TSUKUBA J. MATH.
Vol. 28 No. 1 (2004), 193–213

# ON ALMOST PARA-COSYMPLECTIC MANIFOLDS

By

Piotr Dacko

**Abstract.** An almost para-cosymplectic manifold is by definition an odd-dimensional differentiable manifold endowed with an almost paracontact structure with hyperbolic metric for which the structure forms are closed. The local structure of an almost para-cosymplectic manifold is described. We also treat some special subclasses of this class of manifolds: para-cosymplectic, weakly para-cosymplectic and almost para-cosymplectic with para-Kählerian leaves. Necessary and sufficient conditions for an almost para-cosymplectic manifold to be para-cosymplectic are found. Necessary and sufficient conditions for an almost para-cosymplectic manifold with para-Kählerian leaves to be weakly para-cosymplectic are also established. We construct examples of weakly para-cosymplectic manifolds, which are not para-cosymplectic. It is proved that in dimensions $\geq 5$ an almost para-cosymplectic manifold cannot be of non-zero constant sectional curvature. Main curvature identities which are fulfilled by any almost para-cosymplectic manifold are found.

## 1. Preliminaries

Let $M$ be a $(2n+1)$-dimensional differentiable manifold. Suppose that $(\varphi, \xi, \eta, g)$ is an almost paracontact hyperbolic metric structure on $M$. This means that $(\varphi, \xi, \eta, g)$ is a quadruple consisting of a $(1,1)$-tensor field $\varphi$, a vector field $\xi$, a 1-form $\eta$ and a pseudo-Riemannian metric $g$ on $M$ satisfying the following relations

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y).$$

*Key words and phrases.* Almost para-cosymplectic manifold, para-cosymplectic manifold, weakly para-cosymplectic manifold, manifold of constant curvature.
2000 *Mathematics Subject Classification.* 53C15, 53C25.
Received December 9, 2002.
Revised October 6, 2003.

In the above and in the sequel, $X, Y, \ldots$ denote arbitrary smooth vector fields on $M$ if it is not otherwise stated. As consequences of the above, we additionally have

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(X) = g(X, \xi), \quad g(\varphi X, Y) = -g(\varphi Y, X).$$

Thus, $\Phi(X, Y) = g(\varphi X, Y)$ is a 2-form on $M$, which will be said the fundamental form of the structure.

With the above terminology we follow [8]. In the papers [14], [3], [4], [1] the authors called such structures almost para-coHermitian.

The manifold $M$ endowed with the almost paracontact hyperbolic metric structure will be called

(a) para-cosymplectic if the forms $\eta$ and $\Phi$ are parallel with respect to the Levi-Civita connection $\nabla$ of the metric $g$, that is, $\nabla\eta = 0$ and $\nabla\Phi = 0$;

(b) almost para-cosymplectic if the forms $\eta$ and $\Phi$ are closed, that is, $d\eta = 0$ and $d\Phi = 0$.

The above notions of (almost) para-cosymplectic manifolds are paracontact—with a hyperbolic metric—analogue of (almost) cosymplectic manifolds (for almost cosymplectic manifolds see [2], [9]).

Our definition of the para-cosymplecticity differs from that used in the paper [8], in which this notion concerns even-dimensional indefinite almost Hermitian or almost para-Hermitian manifolds with coclosed fundamental forms.

For an almost para-cosymplectic manifold, define the $(1,1)$-tensor field $A$ by

$$AX = -\nabla_X \xi.$$

PROPOSITION 1. *For an almost para-cosymplectic manifold, we have*

$$\mathcal{L}_\xi\eta = 0, \quad \mathcal{L}_\xi\Phi = 0, \quad g(AX, Y) = g(AY, X), \quad A\xi = 0,$$

$$\eta \circ A = 0, \quad (\mathcal{L}_\xi g)(X, Y) = -2g(AX, Y), \quad \nabla_\xi\varphi = 0,$$

$$A\varphi + \varphi A = 0, \quad g(\varphi AX, Y) = g(\varphi AY, X), \quad Tr(\varphi A) = Tr(A) = 0,$$

*where $\mathcal{L}$ indicates the operator of the Lie differentiation.*

PROOF. By $d\eta = 0$, $d\Phi = 0$, $i_\xi(\eta) = 1$ and $i_\xi\Phi(X) = g(\varphi\xi, X) = 0$, we have

$$\mathcal{L}_\xi\eta = di_\xi\eta + i_\xi\, d\eta = 0, \quad \mathcal{L}_\xi\Phi = di_\xi\Phi + i_\xi\, d\Phi = 0.$$

Moreover, by $(\nabla_X\eta)(Y) = g(\nabla_X\xi, Y) = -g(AX, Y)$,

$$0 = 2\, d\eta(X, Y) = -g(AX, Y) + g(AY, X),$$

that is, $A$ is a symmetric operator. Consequently,

$$(\mathscr{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = -2g(AX, Y),$$

$$g(A\xi, Y) = g(AY, \xi) = -g(\nabla_Y \xi, \xi) = -\frac{1}{2}\nabla_Y g(\xi, \xi) = 0,$$

and the last line of equations implies $A\xi = -\nabla_\xi \xi = 0$ and $\eta \circ A = 0$.

For $\mathscr{L}_\xi$, we have the decomposition $\mathscr{L}_\xi = \nabla_\xi + A$, here $A$ indicates the unique extension of the $(1,1)$-tensor field $A$ to a derivation of the tensor algebra (see e.g. [10], p. 30).

From one hand, since $\mathscr{L}_\xi \Phi = 0$, it holds

$$\begin{aligned} 0 = (\mathscr{L}_\xi \Phi)(X, Y) &= (\nabla_\xi \Phi)(X, Y) - \Phi(AX, Y) - \Phi(X, AY) \\ &= g((\nabla_\xi \varphi)X, Y) - g((A\varphi + \varphi A)X, Y). \end{aligned}$$

Thus, we have $\nabla_\xi \varphi = A\varphi + \varphi A$.

On the other hand, since $A\xi = 0$, applying $\nabla_\xi$ to $\varphi^2 X = X - \eta(X)\xi$, we obtain

$$\varphi(\nabla_\xi \varphi)X + (\nabla_\xi \varphi)\varphi X = g(A\xi, X)\xi + \eta(X)A\xi = 0.$$

We note that $A = \varphi(\varphi A) = (A\varphi)\varphi$, by $A\xi = 0$ and $\eta \circ A = 0$. Hence

$$\begin{aligned} 0 &= \varphi(\varphi(\nabla_\xi \varphi) + (\nabla_\xi \varphi)\varphi) \\ &= \varphi(\varphi(A\varphi + \varphi A) + (A\varphi + \varphi A)\varphi) = 2\varphi(A + \varphi A\varphi) = 2(\varphi A + A\varphi). \end{aligned}$$

Consequently $\nabla_\xi \varphi = 0$. Since $\varphi$ is skew-symmetric and $A$ symmetric, then $\varphi A$ is traceless. Note that the trace of $A = \varphi(\varphi A)$ also vanishes, because $\varphi A + A\varphi = 0$ implies the symmetry of $\varphi A$. □

## 2. The Local Structure

In this section, we establish a local equivalence between almost para-cosymplectic structures and certain special families of almost para-Kählerian structures.

By an almost para-Kählerian manifold it is meant a $2n$-dimensional differentiable manifold $\tilde{M}$ endowed with a pair $(\tilde{J}, \tilde{g})$, where $\tilde{J}$ is an almost para-complex structure $(\tilde{J}^2 = \tilde{I})$, $\tilde{g}$ is a pseudo-Riemannian metric such that $\tilde{g}(\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}) = -\tilde{g}(\tilde{X}, \tilde{Y})$ and the fundamental form $\tilde{\Omega}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{J}\tilde{X}, \tilde{Y})$ is closed. An almost para-Kählerian manifold with integrable almost para-complex structure $\tilde{J}$ (equivalently, $\tilde{\nabla}\tilde{J} = 0$) is said to be para-Kählerian. For almost para-Kählerian structures, we refer the survey articles [5], [6].

Let $(\tilde{J}_t, \tilde{g}_t)$, $t \in (a,b)$, $a < b$, be a 1-parameter family of almost para-Kählerian structures on a $2n$-dimensional manifold $\tilde{M}$ such that $\tilde{\Omega}_t = \tilde{\Omega}$ for any $t \in (a,b)$, $\tilde{\Omega}$ being a fixed closed 2-form on $\tilde{M}$. This family enables us to define an almost para-cosymplectic structure on the product $M = (a,b) \times \tilde{M}$. In fact, it is sufficient to assume that $\varphi, \xi, \eta, g$ are given on $M$ by

$$\varphi_{(t,p)} = (\tilde{J}_t)_p, \quad g = dt \otimes dt + \tilde{g}_t, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = dt. \tag{1}$$

The fundamental form $\Phi(X,Y) = g(\varphi X, Y)$ at any $(t,p) \in M$ is given by $\Phi_{(t,p)} = (\tilde{\Omega}_t)_p = \tilde{\Omega}_p$, and therefore it is closed. Especially, if the family $(\tilde{J}_t, \tilde{g}_t)$ collapses to a single almost para-Käler structure $(\tilde{J}, \tilde{g})$, that is, $(\tilde{J}_t, \tilde{g}_t) = (\tilde{J}, \tilde{g})$ for any $t \in (a,b)$, then we say that the almost para-cosymplectic manifold $M$ is the product of the open interval $(a,b)$ and the almost para-Kählerian manifold $\tilde{M}$.

We will show that any almost para-cosymplectic structure can locally be seen as that in formula (1).

In fact, let $(\varphi, \xi, \eta, g)$ be an almost para-cosymplectic structure on $M$ and $p$ a fixed point of $M$. Since $d\eta = 0$ and $\eta(\xi) = 1$, we choose a coordinate neighbourhood $U$ around $p$, which is diffeomorphic to $(-a,a) \times \tilde{U}$, $a > 0$, $\tilde{U} \subset \boldsymbol{R}^{2n}$, with coordinates $(x^0, x^1, \dots, x^{2n})$, $x^0$ being the coordinate on $(-a,a)$, such that

$$\xi = \frac{\partial}{\partial x^0}, \quad \eta = dx^0.$$

Since $g_{0i} = \delta_{0i}$, $g$ can be written as

$$g = dx^0 \otimes dx^0 + \sum_{i,j=1}^{2n} g_{ij}\, dx^i \otimes dx^j.$$

By $\varphi\xi = 0$ and $\eta(\varphi X) = 0$, we find

$$\varphi = \sum_{i,j=1}^{2n} \varphi_i^j\, dx^i \otimes \frac{\partial}{\partial x^j}.$$

Moreover, $\Phi(\xi, \cdot) = 0$ and $\mathscr{L}_\xi \Phi = 0$ yield for the components of $\Phi$

$$\Phi_{0i} = 0, \quad \frac{\partial \Phi_{ij}}{\partial x^0} = 0.$$

Thus the fundamental form $\Phi$ has the shape

$$\Phi = 2 \sum_{1 \le i < j \le 2n} \Phi_{ij}\, dx^i \wedge dx^j$$

and does not depend on $x^0$. For any fixed $x^0 = t$, define an almost para-Kählerian structure $(\tilde{J}_t, \tilde{g}_t)$ on $\tilde{U}$ by putting

$$\tilde{J}_t = \sum_{i,j=1}^{2n} \varphi_i^j(t,\cdot)\, dx^i \otimes \frac{\partial}{\partial x^j}, \quad \tilde{g}_t = \sum_{i,j=1}^{2n} g_{ij}(t,\cdot)\, dx^i \otimes dx^j,$$

with the fundamental form $\tilde{\Omega}_t = \Phi|\tilde{U}$.

We have just proved the following theorem.

THEOREM 1. *Let $M(\varphi, \xi, \eta, g)$ be an almost para-cosymplectic manifold. Then, for any point $p \in M$,*

(a) *there is a neighbourhood $U = (-a, a) \times \tilde{U}$ of $p$, where $\tilde{U}$ is a $2n$-dimensional differentiable manifold and $a > 0$;*
(b) *there exist a 1-parameter family of almost para-Kählerian structures $(\tilde{J}_t, \tilde{g}_t)$, $t \in (-a, a)$, which are defined on $\tilde{U}$ with the fundamental forms $\tilde{\Omega}_t$ not depending on the parameter $t$, $\tilde{\Omega}_t = \tilde{\Omega}$; and*
(c) *on $(-a, a) \times \tilde{U}$, the structure $(\varphi, \xi, \eta, g)$ is given as in formula* (1). □

Families of almost para-Kählerian structures with the same fundamental form can be constructed in many ways. Below, we present some of them.

EXAMPLE 1. Let $(\tilde{J}, \tilde{g})$ be a fixed almost para-Kähler structure on a $2n$-dimensional differentiable manifold $N$ and $\tilde{\Omega}$ its fundamental form. Let $V$ be an open subset of $N$ endowed with a frame of vector fields $(E_1, \ldots, E_{2n})$ such that $\tilde{J}E_\alpha = E_{\alpha+n}$, $\tilde{J}E_{\alpha+n} = E_\alpha$, $\tilde{g}(E_\alpha, E_\beta) = \delta_{\alpha\beta}$, $\tilde{g}(E_{\alpha+n}, E_{\beta+n}) = -\delta_{\alpha\beta}$, $\tilde{g}(E_\alpha, E_{\beta+n}) = 0$ for $\alpha, \beta = 1, \ldots, n$.

Given a family of functions $f_t : V \to \boldsymbol{R}$, $a < t < b$, define $(\tilde{J}_t, \tilde{g}_t)$ by

$$\tilde{J}_t E_\alpha = \exp(f_t) E_{\alpha+n}, \quad \tilde{J}_t E_{\alpha+n} = \exp(-f_t) E_\alpha,$$

$$\tilde{g}_t(E_\alpha, E_\beta) = \exp(f_t)\tilde{g}(E_\alpha, E_\beta), \quad \tilde{g}_t(E_{\alpha+n}, E_{\beta+n}) = \exp(-f_t)\tilde{g}(E_{\alpha+n}, E_{\beta+n})$$

for any $t \in (a, b)$. One checks that $(\tilde{J}_t, \tilde{g}_t)$ are almost para-Kählerian structures with fundamental forms $\tilde{\Omega}_t = \tilde{\Omega}$. □

EXAMPLE 2. Let $(\tilde{J}, \tilde{g})$ be an almost para-Kählerian structure on a $2n$-dimensional differentiable manifold $N$. Let $V$ be an open subset of $N$ and assume that there exist a 1-parameter family of diffeomorphisms $f_t : V \to f_t(V) \subset N$, $t \in (-a, a)$, $a > 0$, such that the fundamental form $\tilde{\Omega}$ of $N$ is invariant with respect to all $f_t$'s, that is, $f_t^*\tilde{\Omega} = \tilde{\Omega}$. [One should note that any point of $N$ has a neighbourhood $V$ with this property.] Define a family of almost para-Hermitian structures $(\tilde{J}_t, \tilde{g}_t)$ on $V$ as follows

$$\tilde{J}_t = f_{t*}^{-1}\tilde{J}f_{t*}, \quad \tilde{g}_t = f_t^*\tilde{g}.$$

It can be checked that $(\tilde{J}_t, \tilde{g}_t)$ are almost para-Kählerian structures on $V$ with fundamental forms $\tilde{\Omega}_t = \tilde{\Omega}$ for any $t \in (-a, a)$.

The very special case of the above construction can be obtained when $X$ is a vector field on $N$ satisfying $\mathscr{L}_X\tilde{\Omega} = 0$. Then, any point of $N$ has a neighbourhood $V \subset N$ and there exists a 1-parameter group of diffeomorphisms $f_t : V \to f_t(V) \subset N$ generated by $X$. By $\mathscr{L}_X\tilde{\Omega} = 0$, any $f_t$ preserves $\tilde{\Omega}$. □

REMARK 1. An almost para-cosymplectic manifold $M$ possesses a canonical foliation $\mathscr{F}$ generated by the $2n$-dimensional, completely integrable and $\varphi$ invariant distribution $\mathscr{D} = \ker \eta$. A leaf $\tilde{M}$ of $\mathscr{F}$ is a submanifold of $M$ of codimension 1. Since $\xi|\tilde{M}$ is a vector field normal to $\tilde{M}$, we may treat $\tilde{M}$ as a pseudo-Riemannian hypersurface. Then $A = -\nabla\xi$ restricted to $\tilde{M}$ is the shape operator $\tilde{A}$ of $\tilde{M}$.

Let $\tilde{J}$ be the $(1,1)$-tensor field defined by $\tilde{J}\tilde{X} = \varphi\tilde{X}$ and $\tilde{g}$ the induced metric on $\tilde{M}$. Then the pair $(\tilde{J}, \tilde{g})$ is an almost para-Hermitian structure on $\tilde{M}$. In fact, it is almost para-Kählerian since its fundamental form is closed, as it is the pull-back of the fundamental form of $M$.

Fix a point of $M$ and choose a neighbourhood $U = (-a, a) \times \tilde{U}$, on which the structure $(\varphi, \xi, \eta, g)$ is given as in (1), where $(\tilde{J}_t, \tilde{g}_t)$ is a suitable family of almost para-Kählerian structures on $\tilde{U}$. Then $\{t\} \times \tilde{U}$ is an open subset of a leaf. Identifying the set $\tilde{U}$ with $\{t\} \times \tilde{U}$, we note that $(\tilde{J}_t, \tilde{g}_t)$ is just the induced almost para-Kählerian structure $(\tilde{J}, \tilde{g})$ on $\{t\} \times \tilde{U}$. Moreover, by the equality $g(AX, Y) = -(1/2)(\mathscr{L}_\xi g)(X, Y)$ (see Proposition 1), for the second fundamental form $h_t$ of $\{t\} \times \tilde{U}$, we have $h_t = -(1/2)(\partial\tilde{g}_s/\partial s)|_{s=t}$. □

## 3. Basic Structure Identities

LEMMA 1. *For an almost paracontact hyperbolic metric manifold $M(\varphi, \xi, \eta, g)$ with its fundamental 2-form $\Phi$ the following equations hold*

$$(\nabla_X\Phi)(Y, Z) = g((\nabla_X\varphi)Y, Z), \tag{2}$$

$$(\nabla_X\Phi)(Z, \varphi Y) + (\nabla_X\Phi)(Y, \varphi Z) = -\eta(Z)g(AX, Y) - \eta(Y)g(AX, Z), \tag{3}$$

$$(\nabla_X\Phi)(\varphi Y, \varphi Z) - (\nabla_X\Phi)(Y, Z) = -\eta(Z)g(AX, \varphi Y) + \eta(Y)g(AX, \varphi Z), \tag{4}$$

*where $A = -\nabla\xi$.*

PROOF. Equality (2) is obvious. Differentiating the identity $\varphi^2 = I - \eta \otimes \xi$ covariantly, we obtain

$$(\nabla_X \varphi)\varphi Y + \varphi(\nabla_X \varphi)Y = g(AX, Y)\xi + \eta(Y)AX \tag{5}$$

Projecting this equality onto $Z$, we find (3).

To prove (4), we find at first

$$(\nabla_X \varphi)\xi = -\varphi \nabla_X \xi = \varphi AX, \tag{6}$$

whence it follows

$$(\nabla_X \Phi)(Y, \xi) = -g((\nabla_X \varphi)\xi, Y) = -g(\varphi AX, Y).$$

Replacing $Z$ by $\varphi Z$ in (3), and applying the last equality, we find (4). □

PROPOSITION 2. *For any almost para-cosymplectic manifold, we have*

$$(\nabla_{\varphi X} \varphi)\varphi Y - (\nabla_X \varphi)Y - \eta(Y)A\varphi X = 0. \tag{7}$$

PROOF. Let us define $(0,3)$-tensor field $B$ as follows

$$\begin{aligned} B(X, Y, Z) &= g((\nabla_{\varphi X} \varphi)\varphi Y - (\nabla_X \varphi)Y - \eta(Y)A\varphi X, Z) \\ &= (\nabla_{\varphi X} \Phi)(\varphi Y, Z) - (\nabla_X \Phi)(Y, Z) - \eta(Y)g(\varphi X, AZ). \end{aligned}$$

Antisymmetrizing $B$ with respect to $X, Y$ we have

$$\begin{aligned} B(X, Y, Z) - B(Y, X, Z) = {} & (\nabla_{\varphi X} \Phi)(\varphi Y, Z) - (\nabla_{\varphi Y} \Phi)(\varphi X, Z) \\ & - (\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(X, Z) \\ & - \eta(Y)g(\varphi X, AZ) + \eta(X)g(\varphi Y, AZ). \end{aligned}$$

Since the metric connection $\nabla$ is torsionless and $d\Phi = 0$,

$$(\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y) = 0.$$

Applying this in the previous formula, we obtain

$$\begin{aligned} B(X, Y, Z) - B(Y, X, Z) = {} & -(\nabla_Z \Phi)(\varphi X, \varphi Y) + (\nabla_Z \Phi)(X, Y) \\ & - \eta(Y)g(\varphi X, AZ) + \eta(X)g(\varphi Y, AZ). \end{aligned}$$

By (4), the right hand side of this equality vanishes identically, so that $B(X, Y, Z) - B(Y, X, Z) = 0$, i.e. $B$ is symmetric with respect to $X, Y$.

Symmetrizing $B$ with respect to $Y, Z$, we find

$$B(X,Y,Z)+B(X,Z,Y)=(\nabla_{\varphi X}\Phi)(\varphi Y,Z)+(\nabla_{\varphi X}\Phi)(\varphi Z,Y)$$
$$-\eta(Y)g(\varphi X,AZ)-\eta(Z)g(\varphi X,AY).$$

This, with the help of (3), implies $B(X,Y,Z)+B(X,Z,Y)=0$, i.e. $B$ is anti-symmetric with respect to $Y, Z$. The tensor $B$ having such symmetries must vanish identically, which implies (7). □

Lemma 2. *For an almost para-cosymplectic manifold, we also have*

$$(\nabla_{\varphi X}\varphi)Y-(\nabla_X\varphi)\varphi Y+\eta(Y)AX=0, \tag{8}$$

$$(\nabla_{\varphi X}\varphi)Y+\varphi(\nabla_X\varphi)Y-g(AX,Y)\xi=0. \tag{9}$$

Proof. Putting $\varphi Y$ instead of $Y$ in (7), we get

$$(\nabla_{\varphi X}\varphi)Y-\eta(Y)(\nabla_{\varphi X}\varphi)\xi-(\nabla_X\varphi)\varphi Y=0.$$

By (6), $(\nabla_{\varphi X}\varphi)\xi=\varphi A\varphi X=-AX$, which applied to the above gives (8). Now, (9) follows from (8) and (5). □

Proposition 3. *For the curvature of an almost para-cosymplectic manifold, we have the following identities*

$$R_{XY}\xi=-(\nabla_X A)Y+(\nabla_Y A)X, \tag{10}$$

$$R_{\varphi X\varphi Y}\xi+R_{XY}\xi+\varphi R_{\varphi XY}\xi+\varphi R_{X\varphi Y}\xi=-\nabla_{N(X,Y)}\xi, \tag{11}$$

*where* $R_{XY}=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ *and* $N$ *is the Nijenhuis torsion tensor of* $\varphi$,

$$N(X,Y)=\varphi^2[X,Y]+[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y].$$

Proof. By $AX=-\nabla_X\xi$, we have

$$(\nabla_X A)Y=-\nabla_X\nabla_Y\xi+\nabla_{\nabla_X Y}\xi.$$

Hence, we get

$$-(\nabla_X A)Y+(\nabla_Y A)X=[\nabla_X,\nabla_Y]\xi-\nabla_{[X,Y]}\xi=R_{XY}\xi,$$

that is, formula (10).

Proposition 1 leads to

$$\nabla_{\varphi X}\xi=-A\varphi X=\varphi AX=-\varphi\nabla_X\xi. \tag{12}$$

Moreover, using (8), we obtain

$$-(\nabla_{\varphi X}\varphi)\nabla_Y\xi + (\nabla_X\varphi)\varphi\nabla_Y\xi = 0.$$

Hence this, together with (12), gives

$$\nabla_{\varphi X}\nabla_{\varphi Y}\xi + \varphi\nabla_{\varphi X}\nabla_Y\xi + \nabla_X\nabla_Y\xi + \varphi\nabla_X\nabla_{\varphi Y}\xi = 0. \tag{13}$$

Now, using (12) and (13), we find

$$\begin{aligned}
-\nabla_{N(X,Y)}\xi &= \nabla_{\varphi X}\nabla_{\varphi Y}\xi + \varphi\nabla_{\varphi X}\nabla_Y\xi + \nabla_X\nabla_Y\xi + \varphi\nabla_X\nabla_{\varphi Y}\xi \\
&\quad - \nabla_{\varphi Y}\nabla_{\varphi X}\xi - \varphi\nabla_{\varphi Y}\nabla_X\xi - \nabla_Y\nabla_X\xi - \varphi\nabla_Y\nabla_{\varphi X}\xi \\
&\quad - \nabla_{[X,Y]}\xi - \nabla_{[\varphi X,\varphi Y]}\xi - \varphi\nabla_{[\varphi X,Y]}\xi - \varphi\nabla_{[X,\varphi Y]}\xi \\
&= R_{\varphi X\varphi Y}\xi + R_{XY}\xi + \varphi R_{\varphi XY}\xi + \varphi R_{X\varphi Y}\xi,
\end{aligned}$$

that is (11). □.

### 4. Para-cosymplectic Manifolds

In this section, we prove various necessary and sufficient conditions for an almost para-cosymplectic manifold to be para-cosymplectic.

At first, we prove the following proposition

PROPOSITION 4. *For the Nijenhuis torsion tensor $N$ of an almost para-cosymplectic manifold, we have the following*

$$N(X,Y) = 2((\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X) = -2\varphi((\nabla_X\varphi)Y - (\nabla_Y\varphi)X), \tag{14}$$

$$N(\varphi X,\varphi Y) = N(X,Y) - 2\eta(X)AY + 2\eta(Y)AX, \tag{15}$$

$$(\nabla_{\varphi Z}\Phi)(X,Y) = -\frac{1}{2}g(Z,N(X,Y)), \tag{16}$$

$$\eta(N(X,Y)) = 0, \quad N(\xi,X) = 2AX. \tag{17}$$

PROOF. Writing the Nijenhuis torsion tensor of $\varphi$ with the help of the Levi-Civita connection, we get

$$N(X,Y) = -\varphi(\nabla_X\varphi)Y + \varphi(\nabla_Y\varphi)X + (\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X.$$

Formula (14) follows from the above in view of (9). Using (14) and (7), we find (15). Moreover, using (14), we compute

$$\begin{aligned}(\nabla_{\varphi Z}\Phi)(X, Y) &= -g((\nabla_X\varphi)Y - (\nabla_Y\varphi)X, \varphi Z) \\ &= g(\varphi((\nabla_X\varphi)Y - (\nabla_Y\varphi)X), Z) = -\frac{1}{2}g(Z, N(X, Y)),\end{aligned}$$

which gives (16). Formulas (17) are immediate consequences of (14) and (15), respectively. □

THEOREM 2. *For an almost para-cosymplectic manifold $M$, the following conditions are equivalent*

(a) *$M$ is para-cosymplectic,*
(b) $N = 0$,
(c) *$\varphi$ is parallel,*
(d) *$M$ is locally a product of an open interval and a para-Kählerian manifold,*
(e) *the leaves $\tilde{M}$ of the canonical foliation $\mathscr{F}$ are totally geodesic and the induced structures $(\tilde{J}, \tilde{g})$ are para-Kählerian.*

PROOF. (a) $\Rightarrow$ (e): Note that $A = 0$ since $AZ = \varphi(\nabla_Z\varphi)\xi$ and $\nabla\varphi = 0$. Therefore, for the shape operator of a leaf $\tilde{M}$ of $\mathscr{F}$, it holds $\tilde{A} = A|\tilde{M} = 0$. Thus, $\tilde{M}$ is totally geodesic and $\tilde{\nabla} = \nabla|\tilde{M}$ by the Gauss equation. Consequently, $\tilde{\nabla}\tilde{J} = 0$, that is, $(\tilde{J}, \tilde{g})$ is para-Kählerian.

(e) $\Rightarrow$ (d): By Theorem 1, choose a neighbourhood $U = (-a, a) \times \tilde{U}$ on which the structure $(\varphi, \xi, \eta, g)$ is given as in (1), where $(\tilde{J}_t, \tilde{g}_t)$ is a family of almost para-Kählerian structures on $\tilde{U}$ with $\tilde{\Omega}_t$ not depending on $t$. Restrict further considerations to the set $U$. As we have already known, $(\tilde{J}_t, \tilde{g}_t)$'s are the induced structures on leaves. By our assumption, they are para-Kählerian. Since the leaves are also totally geodesic, their second fundamental forms $h_t$ vanish identically and consequently $(\partial/\partial t)\tilde{g}_t = -2h_t = 0$. Hence $(\tilde{J}_t, \tilde{g}_t)$ are independent of $t$. Thus, $M$ is locally a product of an open interval and a para-Kählerian manifold.

(d) $\Rightarrow$ (c): It is obvious.

(c) $\Rightarrow$ (b): It follows from (2) and (16).

(b) $\Rightarrow$ (a): Since $N = 0$, (2) and (16) give $\nabla_{\varphi Z}\Phi = 0$ and $(\nabla_{\varphi Z}\varphi) = 0$. But then, by the virtue of Proposition 1 that $\nabla_\xi\varphi = 0$, we get $\nabla\varphi = 0$ which in turn implies $\nabla\xi = 0$. Also $\nabla_\xi\Phi = 0$ by (2), so we have $\nabla\Phi = 0$. On the other hand, since $(\nabla_X\eta)(Y) = g(\nabla_X\xi, Y)$ and $\nabla\xi = 0$ one gets $\nabla\eta = 0$. Thus (a) follows. □

Let us call an almost para-cosymplectic manifold, satisfying the condition

$$[R_{XY},\varphi] = R_{XY}\circ\varphi - \varphi\circ R_{XY} = 0, \tag{18}$$

weakly para-cosymplectic.

It is obvious that a para-cosymplectic manifold is weakly para-cosymplectic. The converse implication does not hold in general. Indeed, the almost para-cosymplectic manifolds given in the below example fulfill (18) and are not para-cosymplectic.

Example 3. Consider the flat pseudo-Riemannian metric $g$ on $\boldsymbol{R}^3$ of signature $(++-)$,

$$g = dx^1\otimes dx^1 + dx^2\otimes dx^2 - dx^3\otimes dx^3.$$

Let $h = 1 - x^1 - x^3$ and define a frame of vector fields $(E_0, E_1, E_2)$ on $\boldsymbol{R}^3$ by

$$\begin{aligned} E_0 &= h\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - h\frac{\partial}{\partial x^3}, \\ E_1 &= \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^3}, \\ E_2 &= \frac{1}{2}(1-h^2)\frac{\partial}{\partial x^1} - h\frac{\partial}{\partial x^2} + \frac{1}{2}(1+h^2)\frac{\partial}{\partial x^3}. \end{aligned}$$

For these vector fields, we have $g(E_0,E_0) = g(E_1,E_2) = g(E_2,E_1) = 1$, otherwise $g(E_i,E_j) = 0$. Let $(\omega^0,\omega^1,\omega^2)$ be the dual frame of 1-forms. Then $g$ can be written in the form

$$g = \omega^0\otimes\omega^0 + \omega^1\otimes\omega^2 + \omega^2\otimes\omega^1.$$

Let us define $\varphi,\xi,\eta$ by

$$\xi = E_0, \quad \eta = \omega^0, \quad \varphi = \omega^1\otimes E_1 - \omega^2\otimes E_2.$$

Then $M_0 = \boldsymbol{R}^3(\varphi,\xi,\eta,g)$ is a 3-dimensional flat almost para-cosymplectic manifold. By the flatness, $M_0$ realizes (18). Since $AE_2 = -\nabla_{E_2}\xi = \partial/\partial x^1 - \partial/\partial x^3 \neq 0$, then $\nabla\varphi \neq 0$. Thus, $M_0$ is weakly para-cosymplectic but not para-cosymplectic.

It is interesting to point out that the vector fields $E_0, E_1, E_2$ form a basis of a 3-dimensional Lie algebra isomorphic to the Lie algebra of the Heisenberg group $\boldsymbol{H}^3$. Explicitly, the Poisson brackets are the following

$$[E_0,E_1] = 0, \quad [E_0,E_2] = E_1, \quad [E_1,E_2] = 0.$$

Moreover, $E_0, E_1, E_2$ are complete. Thus, there is a unique Lie group structure $G$ on $\boldsymbol{R}^3$ with $(0,0,0) \in \boldsymbol{R}^3$ as the identity element, for which $E_0, E_1, E_2$ are left-invariant [15]. Because the group $G$ is connected and simply connected, $G$ is isomorphic to the Heisenberg group $\boldsymbol{H}^3$. By the above construction, the structure $(\varphi, \xi, \eta, g)$ is left-invariant.

Let $G_0$ be a discrete, cocompact subgroup of $G$ and $M_1 = G_0 \backslash G$ be a compact cosets manifold. Via the canonical projection, we obtain a flat non para-cosymplectic, almost para-cosymplectic structure on $M_1$, which will be denoted also by $(\varphi, \xi, \eta, g)$.

Examples of strictly weakly para-cosymplectic manifolds in higher dimensions can be obtained in the following way. Let $M = M_0$ or $M = M_1$ with the suitable structure $(\varphi, \xi, \eta, g)$ defined in the above and $\tilde{M}(\tilde{J}, \tilde{g})$ be an arbitrary para-Kählerian manifold. On the product manifold $M' = M \times \tilde{M}$, define an almost para-cosymplectic structure $(\varphi', \xi', \eta', g')$ as the product structure

$$\varphi' = (\varphi, \tilde{J}), \quad \xi' = (\xi, 0), \quad \eta' = (\eta, 0), \quad g' = (g, \tilde{g}).$$

Then, clearly, $[R'_{XY}, \varphi'] = 0$ and $\nabla' \varphi' \neq 0$. Thus, $M'$ is weakly para-cosymplectic non para-cosymplectic. If $M = M_1$ and $\tilde{M}$ is compact, then $M'$ is compact too. □

## 5. Manifolds with Para-Kählerian Leaves

In this section, we study almost para-cosymplectic manifolds, whose leaves of the canonical foliation are para-Kählerian submanifolds. We will call such manifolds almost para-cosymplectic with para-Kählerian leaves.

THEOREM 3. *An almost para-cosymplectic manifold $M$ has para-Kählerian leaves if and only if any of the following equivalent conditions holds*

$$N(X, Y) = 2\eta(X)AY - 2\eta(Y)AX, \tag{19}$$

$$(\nabla_X \varphi)Y = g(A\varphi X, Y)\xi - \eta(Y)A\varphi X, \tag{20}$$

PROOF. Note that the Nijenhuis tensors $N$ and $\tilde{N}$ of $\varphi$ and the induced para-complex structure $\tilde{J}$ of a leaf $\tilde{M} \in \mathscr{F}$ are related by $N|_{\tilde{M}} = \tilde{N}$. If the induced structures $(\tilde{J}, \tilde{g})$ are para-Kählerian, then $\tilde{N} = 0$, and consequently $N(X, Y) = 0$ for any vector fields $X, Y$ tangent to $\tilde{M}$. Therefore, $N(\varphi X, \varphi Y) = 0$ for any vector fields on $M$, whence (19) follows by (15).

Let us assume (19) for an almost para-cosymplectic manifold. Then, by (16), we have

$$(\nabla_{\varphi Z}\Phi)(X,Y) = -\eta(X)g(AZ,Y) + \eta(Y)g(AZ,X),$$

and hence

$$(\nabla_{\varphi Z}\varphi)X = g(AZ,X)\xi - \eta(X)AZ.$$

If we replace $Z$ by $\varphi Z$ in the last equation and use $\nabla_\xi\varphi = 0$, we get (20).

Now we prove that (20) implies the leaves of the manifold $M$ are para-Kählerian. Let $(\tilde{J},\tilde{g})$ be the induced almost para-Kählerian structure on a leaf $\tilde{M}$ and $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{g}$. By the Gauss equation

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \nabla_{\tilde{X}}\tilde{Y} - g(A\tilde{X},\tilde{Y})\xi$$

and (20), we find

$$(\tilde{\nabla}_{\tilde{X}}\tilde{J})\tilde{Y} = (\nabla_{\tilde{X}}\varphi)\tilde{Y} + g(\varphi A\tilde{X},\tilde{Y})\xi = g(A\varphi\tilde{X},\tilde{Y})\xi + g(\varphi A\tilde{X},\tilde{Y})\xi = 0,$$

hence $(\tilde{J},\tilde{g})$ is para-Kählerian. □

PROPOSITION 5. *For an almost para-cosymplectic manifold with para-Kählerian leaves, we have the following curvature identity*

$$\begin{aligned}(21)\quad R_{ZX}\varphi Y - \varphi R_{ZX}Y &= g(A\varphi Z,Y)AX - g(A\varphi X,Y)AZ + g(AZ,Y)A\varphi X\\ &\quad - g(AX,Y)A\varphi Z - g(R_{ZX}\xi,\varphi Y)\xi - \eta(Y)\varphi R_{ZX}\xi.\end{aligned}$$

PROOF. By $\varphi A = -A\varphi$, (20) and $\eta(AX) = 0$ we find

$$(\nabla_Z(A\varphi))X = -(\nabla_Z\varphi)AX - \varphi(\nabla_Z A)X = -g(A\varphi Z, AX)\xi - \varphi(\nabla_Z A)X.$$

Differentiating covariantly (20) and using the relation above, we obtain

$$\begin{aligned}(\nabla^2_{ZX}\varphi)Y &= -g(A\varphi X,Y)AZ + g(AZ,Y)A\varphi X\\ &\quad + g((\nabla_Z A)X,\varphi Y)\xi + \eta(Y)\varphi(\nabla_Z A)X.\end{aligned}$$

Now, the result follows if we antisymmetrize the last relation with respect to $Z,X$ and use (10). □

THEOREM 4. *Almost para-cosymplectic manifolds of constant non-zero sectional curvature do not exist in dimensions* $\geq 5$.

PROOF. Let $M$ be an almost para-cosymplectic manifold of non-zero constant sectional curvature $\lambda \neq 0$ of dimension $2n+1 \geq 5$. Then

$$R_{\xi X}\xi = \varphi R_{\xi\varphi X}\xi = \lambda\eta(X)\xi - \lambda X.$$

On the other hand, by (10) and Proposition 1, $R_{\xi X}\xi = A^2X - (\nabla_\xi A)X$ and $\varphi R_{\xi\varphi X}\xi = A^2X + (\nabla_\xi A)X$. Hence, $(\nabla_\xi A)X = 0$ and

$$A^2X = -\lambda X + \lambda\eta(X)\xi. \tag{22}$$

Formula (11) implies

$$2\lambda\eta(Y)X - 2\lambda\eta(X)Y = -\nabla_{N(X,Y)}\xi = AN(X,Y).$$

Applying $A$ to the both sides of the above equation and using (22), (17) and $\lambda \neq 0$, we find

$$2\eta(Y)AX - 2\eta(X)AY = -N(X,Y).$$

Then by the virtue of Theorem 3, $M$ has para-Kählerian leaves. By $Tr(A) = Tr(A\varphi) = Tr(Z \mapsto \varphi R_{ZX}\xi) = 0$, the trace of (21) with respect to $Z$ gives

$$(2n-1)\lambda g(X,\varphi Y) = -g(R_{\xi X}\xi, \varphi Y) = \lambda g(X,\varphi Y).$$

This is a contradiction since $n \geq 2$ and $\lambda \neq 0$. □

Lemma 3. *Let $u, v$ be bilinear symmetric forms on a real $s$-dimensional vector space $W$, $s \geq 2$. If* $\operatorname{rank}(u) = \operatorname{rank}(v) = p$, *$u$ and $v$ have a common diagonalizing basis and*

$$u(z,y)u(x,w) - u(x,y)u(z,w) + v(z,y)v(x,w) - v(x,y)v(z,w) = 0$$

*for any $x, y, z, w \in W$, then $p \leq 2$.*

Proof. Choose a basis $(e_i, i = 1, 2, \dots, s)$, so that $u(e_i, e_j) = a_i\delta_{ij}$, $v(e_i, e_j) = b_i\delta_{ij}$ for certain $a_i, b_i$. We may assume that $a_i \neq 0$, $b_i \neq 0$ for $i = 1, \dots, p$, otherwise $a_i = b_i = 0$. Let us suppose that $p \geq 3$. From (3), for $z = y = e_i$, $x = w = e_j$, $1 \leq i \neq j \leq p$, we obtain $a_ia_j + b_ib_j = 0$. Hence $b_i = -a_ia_1/b_1$ for $2 \leq i \leq p$. This applied in the previous equation gives $a_ia_j = 0$, $2 \leq i \neq j \leq p$, which is a contradiction. □

Theorem 5. *Let $M$ be an almost para-cosymplectic manifold with para-Kählerian leaves. Then $M$ is weakly para-cosymplectic if and only if the following two conditions* (I) *and* (II) *are fulfilled*

(I) *the tensor field $A$ is a Codazzi tensor, that is, $(\nabla_X A)Y = (\nabla_Y A)X$;*
(II) *at any point $p \in M$, the operator $A$ has one of the following shape*

(a) $A = 0$,
(b) $AX = \varepsilon g(X, V)V$, *where* $|\varepsilon| = 1$ *and* $V$ *is a non-zero null vector such that* $\varphi V = V$ *or* $\varphi V = -V$,
(c) $AX = \varepsilon_1 g(X, V_1)V_1 + \varepsilon_2 g(X, V_2)V_2$, *where* $V_1, V_2$ *are non-zero orthogonal null vectors such that* $\varphi V_1 = -V_1$, $\varphi V_2 = V_2$ *and* $|\varepsilon_i| = 1$.

Proof. Let $M$ be the weakly para-cosymplectic. Then $\varphi R_{XY}\xi = R_{XY}\varphi\xi - [R_{XY}, \varphi]\xi = 0$, and hence $R_{XY}\xi = 0$. Now (I) follows by (10). Observe, $0 = \varphi R_{\xi\varphi X}\xi = A^2X + (\nabla_\xi A)X = A^2X + (\nabla_X A)\xi = 2A^2X$. Thus $A^2X = 0$.

By $R_{XY}\xi = 0$ the identity (21) simplifies to

$$g(A\varphi Z, Y)AX - g(A\varphi X, Y)AZ + g(AZ, Y)A\varphi X - g(AX, Y)A\varphi Z = 0.$$

Projecting the last relation onto $\varphi W$ we find

$$\begin{aligned}(23)\qquad & g(A\varphi Z, Y)g(A\varphi X, W) - g(A\varphi X, Y)g(A\varphi Z, W)\\ &+ g(AZ, Y)g(AX, W) - g(AX, Y)g(AZ, W) = 0.\end{aligned}$$

Now, let $(E_0, E_\alpha, E_{\alpha+n})$, $\alpha = 1, \ldots, n$, be a basis of the tangent space at a point $p \in M$, such that

$$\begin{aligned}(24)\qquad & E_0 = \xi_p, \quad \varphi E_\alpha = E_\alpha, \quad \varphi E_{\alpha+n} = -E_{\alpha+n},\\ & g(E_0, E_0) = 1, \quad g(E_\alpha, E_{\alpha+n}) = 1.\end{aligned}$$

For $X = \sum_{i=0}^{2n} X^i E_i$, $Y = \sum_{i=0}^{2n} Y^i E_i$, put $g(AX, Y) = \sum_{i,j=0}^{2n} c_{ij} X^i Y^j$. By $A\xi = 0$, $c_{0i} = g(A\xi_p, E_i) = 0$ and by $\varphi A = -A\varphi$, (24)

$$\begin{aligned}c_{\alpha(\beta+n)} &= g(AE_\alpha, E_{\beta+n}) = -g(\varphi AE_\alpha, \varphi E_{\beta+n})\\ &= g(A\varphi E_\alpha, \varphi E_{\beta+n}) = -g(AE_\alpha, E_{\beta+n}) = -c_{\alpha(\beta+n)} = 0.\end{aligned}$$

Hence

$$(25)\qquad g(AX, Y) = \sum_{\alpha,\beta=1}^{n} (c_{\alpha\beta}X^\alpha Y^\beta + c_{(\alpha+n)(\beta+n)}X^{\alpha+n}Y^{\beta+n}).$$

Observe the following

$$\begin{aligned} g(A\varphi E_\alpha, E_\beta) &= g(AE_\alpha, E_\beta) = c_{\alpha\beta},\\ g(A\varphi E_\alpha, E_{\beta+n}) &= g(AE_\alpha, E_{\beta+n}) = c_{\alpha(\beta+n)} = 0,\\ g(A\varphi E_{\alpha+n}, E_{\beta+n}) &= -g(AE_{\alpha+n}, E_{\beta+n}) = -c_{(\alpha+n)(\beta+n)}.\end{aligned}$$

Thus we have

$$g(A\varphi X, Y) = \sum_{\alpha,\beta=1}^{n} (c_{\alpha\beta} X^{\alpha} Y^{\beta} - c_{(\alpha+n)(\beta+n)} X^{\alpha+n} Y^{\beta+n}). \tag{26}$$

The forms $g(A\cdot,\cdot), g(A\varphi\cdot,\cdot)$ have the same rank $r$, common diagonalizing basis (by (25) and (26)) and fulfill (23). That means, they realize the assumptions of Lemma 3, therefore it must hold $r \leq 2$. Note that $r = \mathrm{rank}(c_{\alpha\beta}) + \mathrm{rank}(c_{(\alpha+n)(\beta+n)})$. If $r = 0$, then $A = 0$. Let $r = 1$. We will show the assertion (II)(b). At first, consider the case $\mathrm{rank}(c_{\alpha\beta}) = 1$. Then $c_{(\alpha+n)(\beta+n)} = 0$, and $c_{\alpha\beta} = \varepsilon d_\alpha d_\beta$ for $|\varepsilon| = 1$ and certain $d_\alpha$. Define a 1-form $\omega$ and a vector $V$ by assuming $\omega(X) = \sum_{\alpha=1}^{n} d_\alpha X^\alpha$ and $V = \sum_{\alpha=1}^{n} d_\alpha E_{\alpha+n}$. One checks that $A = \varepsilon\omega \otimes V$, $\varphi V = -V$, $\omega(X) = g(X, V)$ and $\omega(V) = g(V, V) = 0$. Similarly, in the case $\mathrm{rank}(c_{\alpha\beta}) = 0$ and $\mathrm{rank}(c_{(\alpha+n)(\beta+n)}) = 1$, we find a 1-form $\omega$ and a vector $V$ for which $A = \varepsilon\omega \otimes V$, $\varphi V = V$, $\omega(X) = g(X, V)$ and $g(V, V) = 0$. Let now $r = 2$. Suppose $\mathrm{rank}(c_{\alpha\beta}) = 2$. Then $c_{(\alpha+n)(\beta+n)} = 0$, which together with (25) and (26) implies $g(A\varphi X, Y) = g(AX, Y)$. Applying the last relation into (23), we find

$$g(AZ, Y)g(AX, W) - g(AX, Y)g(AZ, W) = 0,$$

which clearly yields $\mathrm{rank}(g(A\cdot,\cdot)) = \mathrm{rank}(c_{\alpha\beta}) \leq 1$, a contradiction. Hence $\mathrm{rank}(c_{\alpha\beta}) < 2$. Similar arguments show that $\mathrm{rank}(c_{(\alpha+n)(\beta+n)}) < 2$. Thus

$$\mathrm{rank}(c_{\alpha\beta}) = \mathrm{rank}(c_{(\alpha+n)(\beta+n)}) = 1,$$

that is, $c_{\alpha\beta} = \varepsilon_1 d_\alpha d_\beta$, $c_{(\alpha+n)(\beta+n)} = \varepsilon_2 h_\alpha h_\beta$, $|\varepsilon_i| = 1$. Define 1-forms $\omega_1, \omega_2$ and vectors $V_1, V_2$ by assuming

$$\omega_1(X) = \sum_{\alpha=1}^{n} d_\alpha X^\alpha, \quad \omega_2(X) = \sum_{\alpha=1}^{n} h_\alpha X^{\alpha+n}, \quad V_1 = \sum_{\alpha=1}^{n} d_\alpha E_{\alpha+n}, \quad V_2 = \sum_{\alpha=1}^{n} h_\alpha E_\alpha.$$

We verify that

$$A = \varepsilon_1\omega_1 \otimes V_1 + \varepsilon_2\omega_2 \otimes V_2, \quad \omega_1(X) = g(X, V_1), \quad \omega_2(X) = g(X, V_2),$$

$$\varphi V_1 = -V_1, \quad \varphi V_2 = V_2, \quad g(V_1, V_1) = g(V_2, V_2) = 0.$$

Finally, $A^2 = 0$ implies $A^2 V_1 = \varepsilon_1\varepsilon_2 g(V_1, V_2)^2 V_1 = 0$, hence $g(V_1, V_2) = 0$. Thus, the assertion (II)(c) holds.

Conversely, by (10) and (I), $R_{XY}\xi = 0$. Moreover, (II) implies (23). Consequently, $R_{ZX}\varphi Y - \varphi R_{ZX} Y = 0$ follows from (21). □

EXAMPLE 4. Let $M = \boldsymbol{R}^5$ with coordinates $(z, u_1, u_2, v_1, v_2)$. Define a $(1,1)$-tensor field $\varphi$, a vector field $\xi$, a 1-form $\eta$ and a metric $g$ as follows

$$\varphi\frac{\partial}{\partial z} = 2u_1\frac{\partial}{\partial v_1} - 2u_2\frac{\partial}{\partial v_2},$$

$$\varphi\frac{\partial}{\partial u_1} = -2u_1\frac{\partial}{\partial z} - \frac{\partial}{\partial u_1} + 4u_1u_2\frac{\partial}{\partial v_2},$$

$$\varphi\frac{\partial}{\partial u_2} = 2u_2\frac{\partial}{\partial z} + \frac{\partial}{\partial u_2} - 4u_1u_2\frac{\partial}{\partial v_1},$$

$$\varphi\frac{\partial}{\partial v_1} = \frac{\partial}{\partial v_1}, \quad \varphi\frac{\partial}{\partial v_2} = -\frac{\partial}{\partial v_2},$$

$$\xi = \frac{\partial}{\partial z} - 2u_1\frac{\partial}{\partial v_1} - 2u_2\frac{\partial}{\partial v_2},$$

$$\eta = dz - 2u_1\, du_1 - 2u_2\, du_2,$$

$$g = dz^2 + 2\, du_1dv_1 + 2\, du_2dv_2.$$

By straightforward computations we verify that $(\varphi, \xi, \eta, g)$ is the almost para-cosymplectic structure on $M$ with the fundamental form $\Phi$ given by

$$\Phi = 4u_1\, dz \wedge du_1 - 4u_2\, dz \wedge du_2 + 8u_1u_2\, du_1 \wedge du_2 - 2\, du_1 \wedge dv_1 + 2\, du_2 \wedge dv_2.$$

For the tensor field $A = -\nabla\xi$, we have

$$A = 2\, du_1 \otimes \frac{\partial}{\partial v_1} + 2\, du_2 \otimes \frac{\partial}{\partial v_2},$$

so that $A$ is of rank 2 everywhere on $M$. The covariant derivative $\nabla\varphi$, which is nonzero, satisfies the relation (20) and therefore by Theorem 3, $M$ has para-Kählerian leaves. $M$ is weakly para-cosymplectic since the metric $g$ is flat. □

## 6. Curvature Identities

PROPOSITION 6. *For any almost para-cosymplectic manifold, we have*

$$[R_{Z\varphi X}, \varphi] + [R_{\varphi ZX}, \varphi] - [R_{\varphi Z\varphi X}, \varphi]\varphi - [R_{ZX}, \varphi]\varphi = \nabla_{\varphi N(Z,X)}\varphi + \eta \otimes (R_{\varphi Z\varphi X}\xi + R_{ZX}\xi). \tag{27}$$

PROOF. At first, by (8), we have

$$(\nabla_X\varphi)\varphi Y - (\nabla_{\varphi X}\varphi)Y = \eta(Y)AX.$$

Next, differentiating this covariantly, we obtain

$$(28)\quad (\nabla^2_{ZX}\varphi)\varphi Y - (\nabla^2_{Z\varphi X}\varphi)Y$$
$$= (\nabla_{(\nabla_Z\varphi)X}\varphi)Y - (\nabla_X\varphi)(\nabla_Z\varphi)Y - g(AZ, Y)AX + \eta(Y)(\nabla_Z A)X.$$

Replacing in (28) $Z, X, Y$ by $\varphi V, U, \varphi W$, respectively, we obtain

$$(29)\quad (\nabla^2_{\varphi VU}\varphi)W - \eta(W)(\nabla^2_{\varphi VU}\varphi)\xi - (\nabla^2_{\varphi V\varphi U}\varphi)\varphi W$$
$$= (\nabla_{(\nabla_{\varphi V}\varphi)U}\varphi)\varphi W - (\nabla_U\varphi)(\nabla_{\varphi V}\varphi)\varphi W - g(AV, W)AU.$$

On the other hand, using (7) and (9), we find

$$(\nabla_U\varphi)(\nabla_{\varphi V}\varphi)\varphi W = (\nabla_U\varphi)(\nabla_V\varphi)W + \eta(W)(\nabla_U\varphi)A\varphi V,$$
$$(\nabla_{(\nabla_{\varphi V}\varphi)U}\varphi)\varphi W = (\nabla_{-\varphi(\nabla_V\varphi)U+g(AV,U)\xi}\varphi)\varphi W = -(\nabla_{\varphi(\nabla_V\varphi)U}\varphi)\varphi W$$
$$= -(\nabla_{(\nabla_V\varphi)U}\varphi)W - \eta(W)A\varphi(\nabla_V\varphi)U.$$

By these relations, (29) turns into

$$(30)\quad (\nabla^2_{\varphi VU}\varphi)W - (\nabla^2_{\varphi V\varphi U}\varphi)\varphi W$$
$$= -(\nabla_{(\nabla_V\varphi)U}\varphi)W - (\nabla_U\varphi)(\nabla_V\varphi)W - g(AV, W)AU$$
$$+ \eta(W)((\nabla^2_{\varphi VU}\varphi)\xi - (\nabla_U\varphi)A\varphi V - A\varphi(\nabla_V\varphi)U).$$

Putting $V = X$, $U = Z$, $W = Y$ in (30) and adding the obtained relation to (28), we have

$$-[R_{Z\varphi X}, \varphi]Y - (\nabla^2_{\varphi X\varphi Z}\varphi)\varphi Y + (\nabla^2_{ZX}\varphi)\varphi Y$$
$$= -(\nabla_{(\nabla_X\varphi)Z}\varphi)Y + (\nabla_{(\nabla_Z\varphi)X}\varphi)Y - (\nabla_X\varphi)(\nabla_Z\varphi)Y - (\nabla_Z\varphi)(\nabla_X\varphi)Y$$
$$- g(AX, Y)AZ - g(AZ, Y)AX + \eta(Y)((\nabla^2_{\varphi XZ}\varphi)\xi - (\nabla_Z\varphi)A\varphi X$$
$$- A\varphi(\nabla_X\varphi)Z + (\nabla_Z A)X).$$

Antisymmetrization of the last equality with respect to $Z, X$ and application of (14), gives

$$(31)\quad [R_{Z\varphi X}, \varphi]Y + [R_{\varphi ZX}, \varphi]Y - [R_{\varphi Z\varphi X}, \varphi]\varphi Y - [R_{ZX}, \varphi]\varphi Y$$
$$= -2(\nabla_{(\nabla_Z\varphi)X}\varphi)Y + 2(\nabla_{(\nabla_X\varphi)Z}\varphi)Y - \eta(Y)S(Z, X)$$
$$= (\nabla_{\varphi N(Z,X)}\varphi)Y - \eta(Y)S(Z, X),$$

where $S$ is a $(1,2)$ skew-symmetric tensor field. Put $Y=\xi$ in (31) and find

$$[R_{Z\varphi X},\varphi]\xi+[R_{\varphi ZX},\varphi]\xi-(\nabla_{\varphi N(Z,X)}\varphi)\xi=-S(Z,X).$$

This implies $g(S(Z,X),\xi)=0$. Having this in mind and projecting (31) onto $\xi$, we find

$$[R_{Z\varphi X},\varphi]\xi+[R_{\varphi ZX},\varphi]\xi+\varphi[R_{\varphi Z\varphi X},\varphi]\xi+\varphi[R_{ZX},\varphi]\xi=(\nabla_{\varphi N(Z,X)}\varphi)\xi,$$

which having substituted to the previous equation, gives

$$S(Z,X)=\varphi[R_{\varphi Z\varphi X},\varphi]\xi+\varphi[R_{ZX},\varphi]\xi=-R_{\varphi Z\varphi X}\xi-R_{ZX}\xi.$$

But then this reduces (31) to (27). □

Let $Ric$ and $Ric^*$ be the Ricci and $*$-Ricci tensors defined by

$$Ric(X,Y)=Tr\{Z\mapsto R_{ZX}Y\},\quad Ric^*(X,Y)=Tr\{Z\mapsto \varphi R_{ZX}\varphi Y\}.$$

Let $\widetilde{Ric},\widetilde{Ric^*}$ be the Ricci and $*$-Ricci operators and $r,r^*$ be the scalar and $*$-scalar curvatures given by

$$Ric(X,Y)=g(\widetilde{Ric}X,Y),\quad Ric^*(X,Y)=g(\widetilde{Ric^*}X,Y),$$
$$r=Tr(\widetilde{Ric}),\quad r^*=Tr(\widetilde{Ric^*}).$$

THEOREM 6. *For an almost para-cosymplectic manifold, we have*

(32) $$R_{Z\varphi X}\varphi Y-\varphi R_{Z\varphi X}Y+R_{\varphi ZX}\varphi Y-\varphi R_{\varphi ZX}Y$$
$$-R_{\varphi Z\varphi X}Y+\varphi R_{\varphi Z\varphi X}\varphi Y-R_{ZX}Y+\varphi R_{ZX}\varphi Y=(\nabla_{\varphi N(Z,X)}\varphi)Y,$$

(33) $$Ric^*(X,Y)+Ric^*(Y,X)-Ric(X,Y)+Ric(\varphi X,\varphi Y)$$
$$+\frac{1}{2}(R_{\xi XY\xi}-R_{\xi\varphi X\varphi Y\xi})+\sum_{j=0}^{2n}\varepsilon_j g((\nabla_{E_j}\varphi)X,(\nabla_{E_j}\varphi)Y)=0,$$

(34) $$r^*-r+Ric(\xi,\xi)+\frac{1}{2}g(\nabla\varphi,\nabla\varphi)=0.$$

*where* $(E_j, 0\le j\le 2n)$ *is an orthonormal frame,* $\varepsilon_j$*'s are the indicators of* $E_j$*'s and*

$$g(\nabla\varphi,\nabla\varphi)=\sum_{i,j=0}^{2n}\varepsilon_i\varepsilon_j g((\nabla_{E_j}\varphi)E_i,(\nabla_{E_j}\varphi)E_i).$$

PROOF. Formula (32) is in fact a direct consequence of the identity (27).

Taking the trace of (32) with respect to $Z$, we find

$$(35)\qquad 2Ric^*(X,Y) - 2Ric(X,Y) + 2Ric(\varphi X, \varphi Y) - 2Tr\{Z \mapsto \varphi R_{Z\varphi X} Y\} + R_{\xi XY\xi} - R_{\xi\varphi X\varphi Y\xi} = Tr\{Z \mapsto (\nabla_{\varphi N(Z,X)}\varphi)Y\}.$$

Let $(E_i)$ be a local orthonormal frame and compute

$$Tr\{Z \mapsto \varphi R_{Z\varphi X} Y\} = -\sum_{i=0}^{2n} \varepsilon_i R_{\varphi E_i Y \varphi X E_i} = -Ric^*(Y, X).$$

By (16), we have

$$Tr\{Z \mapsto (\nabla_{\varphi N(Z,X)}\varphi)Y\} = \sum_{i=0}^{2n} \varepsilon_i g((\nabla_{\varphi N(E_i,X)}\varphi)Y, E_i)$$

$$= \frac{1}{2}\sum_{i=0}^{2n} \varepsilon_i g(N(E_i,X), N(E_i,Y)) = -\frac{1}{2}\sum_{i=0}^{2n} \varepsilon_i g(\varphi N(E_i,X), \varphi N(E_i,Y))$$

$$= -\frac{1}{2}\sum_{i,j=0}^{2n} \varepsilon_i\varepsilon_j g(E_j, \varphi N(E_i,X)) g(E_j, \varphi N(E_i,Y))$$

$$= -2\sum_{i,j=0}^{2n} \varepsilon_i\varepsilon_j g((\nabla_{E_j}\varphi)X, E_i) g((\nabla_{E_j}\varphi)Y, E_i) = -2\sum_{j=0}^{2n} \varepsilon_j g((\nabla_{E_j}\varphi)X, (\nabla_{E_j}\varphi)Y).$$

Applying these relations into (35), we find (33).

Taking the trace of (33) with respect to $g$, we obtain

$$(36)\qquad 2r^* - r + Tr_g\{(X,Y) \mapsto Ric(\varphi X, \varphi Y)\} + \frac{1}{2}Ric(\xi,\xi) - \frac{1}{2}Tr_g\{(X,Y) \mapsto R_{\xi\varphi X\varphi Y\xi}\} + g(\nabla\varphi, \nabla\varphi) = 0.$$

On the other hand, we find

$$Tr_g\{(X,Y) \mapsto Ric(\varphi X, \varphi Y)\} = -Tr\{X \mapsto \varphi\widetilde{Ric}\varphi X\} = -Tr\{X \mapsto \widetilde{Ric}\varphi^2 X\} = -r + Ric(\xi,\xi).$$

Moreover

$$Tr_g\{(X,Y) \mapsto R_{\xi\varphi X\varphi Y\xi}\} = -Tr\{X \mapsto \varphi R_{\varphi X\xi}\xi\} = -Tr\{X \mapsto R_{X\xi}\xi\} = -Ric(\xi,\xi).$$

The last two relations reduce (36) to (34). □

FINAL REMARKS. Certain of our results are para-cosymplectic analogies of theorems concerning almost cosymplectic manifolds proved in [12], [13], [7].

## References


[1] C.-L. Bejan, Almost parahermitian structures on the tangent bundle of an almost para-cohermitian manifold, In: The Proceedings of the Fifth National Seminar of Finsler and Lagrange Spaces (Braşov, 1988), pp. 105–109, Soc. Ştiinţe Mat. R.S. România, Bucharest, 1989.

[2] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress Math. Vol. **203**, Birkhäuser, Boston-Basel-Berlin, 2002.

[3] K. Buchner and R. Rosca, Variétés para-coKähleriennes à champ concirculaire horizontale, C. R. Acad. Sci. Paris **285** (1977), Ser. A, 723–726.

[4] K. Buchner and R. Rosca, Co-isotropic submanifolds of a para-coKählerian manifold with concircular structure vector field, J. Geometry **25** (1985), 164–177.

[5] V. Cruceanu, P. Fortuny and P. M. Gadea, A survey on paracomplex geometry, Rocky Mountain J. Math. **26** (1996), 83–115.

[6] V. Cruceanu, P. M. Gadea and J. Muñoz Masqué, Para-Hermitian and para-Kähler manifolds, Quaderni dell'Istituto di Matematica, Facoltà di Economia, Università di Messina, No. **1** (1995), 1–72.

[7] P. Dacko and Z. Olszak, On conformally flat almost cosymplectic manifolds with Kählerian leaves, Rend. Sem. Mat. Univ. Pol. Torino, Vol. **56** (1998), 89–103.

[8] S. Erdem, On almost (para)contact (hyperbolic) metric manifolds and harmonicity of $(\varphi, \varphi')$-holomorphic maps between them, Houston J. Math. **28** (2002), 21–45.

[9] S. I. Goldberg and K. Yano, Integrability of almost cosymplectic structures, Pacific J. Math. **31** (1969), 373–382.

[10] S. Kobayashi and K. Nomizu, Foundations of differentiable manifolds, Vol. I, Interscience Publishing, New York-London, 1963.

[11] I. Mihai, R. Rosca and L. Verstraelen, Some aspects of the differential geometry of vector fields. On skew symmetric Killing and conformal vector fields, and their relations to various geometrical structures, Centre for Pure and Applied Differential Geometry, Katholieke Universiteit Leuven/Brussel, 1996.

[12] Z. Olszak, On almost cosymplectic manifolds, Kodai Math. J. **4** (1981), 239–250.

[13] Z. Olszak, Almost cosymplectic manifolds with Kählerian leaves, Tensor N.S. **46** (1987), 117–124.

[14] R. Roşca and L. Vanhecke, Sür une variété presque paracokählérienne munie d'une connexion self-orthogonale involutive, Ann. Şti. Univ. "Al. I. Cuza" Iaşi **22** (1976), 49–58.

[15] F. Tricerri and L. Vanhecke, Homogeneous structures on Riemannian manifolds, London Math. Soc. Lect. Notes Ser. **83**, Cambridge Univ. Press, Cambridge, 1983.



Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław
Poland