U of $u^{-1}(P)$ such that $F|U \cong \boldsymbol{O}_U^{\oplus r}$.

Claim: The sheaf $\pi_*(F)$ is a locally free sheaf on $Q \cap H$ with $\operatorname{rank}(\pi_*(F)) = r$ and the natural map $\alpha : \pi^*(\pi_*(F)) \to F$ is an isomorphism.

Proof. By [H], Example 5 at p. 38, Th. 1′ at p. 46 and Th. 2 at p. 50, the sheaf $\pi_*(F)$ is pseudo-coherent in the sense of [H]; here we use that $Q \cap H$ is locally paracompact. By Lemmas 2.1 and 3.4 for every line $R \subset Q$ we have $E|R \cong \boldsymbol{O}_R^{\oplus r}$. Hence for every fiber T of π we have $F|T \cong \boldsymbol{O}_T^{\oplus r}$. The morphism π is a locally trivial $\boldsymbol{P}^1$-bundle and in particular it has locally many sections. Fix one such section σ, a point $A \in M \cap H$ and an open neighborhood Ω of A on which σ is defined and such that $\pi^{-1}(\Omega) \cong \Omega \times \boldsymbol{P}^1$. Since F is locally trivial, we may also assume that F is trivial to arbitrary order in the sense of [L1], line 10 of p. 505. Hence we may apply [L1], Prop. 5.7, and obtain that $\pi_*(F)$ is a rank r vector bundle on U; notice that [L1], Prop. 5.7, states that $\pi_*(F)$ is holomorphically locally trivial. Furthermore, by [L1], Prop. 5.7, the map α is fiberwise injective with a subbundle of F as image. Since $\operatorname{rank}(\pi_*(F)) = \operatorname{rank}(F)$, α is an isomorphism, proving the claim.

Since F is trivial in an open neighborhood of $u^{-1}(P)$ and $\pi^*(\pi_*(F))\,|\,u^{-1}(P) \cong \pi_*(F)$, $\pi_*(F)$ is trivial. Hence the bijectivity of α implies the triviality of F. We have $\pi_*(\boldsymbol{O}_Z) = \boldsymbol{O}_Q$ and $\pi_*((\pi^*(A)) \cong A$ for every holomorphic vector bundle A on Q. Thus $E \cong \pi_*(F)$. Since $F \cong \boldsymbol{O}_Z^{\oplus r}$, we obtain $E \cong \boldsymbol{O}_Q^{\oplus r}$, proving the first assertion of the lemma. The last assertion of the lemma follows from the triviality of E and the vanishing theorem proven in [B2], Cor. 2.8, because [B2], Remark 2.11, shows that the linear projection of Q from a general point of $\boldsymbol{P}(V)$ onto a closed hyperplane of $\boldsymbol{P}(V)$ is c-flat in the sense of [B2].

Lemma 4.2. *Let V be a Banach space with countable unconditional basis and the localizing property and $Q \subset \boldsymbol{P}(V)$ a smooth quadric hypersurface. Let E be a rank r holomorphic vector bundle on Q. Assume the existence of a line $D \subset Q$ such that $E|D$ is trivial. Then E is trivial and $h^1(Q, E(t)) = 0$ for every integer t.*

Proof. By Lemmas 2.1 and 3.4 for every line $R \subset Q$ the holomorphic vector bundle $E|R$ is trivial. Fix $P \in Q$ and let $T_PQ \subset \boldsymbol{P}(V)$ the tangent space to Q at P. Let $E|\{P\} \cong \boldsymbol{C}^r$ be the fiber of E at P. Thus T_PQ is a codimension one closed linear projective subspace of $\boldsymbol{P}(V)$. Set $Y := Q \cap T_PQ$. Thus Y is an irreducible quadric hypersurface of T_PQ and P is a singular point of Y. Since Q is smooth, T_PQ is tangent to Q only at P. Thus P is the only singular point of Y. For any line $R \subset Y$ the vector bundle $E|R$ is trivial. Hence by Lemma 4.1 the vector bundle $E|Y$ is trivial. Consider the exact sequence

$$0 \to E(-1) \to E \to E|Y \to 0 \tag{3}$$

We have $H^1(Q, E(-1)) = 0$ by [B2], Cor. 2.8, because [B2], Remark 2.11, shows that the linear projection of Q from a general point of $\boldsymbol{P}(V)$ onto a closed hyperplane of $\boldsymbol{P}(V)$ is c-flat in the sense of [B2]. Since $E|Y \cong \boldsymbol{O}_Y^{\oplus r}$, $h^0(T, E(-1) \mid T) = 0$ for all lines $T \subset Q$, we obtain $h^0(Q, E) = r$ and that the restriction map $H^0(Q, E) \to H^0(Y, E|Y)$ is bijective. In particular we see that the evaluation map $H^0(Q, E) \otimes \boldsymbol{O}_Q \to E|\{P\}$ is bijective. Since P is an arbitrary point of Q, this is true for every point of Q, i.e. $E \cong \boldsymbol{O}_Q^{\oplus r}$. Since E is trivial and Q is a smooth hypersurface, the last assertion was also proved in [K], Th. 8.7, at least if V admits smooth partitions of unity.

LEMMA 4.3. *Let V be a Banach space with countable unconditional basis and the localizing property and $Q \subset \boldsymbol{P}(V)$ a quadric hypersurface whose singular locus is finite-dimensional. Let E be a rank r holomorphic vector bundle on Q. Assume the existence of a line $D \subset Q$ such that $E|D$ is trivial. Then E is trivial and $h^1(Q, E(t)) = 0$ for every integer t.*

PROOF. By [B2], Cor. 2.8 and Remark 2.11, we have $H^1(Q, E(t)) = 0$ for every integer t and in particular $H^1(Q, E(-1)) = 0$. Set $b := \dim(\mathrm{Sing}(Q))$. The case $b = -1$ (i.e. Q smooth), is just Lemma 4.2. The case $b = 0$ is just Lemma 4.2. Hence we may assume $b \geq 1$ and that the result is true for hyperquadrics whose singular locus has dimension $b - 1$. Notice that if $V = W \oplus C^x$ (topological direct sum) for some integer $x > 0$ and some closed linear subspace W of the Banach space V, V has the localizing property if and only if W has the localizing property (see e.g. [K], p. 28). Obviously, the same is true for the property of having a countable unconditional basis. The singular set of Q is a b-dimensional subspace M of $\boldsymbol{P}(V)$. Let $H \subset \boldsymbol{P}(V)$ be a closed linear subspace not containing M and $P \in M \backslash M \cap H$. It is easy to check that $Q \cap H$ is a quadric cone whose vertex is exactly $H \cap M$ and that Q is a cone with vertex P and $Q \cap H$ as a basis. By the inductive assumption on b the bundle $E \mid Q \cap H$ is trivial. Apply the exact sequence (3) with $Q \cap H$ instead of Y and repeat the proof of Lemma 4.1.

PROOF OF THEOREM 1.1. The last assertion is just [B2], Cor. 2.8 and Remark

2.11, because the projection of Q from a general points of $\boldsymbol{P}(V)$ onto a closed hyperplane of $\boldsymbol{P}(V)$ is c-flat in the sense of [B2]. For the same reason for any finite-codimensional closed linear subspace A of $\boldsymbol{P}(V)$ we have $h^1(A,(E|A)(t))=0$ for every integer t. Fix any line $D\subset Q$. There is an integer s such that $1\le s\le r$ and uniquely determined integers $b_1>\cdots>b_s$ and $m_j>0$, $1\le i\le s$, such that $m_1+\cdots+m_s=r$ and $E|D\cong\bigoplus_{1\le j\le s}\boldsymbol{O}_D(b_j)^{\oplus m_j}$ (the Harder-Narasimhan filtration of $E|D$) because $D\cong\boldsymbol{P}^1$, every vector bundle on $\boldsymbol{P}^1$ is a direct sum of line bundles and $\deg(D)=1$. We will show that $E\cong\bigoplus_{1\le j\le s}\boldsymbol{O}_Q(b_j)^{\oplus m_j}$. By Lemma 4.3 this is true (just twisting with $\boldsymbol{O}_Q(-b_1)$) if $s=1$. Hence we may assume $s\ge 2$, i.e. $m_1<r$. Let b be the dimension of the singular locus of Q, with the convention $b=-1$ if and only if Q is smooth.

(a) First assume $b=0$. Hence $\mathrm{Sing}(Q)$ is one point, P. Take a closed hyperplane H of $\boldsymbol{P}(V)$ such that $Q\cap H$ is a smooth quadric hypersurface of H. Let $u:Z\to Q$ be the blowing-up of Q at P, i.e. the closure of $v^{-1}(Q\backslash\{P\})$ in $\mathrm{Bl}_P(\boldsymbol{P}(V))$, where $v:\mathrm{Bl}_P(\boldsymbol{P}(V))\to\boldsymbol{P}(V)$ is the blowing-up of $\boldsymbol{P}(V)$ at P considered in [L1], §7. Z is a smooth manifold and there is a holomorphic map $\pi:Z\to Q\cap H$ such that $Z\cong\boldsymbol{P}(\boldsymbol{O}_{Q\cap H}\oplus O_{Q\cap H}(-1))$ and π is the associated $\boldsymbol{P}^1$-bundle. The closed set $u^{-1}(P)$ is a smooth manifold isomorphic to $Q\cap H$ and $\pi\,|\,u^{-1}(P)$ induces an isomorphism between $u^{-1}(P)$ and $Q\cap H$. Furthermore, $u^{-1}(P)$ is a Cartier divisor of Z. Set $F:=u^*(E)$. Thus F is a rank r holomorphic vector bundle on F. Since E is locally trivial, there is an open neighborhood U of $u^{-1}(P)$ such that $F|U\cong\boldsymbol{O}_U^{\oplus r}$. Twisting E with $\boldsymbol{O}_Q(-b_1)$ we reduce to the case $b_1=0$. The construction in the proof of Lemma 4.1 is the same as the construction given in the proof of Proposition 2.4 and this construction commutes with taking a linear subspace of H. Thus $\pi_*(F)$ is a rank m_1 subbundle of $E\,|\,Q\cap H$, i.e. the quotient sheaf $(E\,|\,Q\cap H)/\pi_*(F)$ is a locally free sheaf with rank $r-m_1$. Fix any finite-dimensional linear subspace $B\subset H$ such that $B\cap H$ is smooth and $\dim(B)>2r$. Let $A\subset\boldsymbol{P}(V)$ be the linear span of B and P. The construction given in the proof of Proposition 2.4 applied to $Q\cap A$ is the same as the blowing-up just given and hence (calling π_A the map in that proof and F_A the corresponding bundle) we have $\pi_*(F)\,|\,B\cong\pi_{A_*}(F_A)$ and $(E\,|\,Q\cap H)/\pi_*(F)\,|\,B\cong(E|B)/\pi_{A_*}(F_A)$. Thus for any line $R\subset B$ we have $\pi_*(F)\,|\,R\cong\boldsymbol{O}_R^{\oplus m_1}$ and $((E\,|\,Q\cap H)/\pi_*(F))\,|\,R\cong\bigoplus_{2\le j\le s}\boldsymbol{O}_D(b_j)^{\oplus m_j}$. Hence the Harder-Narasimhan filtration of $\pi_*(F)$ has $s'=1$ blocks, while the Harder-Narasimhan filtration of $(E\,|\,Q\cap H)/\pi_*(F)$ has $s'=s-1$ blocks. By the inductive assumption on the integer s we have $\pi_*(F)\cong\boldsymbol{O}_{Q\cap H}^{\oplus m_1}$ and $(E\,|\,Q\cap H)/\pi_*(F)\cong\bigoplus_{2\le j\le s}\boldsymbol{O}_{Q\cap H}(b_j)^{\oplus m_j}$. Since $h^1(Q\cap H,\boldsymbol{O}_{Q\cap H}(t))=0$ for every integer t ([B2], Cor. 2.8 and Remark 2.11), we obtain $h^1(Q\cap H,\mathrm{Hom}((E\,|\,Q\cap H)/\pi_*(F)),\pi_*(F))=0$. Hence any extension of $(E\,|\,Q\cap H)/\pi_*(F)$ by $\pi_*(F)$ splits. In particular we have $E\,|\,Q\cap H \cong (E\,|\,Q\cap H)/\pi_*(F)\oplus\pi_*(F)$ and hence $E\,|\,Q\cap H\cong\bigoplus_{1\le j\le s}\boldsymbol{O}_{Q\cap H}(b_j)^{\oplus m_j}$. As in the proof of Lemma 4.2 we obtain $E\cong\bigoplus_{1\le j\le s}\boldsymbol{O}_Q(b_j)^{\oplus m_j}$.

(b) Now assume Q smooth, i.e. $b=-1$. The proof of Lemma 4.2 and the part $b=0$ just proven gives a proof of Theorem 1.1 in this case. Now assume $b>0$. The proof of Lemma 4.3 gives by induction on b the general case, concluding the proof.

5. Proof of Theorem 1.2

Proof of Theorem 1.2. By assumption $X=H\cup M$ with H and M. closed hyperplanes of $\boldsymbol{P}(V)$ and $H\neq M$. Let z (resp. w) be the homogeneous equation of H (resp. M). At each point $P\in H$ (resp. $P\in M$) the germ of z (resp. w) generates the ideal sheaf of H (resp. M) in $\boldsymbol{P}(V)$. At each $P\in H\cap M$ the germs of z and w generate the ideal sheaf of $M\cap H$ in $\boldsymbol{P}(V)$. Thus we have a Mayer-Vietoris exact sequence

$$0\to\boldsymbol{O}_X(t)\to\boldsymbol{O}_H(t)\oplus\boldsymbol{O}_M(t)\to\boldsymbol{O}_{H\cap M}(t)\to 0 \tag{4}$$

Since $H^0(H,\boldsymbol{O}_H(t))$ (resp. $H^0(H\cap M,\boldsymbol{O}_{H\cap M}(t))$) is the set of all degree t continuous homogeneous polynomials on H (resp. $H\cap M$), the restriction map $H^0(H,\boldsymbol{O}_H(t))\to H^0(H\cap M,\boldsymbol{O}_{H\cap M}(t))$ is surjective. Since $h^1(H,\boldsymbol{O}_H(t))=h^1(M,\boldsymbol{O}_M(t))=0$ for every integer t ([L1], Th. 7.3 and 8.2), the exact sequence (4) gives $h^1(X,\boldsymbol{O}_X(t))=0$ for all t. By [L1], Th. 8.5 and Th. 7.1, there are two non-increasing sequences of r integers $a_1\ge\cdots\ge a_r$ and $b_1\ge\cdots\ge b_r$ such that $E|H\cong\bigoplus_{1\le i\le r}\boldsymbol{O}_H(a_i)$ and $E|M\cong\bigoplus_{1\le i\le r}\boldsymbol{O}_H(b_i)$. Since $E\,|\,H\cap M\cong E\,|\,M\cap H$, we have $b_i=a_i$ for every i.

Since E is locally free, by tensoring (4) with E we obtain an exact sequence

$$0\to E\to E|H\oplus E|M\to E\,|\,H\cap M\to 0 \tag{5}$$

If $a_1=a_r$, set $k:=r$. If $a_1>a_r$, let k be the first integer with $1\le k\le r$ and $a_k>a_{k+1}$. By [L1], Th. 8.4, for all integers t, we have $h^1(H,E(t)\,|\,H)=h^1(M,E(t)\,|\,H)=h^1(H\cap M,E(t)\,|\,H\cap M)=0$. Notice that $h^0(H,E(-a_1)\,|\,H)=h^0(M,E(-a_1)\,|\,M)=h^0(H\cap M,E(-a_1)\,|\,H\cap M)=k$. Furthermore, since $E(t)\,|\,H\cong\bigoplus_{1\le i\le r}\boldsymbol{O}_H(a_i+t)$, $E\,|\,H\cap M\cong\bigoplus_{1\le i\le r}\boldsymbol{O}_{H\cap M}(a_i+t)$ and $h^1(H,\boldsymbol{O}_H(z))=0$ for every integer z, the restriction map $H^0(H,E(t)\,|\,H)\to H^0(H\cap M,E(t))$ is surjective. Hence from (6) we obtain $h^1(X,E(t))=0$ for every integer t and $h^0(X,E(-a_1))=k$. The last equality and the definition of the integer k imply that for any line $D\subset X$ the restriction map $H^0(C,E(-a_1))\to$

$H^0(D, E(-a_1) \mid D)$ is an isomorphism. Since any point of X is contained in a line contained in X, we obtain that the natural map $H^0(X, E(-a_1)) \otimes \boldsymbol{O}_X$ is injective and it has as image a rank k trivial subbundle, F, of $E(-a_1)$. If $k = r$ we obtain $E(-a_1) \cong \boldsymbol{O}_X^{\oplus a_1}$, proving the theorem in this case. If $k < r$, we obtain that $E/F(a_1)$ is a rank $r - k$ vector bundle such that its restriction to any line D of X has splitting type $a_{k+1} \geq \cdots \geq a_r$. By induction on the rank we obtain $E/F(a_1) \cong \bigoplus_{k+1 \leq i \leq r} \boldsymbol{O}_X(a_i)$. Since $h^1(X, \boldsymbol{O}_X(t)) = 0$ for every $t \in \boldsymbol{Z}$, every extension of $E/F(a_1)$ by $F(a_1)$ splits. Thus $E \cong F(a_1) \oplus E/F(a_1) \cong \bigoplus_{1 \leq i \leq r} \boldsymbol{O}_X(a_i)$, as wanted. The uniqueness part in the statement of Theorem 1.2 is obvious because for any line $D \subset X$, the non-increasing sequence of r integers $a_1 \geq \cdots \geq a_r$ is the splitting type of $E|D$.

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Dept. of Mathematics, University of Trento
38050 Povo (TN)-Italy
fax: italy +0461881624
e-mail: ballico@science.unitn.it